1. (a) The local truncation error is given by

$$\tau_{n+1}(h) = \frac{y_{n+1} - z_{n+1}}{h},$$

in which we determine $y_{n+1}$ by the use of Taylor expansions around $t_n$:

$$y_{n+1} = y_n + hy'(t_n) + \frac{h^2}{2} y''(t_n) + O(h^3).$$

We bear in mind that

$$y'(t_n) = f(t_n, y_n)$$

$$y''(t_n) = \frac{df(t_n, y_n)}{dt} = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y} y'(t_n) = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y} f(t_n, y_n).$$

Hence

$$y_{n+1} = y_n + hy'(t_n) + \frac{h^2}{2} \left( \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y} f(t_n, y_n) \right) + O(h^3).$$

After substitution of the predictor $z^*_n = y_n + hf(t_n, y_n)$ into the corrector, and after using a Taylor expansion around $(t_n, y_n)$, we obtain for $z_{n+1}$

$$z_{n+1} = y_n + \frac{h}{2} (f(t_n, y_n) + f(t_n + h, y_n + hf(t_n, y_n))) =

y_n + \frac{h}{2} \left( f(t_n, y_n) + f(t_n, y_n) + h \left( \frac{\partial f(t_n, y_n)}{\partial t} + f(t_n, y_n) \frac{\partial f(t_n, y_n)}{\partial y} \right) + O(h^2) \right).$$

Herewith, one obtains

$$y_{n+1} - z_{n+1} = O(h^3), \text{ and hence } \tau_{n+1}(h) = \frac{O(h^3)}{h} = O(h^2).$$
(b) Let \( x_1 = y \) and \( x_2 = y' \), then \( y'' = x'_2 \), and hence
\[
\begin{align*}
x'_2 + 4x_2 + 3x_1 &= \cos(t), \\
x_2 &= x'_1.
\end{align*}
\]  
We write this as
\[
\begin{align*}
x'_1 &= x_2, \\
x'_2 &= -3x_1 - 4x_2 + \cos(t).
\end{align*}
\]  
Finally, this is represented in the following matrix-vector form:
\[
\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos(t) \end{pmatrix}.
\]  
In which, we have the following matrix \( A = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} \) and \( f = \begin{pmatrix} 0 \\ \cos(t) \end{pmatrix} \). The initial conditions are defined by \( \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \).

(c) Application of the integration method to the system \( x' = Ax + f \), gives
\[
\begin{align*}
w^*_1 &= w_0 + h \left(Aw_0 + f_0\right), \\
w_1 &= w_0 + \frac{h}{2} \left(Aw_0 + f_0 + Aw^*_1 + f_1\right).
\end{align*}
\]  
With the initial condition \( w_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \) and \( h = 0.1 \), this gives the following result for the predictor
\[
w^*_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{10} \left(\begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 6/5 \\ 1 \end{pmatrix} .
\]  
The corrector is calculated as follows
\[
w_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{20} \left(\begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} 6/5 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 0 \\ \cos(1/10) \end{pmatrix} = \\
\begin{pmatrix} 1.1500 \\ 1.1698 \end{pmatrix} \]  

(d) Consider the test equation \( y' = \lambda y \), then one gets
\[
\begin{align*}
w^{*}_{n+1} &= w_n + h\lambda w_n = (1 + h\lambda)w_n, \\
w_{n+1} &= w_n + \frac{h}{2} (\lambda w_n + \lambda w^{*}_{n+1}) = \\
&= w_n + \frac{h}{2} (\lambda w_n + \lambda (w_n + h\lambda w_n)) = (1 + h\lambda + \frac{(h\lambda)^2}{2})w_n.
\end{align*}
\]  
Hence the amplification factor is given by
\[
Q(h\lambda) = 1 + h\lambda + \frac{(h\lambda)^2}{2}.
\]
(e) First, we determine the eigenvalues of the matrix $A$. Subsequently, the eigenvalues are substituted into the amplification factor. The eigenvalues of the matrix $A$ are given by $\lambda_1 = -1$ and $\lambda_2 = -3$. We first check the amplification factor of $\lambda_1 = -1$:

$$-1 \leq 1 - h + \frac{1}{2}h^2 \leq 1 \quad (15)$$

The first inequality leads to

$$0 \leq 2 - h + \frac{1}{2}h^2$$

Since the discriminant of this equation is equal to $1 - 4 \times \frac{1}{2} \times 2 = -3$ the inequality always holds. The second inequality leads to

$$-h + \frac{1}{2}h^2 \leq 0$$

so

$$\frac{1}{2}h^2 \leq h$$

which implies

$$h \leq 2$$

Now we check the amplification factor of $\lambda_2 = -3$:

$$-1 \leq 1 - 3h + \frac{1}{2}9h^2 \leq 1 \quad (16)$$

The first inequality leads to

$$0 \leq 2 - 3h + \frac{1}{2}9h^2$$

Since the discriminant of this equation is equal to $9 - 4 \times \frac{9}{2} \times 2 = -27$ the inequality always holds. The second inequality leads to

$$-3h + \frac{9}{2}h^2 \leq 0$$

so

$$\frac{3}{2}h^2 \leq h$$

which implies

$$h \leq \frac{2}{3}$$

So the integration method is stable if $h \leq \frac{2}{3}$.
2. (a) The first order backward difference formula for the first derivative is given by
\[ d'(t) \approx \frac{d(t) - d(t - h)}{h}. \]

