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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU AESB2210)
Thursday January 29 2015, 18:30-21:30

1. (a) The local truncation error is given by

$$\tau_{n+1}(h) = \frac{y_{n+1} - z_{n+1}}{h}, \tag{1}$$

in which we determine y_{n+1} by the use of Taylor expansions around t_n:

$$y_{n+1} = y_n + hy'(t_n) + \frac{h^2}{2}y''(t_n) + O(h^3). \tag{2}$$

We bear in mind that

$$\begin{aligned} y'(t_n) & = f(t_n, y_n) \\ y''(t_n) & = \frac{df(t_n, y_n)}{dt} = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y}y'(t_n) = \\ & \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y}f(t_n, y_n). \end{aligned} \tag{3}$$

Hence

$$y_{n+1} = y_n + hy'(t_n) + \frac{h^2}{2}\left(\frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y}f(t_n, y_n)\right) + O(h^3). \tag{4}$$

After substitution of the predictor $z^*_{n+1} = y_n + hf(t_n, y_n)$ into the corrector, and after using a Taylor expansion around (t_n, y_n), we obtain for z_{n+1}

$$\begin{aligned} z_{n+1} &= y_n + \tfrac{h}{2}\left(f(t_n, y_n) + f(t_n + h, y_n + hf(t_n, y_n))\right) = \\ & y_n + \frac{h}{2}\left(f(t_n, y_n) + f(t_n, y_n) + h(\frac{\partial f(t_n, y_n)}{\partial t} + f(t_n, y_n)\frac{\partial f(t_n, y_n)}{\partial y}) + O(h^2)\right). \end{aligned} \tag{5}$$

Herewith, one obtains

$$y_{n+1} - z_{n+1} = O(h^3), \text{ and hence } \tau_{n+1}(h) = \frac{O(h^3)}{h} = O(h^2). \tag{6}$$

(b) Let $x_1 = y$ and $x_2 = y'$, then $y'' = x_2'$, and hence

$$\begin{aligned} x_2' + 4x_2 + 3x_1 &= \cos(t), \\ x_2 &= x_1'. \end{aligned} \tag{7}$$

We write this as

$$\begin{aligned} x_1' &= x_2, \\ x_2' &= -3x_1 - 4x_2 + \cos(t). \end{aligned} \tag{8}$$

Finally, this is represented in the following matrix-vector form:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos(t) \end{pmatrix}. \tag{9}$$

In which, we have the following matrix $A = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}$ and $f = \begin{pmatrix} 0 \\ \cos(t) \end{pmatrix}$.

The initial conditions are defined by $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(c) Application of the integration method to the system $\underline{x}' = A\underline{x} + \underline{f}$, gives

$$\begin{aligned} \underline{w}_1^* &= \underline{w}_0 + h\left(A\underline{w}_0 + \underline{f}_0\right), \\ \underline{w}_1 &= \underline{w}_0 + \tfrac{h}{2}\left(A\underline{w}_0 + f_0 + A\underline{w}_1^* + \underline{f}_1\right). \end{aligned} \tag{10}$$

With the initial condition $\underline{w}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $h = 0.1$, this gives the following result for the predictor

$$\underline{w}_1^* = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{10}\left(\begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 6/5 \\ 1 \end{pmatrix}. \tag{11}$$

The corrector is calculated as follows

$$\begin{aligned} \underline{w}_1 &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \tfrac{1}{20}\left(\begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} 6/5 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos(\frac{1}{10}) \end{pmatrix}\right) = \\ &= \begin{pmatrix} 1.1500 \\ 1.1698 \end{pmatrix} \end{aligned} \tag{12}$$

(d) Consider the test equation $y' = \lambda y$, then one gets

$$\begin{aligned} w_{n+1}^* &= w_n + h\lambda w_n = (1 + h\lambda)w_n, \\ w_{n+1} &= w_n + \frac{h}{2}(\lambda w_n + \lambda w_{n+1}^*) = \\ &= w_n + \frac{h}{2}(\lambda w_n + \lambda(w_n + h\lambda w_n)) = (1 + h\lambda + \frac{(h\lambda)^2}{2})w_n. \end{aligned} \tag{13}$$

Hence the amplification factor is given by

$$Q(h\lambda) = 1 + h\lambda + \frac{(h\lambda)^2}{2}. \tag{14}$$

(e) First, we determine the eigenvalues of the matrix A. Subsequently, the eigenvalues are substituted into the amplification factor. The eigenvalues of the matrix A are given by $\lambda_1 = -1$ and $\lambda_2 = -3$. We first check the amplification factor of $\lambda_1 = -1$:

$$-1 \leq 1 - h + \frac{1}{2}h^2 \leq 1 \tag{15}$$

The first inequality leads to

$$0 \leq 2 - h + \frac{1}{2}h^2$$

Since the discriminant of this equation is equal to $1 - 4 * \frac{1}{2} * 2 = -3$ the inequality always holds. The second inequality leads to

$$-h + \frac{1}{2}h^2 \leq 0$$

so

$$\frac{1}{2}h^2 \leq h$$

which implies

$$h \leq 2$$

Now we check the amplification factor of $\lambda_2 = -3$:

$$-1 \leq 1 - 3h + \frac{1}{2}9h^2 \leq 1 \tag{16}$$

The first inequality leads to

$$0 \leq 2 - 3h + \frac{1}{2}9h^2$$

Since the discriminant of this equation is equal to $9 - 4 * \frac{9}{2} * 2 = -27$ the inequality always holds. The second inequality leads to

$$-3h + \frac{9}{2}h^2 \leq 0$$

so

$$\frac{3}{2}h^2 \leq h$$

which implies

$$h \leq \frac{2}{3}$$

So the integration method is stable if $h \leq \frac{2}{3}$.

