DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU and CTB2400)
Thursday July 3 2014, 18:30-21:30

1. We consider the following method for the integration of the initial value problem $y' = f(t,y)$, $y(t_0) = y_0$

$$\begin{cases} w_{n+1}^* = w_n + hf(t_n, w_n) \\ \\ w_{n+1} = w_n + h\left(a_1 f(t_n, w_n) + a_2 f(t_{n+1}, w_{n+1}^*)\right) \end{cases} \tag{1}$$

a Show that the local truncation error of the above method has order $O(h)$ if $a_1 + a_2 = 1$. Which value for a_1 and a_2 will give a local truncation error of order $O(h^2)$? (3 pt.)

b Demonstrate that for general values of a_1 and a_2 the amplification factor is given by

$$Q(h\lambda) = 1 + (a_1 + a_2)h\lambda + a_2(h\lambda)^2. \tag{2}$$

(2 pt.)

c Consider $\lambda < 0$ and $(a_1 + a_2)^2 - 8a_2 < 0$. Derive the condition for stability, to be fullfilled by h. (2 pt.)

d Consider the following system

$$\begin{cases} y_1' = -2_1 - y_1 y_2, \\ y_2' = 2y_1 y_2 - y_2^2, \end{cases} \tag{3}$$

Show that the Jacobian of the right hand side of the above system (which is used for the linearization of the above system) for the initial condition $y_1(0) = 2$ and $y_2(0) = 2$ is given by

$$\begin{pmatrix} -4 & -2 \\ 4 & 0 \end{pmatrix}.$$

(1.5 pt.)

e We apply the numerical method in equation (1) for the case that $a_1 = a_2 = 1/2$ to system (3). Is the method stable near the initial condition $y_1(0) = 2$ and $y_2(0) = 2$, and step size $h = \frac{1}{2}$ (+ motivation)? (1.5 pt.)

[0]**please turn over, For the answers of this test we refer to: http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html**

2. Consider the following boundary value problem:

$$\begin{cases} -\frac{d^2y(x)}{dx^2} + y(x) = 2e^x, & x \in (0,1), \\ y(0) = 2, & y'(1) = 0. \end{cases} \tag{4}$$

a Show that

$$y(x) = e^x(2-x)$$

is the exact solution to the boundary value problem (4). (1 pt.)

We apply the finite difference method to approximate the solution to the above boundary value problem. Let the gridnodes be given by $x_j = jh$, with h as stepsize. Let $x_n = nh = 1$.

b Give a finite differences scheme (+ proof) for which the local truncation error is of order $O(h^2)$. *Hint:* Use a virtual gridnode for the boundary condition at $x = 1$. The discretisation matrix must be symmetric. (3pt.)

c Give the linear system of equations $A\mathbf{w} = \mathbf{b}$ that results from applying the finite difference discretisation with three (after processing the virtual gridnode) unknowns ($h = 1/3$). (2pt.)

d Answer the following questions **without** computing the numerical solution! The numerical solution is called *nodally exact* if

$$e_j = w_j - y(x_j) = 0 \quad \text{for all } j.$$

Can the numerical solution $\mathbf{w} = (w_1, w_2, w_3)$ resulting from the second-order finite difference scheme be nodally exact? Is it possible to find a higher order finite difference formula so that the resulting numerical solution becomes nodally exact? (1pt.)

Consider the Trapezoidal rule for numerical integration.

e Give the linear Lagrange interpolatory polynomial $p_1(x)$ with nodes a and b and derive the Trapezoidal rule to approximate $\int_a^b f(x)\,\mathrm{d}x$ by the use of $p_1(x)$. (1.5pt.)

f Derive that an upper bound of the truncation error of the Trapezoidal rule applied to the interval $[a,b]$ is given by

$$\frac{1}{12}(b-a)^3 \max_{x\in[a,b]} |f''(x)|, \tag{5}$$

if the second order derivative of f is continuous over $[a,b]$. *Hint:* The error for linear interpolation over nodes a and b is given by

$$f(x) - p_1(x) = \frac{1}{2}(x-a)(x-b)f''(\chi), \text{ for some } \chi \in (a,b).$$

(1.5pt.)