Using \( t = 20 \), and \( h = 10 \) the approximation of the velocity is
\[ \frac{d(20) - d(10)}{10} = \frac{100 - 40}{10} = 6 \text{ (m/s)}. \]

(b) Taylor polynomials are:
\[
\begin{align*}
d(0) &= d(2h) - 2hd'(2h) + 2h^2d''(2h) - \frac{(2h)^3}{6}d'''(\xi_0), \\
d(h) &= d(2h) - hd'(2h) + \frac{h^2}{2}d''(2h) - \frac{h^3}{6}d'''(\xi_1), \\
d(2h) &= d(2h).
\end{align*}
\]

We know that \( Q(h) = \frac{\alpha_0}{h}d(0) + \frac{\alpha_1}{h}d(h) + \frac{\alpha_2}{h}d(2h) \), which should be equal to \( d'(2h) + O(h^2) \). This leads to the following conditions:
\[
\begin{align*}
\frac{\alpha_0}{h} + \frac{\alpha_1}{h} + \frac{\alpha_2}{h} &= 0, \\
-2\alpha_0 - \alpha_1 &= 1, \\
2\alpha_0 h + \frac{1}{2}\alpha_1 h &= 0.
\end{align*}
\]

(c) The truncation error follows from the Taylor polynomials:
\[
d'(2h) - Q(h) = d'(2h) - \frac{d(0) - 4d(h) + 3d(2h)}{2h} = \frac{8h^3}{6}d'''(\xi_0) - 4(\frac{h^3}{6}d'''(\xi_1)) = \frac{1}{3}h^2d'''(\xi).
\]

Using the new formula with \( h = 10 \) we obtain the estimate:
\[
\frac{d(0) - 4d(10) + 3d(20)}{20} = \frac{0 - 4 \times 40 + 3 \times 100}{20} = 7 \text{ (m/s)}. \]

(d) To estimate the measuring error we note that
\[
|Q(h) - \hat{Q}(h)| = \left| \frac{d(0) - \hat{d}(0) - 4(d(h) - \hat{d}(h)) + 3(d(2h) - \hat{d}(2h))}{2h} \right|
\leq \frac{|d(0) - \hat{d}(0)| + 4|d(h) - \hat{d}(h)| + 3|d(2h) - \hat{d}(2h)|}{2h} = \frac{4\epsilon}{h},
\]
so \( C_1 = 4 \).
(e) The method of Newton-Raphson is based on linearization around the iterate \( p_n \). This is given by

\[
L(x) = f(p_n) + (x - p_n)f'(p_n).
\]

Next, we determine \( p_{n+1} \) such that \( L(p_{n+1}) = 0 \), that is

\[
f(p_n) + (p_{n+1} - p_n)f'(p_n) = 0 \iff p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}, \quad f'(p_n) \neq 0.
\]

This result can also be proved graphically, see book, chapter 4.

(f) We have \( f(x) = e^{\sin(x)} - \frac{1}{e} \), so \( f'(x) = \cos(x) e^{\sin(x)} \) and hence

\[
p_{n+1} = p_n - \frac{e^{\sin(p_n)} - \frac{1}{e}}{\cos(p_n) e^{\sin(p_n)}}.
\]

With the initial value \( p_0 = \pi \), this gives

\[
p_1 = \pi - \frac{e^0 - \frac{1}{e}}{-1 \times e^0} = \pi + 1 - \frac{1}{e} \approx 3.77.
\]

With the initial value \( p_0 = \frac{3}{2}\pi \), this gives

\[
p_1 = \frac{3}{2}\pi - \frac{e^{-1} - \frac{1}{e}}{0} = \frac{3}{2}\pi - \frac{0}{0}.
\]

In the recursion, one divides by zero. Division by zero does not make any sense, so \( p_0 = \frac{3}{2}\pi \) is not a suitable starting value. Geometrically, one may remark that the tangent is horizontal for \( p_0 = \frac{3}{2}\pi \). However, \( f(p_0) = f\left(\frac{3}{2}\pi\right) = 0 \) so that a practical Newton-Raphson method would not start iterating but return \( p_0 = \frac{3}{2}\pi \) as root.