2. (a) The first order backward difference formula for the first derivative is given by

$$d'(t) \approx \frac{d(t) - d(t-h)}{h}.$$

Using $t = 20$, and $h = 10$ the approximation of the velocity is

$$\frac{d(20) - d(10)}{10} = \frac{100 - 40}{10} = 6 \text{ (m/s)}.$$

(b) Taylor polynomials are:

$$\begin{aligned} d(0) &= d(2h) - 2hd'(2h) + 2h^2 d''(2h) - \frac{(2h)^3}{6} d'''(\xi_0)\,, \\ d(h) &= d(2h) - hd'(2h) + \frac{h^2}{2} d''(2h) - \frac{h^3}{6} d'''(\xi_1)\,, \\ d(2h) &= d(2h). \end{aligned}$$

We know that $Q(h) = \frac{\alpha_0}{h} d(0) + \frac{\alpha_1}{h} d(h) + \frac{\alpha_2}{h} d(2h)$, which should be equal to $d'(2h) + O(h^2)$. This leads to the following conditions:

$$\begin{array}{rcrcrcl} \frac{\alpha_0}{h} & + & \frac{\alpha_1}{h} & + & \frac{\alpha_2}{h} & = & 0\,, \\ -2\alpha_0 & - & \alpha_1 & & & = & 1\,, \\ 2\alpha_0 h & + & \frac{1}{2}\alpha_1 h & & & = & 0\,. \end{array}$$

(c) The truncation error follows from the Taylor polynomials:

$$d'(2h) - Q(h) = d'(2h) - \frac{d(0) - 4d(h) + 3d(2h)}{2h} = \frac{\frac{8h^3}{6} d'''(\xi_0) - 4(\frac{h^3}{6} d'''(\xi_1))}{2h} = \frac{1}{3} h^2 d'''(\xi).$$

Using the new formula with $h = 10$ we obtain the estimate:

$$\frac{d(0) - 4d(10) + 3d(20)}{20} = \frac{0 - 4 \times 40 + 3 \times 100}{20} = 7 \text{ (m/s)}.$$

(d) To estimate the measuring error we note that

$$|Q(h) - \hat{Q}(h)| = |\frac{(d(0) - \hat{d}(0)) - 4(d(h) - \hat{d}(h)) + 3(d(2h) - \hat{d}(2h))}{2h}|$$

$$\leq \frac{|d(0) - \hat{d}(0)| + 4|d(h) - \hat{d}(h)| + 3|d(2h) - \hat{d}(2h)|}{2h} = \frac{4\epsilon}{h},$$

so $C_1 = 4$.

(e) The method of Newton-Raphson is based on linearization around the iterate p_n. This is given by

$$L(x) = f(p_n) + (x - p_n)f'(p_n).$$

Next, we determine p_{n+1} such that $L(p_{n+1}) = 0$, that is

$$f(p_n) + (p_{n+1} - p_n)f'(p_n) = 0 \Leftrightarrow p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}, \qquad f'(p_n) \neq 0.$$

This result can also be proved graphically, see book, chapter 4.

(f) We have $f(x) = e^{\sin(x)} - \frac{1}{e}$, so $f'(x) = \cos(x)e^{\sin(x)}$ and hence

$$p_{n+1} = p_n - \frac{e^{\sin(p_n)} - \frac{1}{e}}{\cos(p_n)e^{\sin(p_n)}}.$$

With the initial value $p_0 = \pi$, this gives

$$p_1 = \pi - \frac{e^0 - \frac{1}{e}}{-1 \times e^0} = \pi + 1 - \frac{1}{e} \approx 3.77.$$

With the initial value $p_0 = \frac{3}{2}\pi$, this gives

$$p_1 = \frac{3}{2}\pi - \frac{e^{-1} - \frac{1}{e}}{0} = \frac{3}{2}\pi - \frac{0}{0}.$$

In the recursion, one divides by zero. Division by zero does not make any sense, so $p_0 = \frac{3}{2}\pi$ is not a suitable starting value. Geometrically, one may remark that the tangent is horizontal for $p_0 = \frac{3}{2}\pi$. However, $f(p_0) = f(\frac{3}{2}\pi) = 0$ so that a practical Newton-Raphson method would not start iterating but return $p_0 = \frac{3}{2}\pi$ as root.