DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU AESB2210 CTB2400)

Thursday July 2 2015, 18:30-21:30

1. We consider the following predictor-corrector method for the integration of the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$:

$$\begin{aligned} w^*_{n+1} &= w_n + \Delta t f(t_n, w_n), \\ w_{n+1} &= w_n + \Delta t \left((1-\mu) f(t_n, w_n) + \mu f(t_{n+1}, w^*_{n+1}) \right), \end{aligned} \tag{1}$$

where Δt, μ and w_n respectively denote the time step, a real number ($0 \leq \mu \leq 1$), and the numerical solution at time t_n.

(a) Show that the local truncation error of the abovementioned method is of order $O(\Delta t)$, if $0 \leq \mu \leq 1$ and of order $O((\Delta t)^2)$, if $\mu = \frac{1}{2}$ (*Note that this has to be demonstrated for the general differential equation* $y' = f(t, y)$). (3 pt)

(b) Demonstrate that the amplification factor of the abovementioned method, is given by

$$Q(\lambda \Delta t) = 1 + \lambda \Delta t + \mu (\lambda \Delta t)^2.$$

(2 pt)

(c) We consider the following system of non linear differential equations:

$$\begin{aligned} x_1' &= -x_1 + \cos x_1 + 2x_2 + t, \; x_1(0) = 0, \\ x_2' &= x_1 - x_2^2, \; x_2(0) = 1. \end{aligned} \tag{2}$$

Do one step with the method given in (1) with $\Delta t = \frac{1}{2}$ and $\mu = \frac{1}{2}$. (1 pt)

(d) Show that the Jacobian of the right-hand side of (2) at $t = 0$ is given by:

$$\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}.$$

(1 pt)

(e) Choose $\mu = 0$. For which values of Δt is the method applied to (2) stable at $t = 0$? Answer the same question for $\mu = \frac{1}{2}$. (3 pt)

[0] **please turn over, For the answers of this test we refer to: http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html**

2. We consider the one-dimensional convection–diffusion equation with Dirichlet boundary conditions:

$$\begin{cases} -u'' + u' = 1, & 0 < x < 1, \\ u(0) = 0, & u(1) = 0, \end{cases} \tag{3}$$

where $u = u(x)$, $u' = \frac{du}{dx}$ and $u'' = \frac{d^2u}{dx^2}$

(a) Show that

$$u(x) = x - \frac{1 - e^x}{1 - e} \tag{4}$$

is the exact solution to the boundary value problem (3). (1 pt.)

(b) We solve the boundary value problem (3) using finite differences, upon setting $x_j = j\Delta x$, $(n+1)\Delta x = 1$, where Δx denotes the uniform stepsize. Give a discretization method (+proof) where the truncation error is of order $O((\Delta x)^2)$. Take the boundary conditions into account. (2 pt.)

(c) Give a (physical or mathematical) motivation why oscillatory numerical solutions to (3) should be considered unreliable. (1pt.)

(d) Use a step size of $\Delta x = 1/4$ to derive the system of equations $Ay = b$. Take care of the boundary conditions. The system must have three unknowns and three equations, i.e. A is a 3×3 matrix and y and b are 1×3 column vectors. You do **not** have to solve this system. (2 pt.)

(e) The following iteration process is given $x_{n+1} = g(x_n)$, with

$$g(x_n) = x_n + h(x_n)(x_n^3 - 27),$$

where h is a continuous function with $h(x) \neq 0$ for each $x \neq 0$. If this process converges, to which limit p does it converge? (1 pt.)

(f) Consider three possible choices for $h(x)$:

i. $h_1(x) = -\frac{1}{x^4}$
ii. $h_2(x) = -\frac{1}{x^2}$
iii. $h_3(x) = -\frac{1}{3x^2}$

For which choice does the process not converge? For which choice is the convergence the fastest? Motivate your answer. (2 pt.)

(g) p is the root of a given function f. $\hat{f}$ is the function perturbed by measurement errors. It is given that $|\hat{f}(x) - f(x)| \leq \epsilon_{max}$ for all x. Show that the root $\hat{p}$ from $\hat{f}$ satisfies the following inequality $|\hat{p} - p| \leq \frac{\epsilon_{max}}{|f'(p)|}$. (1 pt.)