1. (a) The fourth-order Runge-Kutta method (RK4) for the differential equation \( y' = f(t, y) \) is given by the following formulas:

\[
\begin{align*}
    k_1 &= \Delta t f(t_n, w_n) \\
    k_2 &= \Delta t f(t_n + \frac{1}{2}\Delta t, w_n + \frac{1}{2}k_1) \\
    k_3 &= \Delta t f(t_n + \frac{1}{2}\Delta t, w_n + \frac{1}{2}k_2) \\
    k_4 &= \Delta t f(t_n + \Delta t, w_n + k_3)
\end{align*}
\]

\[
w_{n+1} = w_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).
\]

Derive the amplification factor \( Q(\lambda \Delta t) \) of RK4 by applying the method to the homogeneous test equation \( y' = \lambda y \). (2 pt.)

(b) Use the fact that \( y(t_{n+1}) = e^{\lambda \Delta t} y(t_n) \) holds for the exact solution of \( y' = \lambda y \) to show that RK4 solves the homogeneous test equation with a local truncation error of \( O(\Delta t^4) \). (Hint: \( e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots \)) (2 pt.)

In the following you may assume without proof that the last result also holds for systems and that, as a consequence, the global error of RK4 applied to (1) is \( O(\Delta t^4) \).

We consider the following second-order initial-value problem:

\[
y'' + py' + qy = \sin t, \quad y(0) = y_0, \quad y'(0) = y'_0.
\] (1)

(c) Write (1) as a system of first order differential equations of type

\[
y' = Ay + g(t)
\]

Give \( A \) and \( g \), and determine the eigenvalues of \( A \), for arbitrary \( p \) and \( q \). (2 pt.)

(d) Let us consider \( p=1000 \) and \( q=250001 \). Give an approximate stability condition for this case. Hint: use the drawing of the RK4-stability region on page 3. (2 pt.)

(e) Suppose you have the choice between the Trapezoidal rule and RK4 to integrate the problem given in (d). Motivate your choice as good as possible, taking into account the stability condition and the order of magnitude of the global error. (2 pt.)
2. We are interested in the numerical integration of ∫₀¹ y(x)dx with y(x) = x².

(a) Give the Rectangle Rule IR, the corresponding composed integration rule IR(h) and compute the approximate integral ∫₀¹ y(x)dx with h = 1/3. (1 pt.)
(b) Repeat part (a) for the Trapezoidal Rule (IT and IT(h)) with h = 1/3. (1 pt.)
(c) If one approximates ∫₀¹ y(x)dx, the magnitude of the error of the composed integration rules (ε_R and ε_T for the Rectangle and Trapezoidal Rule, respectively) is bounded by

\[ ε_R \leq \frac{h}{2} \max_{x\in[0,1]} |y'(x)|, \quad ε_T \leq \frac{h^2}{12} \max_{x\in[0,1]} |y''(x)|. \] (2)

Which method do you recommend if the number of integration points is large? Give a proper motivation. (2 pt.)

3. We derive and use Newton-Raphson’s method to solve a nonlinear problem.

(a) Given is the scalar nonlinear problem:

Find \( p \in \mathbb{R} \) such that \( f(p) = 0 \). (3)

Derive Newton–Raphson’s formula, given by

\[ p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \text{ for } n \geq 1 \] (4)

with initial guess \( p_0 \) to solve the above problem. (2 pt.)

(b) To prove convergence of Newton-Raphson’s method it can be reformulated as fixed-point method for which a rigorous convergence proof exists:

\[ g(x) = x - \frac{f(x)}{f'(x)}. \] (5)

One of the prerequisites of the convergence proof is that \( g'(x) \) exists and satisfies

\[ |g'(x)| \leq k < 1. \] (6)

Show that Newton-Raphson’s method applied to \( f(x) = \sin(x) \) converges to the root \( p = 0 \) for any initial guess \( p_0 \in (-\sqrt{2}/2, \sqrt{2}/2) \). (1 pt.)

(c) Formulate Newton-Raphson’s method for the general nonlinear problem:

Find \( \mathbf{p} \in \mathbb{R}^m \) such that \( f(\mathbf{p}) = 0 \). (7)

(d) Perform one step of Newton-Raphson’s method applied to the following nonlinear problem for \( w_1 \) and \( w_2 \):

\[ \begin{align*}
18w_1 - 9w_2 + w_1^2 &= 0, \\
-9w_1 + 18w_2 + w_2^2 &= 9.
\end{align*} \] (8)

Use \( w_1 = w_2 = 0 \) as the initial estimate. (2 pt.)

For the answers of this test we refer to:
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html
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Figure 1: Stability region for RK$_4$

Figure 2: Stability region for the Trapezoidal rule