DELFT UNIVERSITY OF TECHNOLOGY
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TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU/Minor AESB2210)
Thursday April 19th 2018, 18:30-21:30

Number of questions: This is an exam with 10 open questions, subdivided in 3 main questions.
Answers All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will not give points.
Tools Only a non-graphical calculator is permitted. All other tools are not permitted.
Assessment In total 20 points can be earned. The final not-rounded grade is given by $P/20$, where P is the number of points earned.

1. A method to integrate the initial value problem defined by $y' = f(t, y)$, $y(t_0) = y_0$, is given by

$$\begin{cases} k_1 &= \Delta t\, f(t_n, w_n) \\ k_2 &= \Delta t\, f(t_n + \Delta t, w_n + k_1) \\ w_{n+1} &= w_n + \frac{1}{2}(k_1 + k_2) \end{cases}$$

where Δt denotes the time–step and w_n represents the numerical solution at time t_n.

   (a) Show that the *local truncation error* of the given method is $\mathcal{O}(\Delta t^2)$. (3 pt.)

   (b) The *amplification factor* of this method is given by

$$Q(\lambda\Delta t) = 1 + \lambda\Delta t + \frac{1}{2}(\lambda\Delta t)^2 .$$

   Derive this amplification factor for the given method. (2 pt.)

   (c) Given is the initial value problem

$$\underline{y}' = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}' = \begin{bmatrix} -2 & 0 & 0 \\ 2 & -3 & 0 \\ 0 & 3 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = A\underline{y} + \underline{f}, \qquad (1)$$

   with the initial conditions $y_1(0) = y_2(0) = y_3(0) = 0$.
   For which value of Δt is the given time-integration method *stable* for this initial-value problem? ($3\frac{1}{2}$ pt.)

   (d) Perform *one step* with the given method with $\Delta t = \frac{1}{2}$ and $t_0 = 0$ for the initial-value problem (1) and the given initial conditions. ($1\frac{1}{2}$ pt.)

2. Consider the following boundary value problem:

$$\begin{cases} -\frac{d^2y(x)}{dx^2} + y(x) = 2e^x, & x \in (0,1), \\ \\ y(0) = 2, & y'(1) = 0. \end{cases} \tag{2}$$

We apply the finite difference method to approximate the solution to the above boundary value problem. Let the gridnodes be given by $x_j = jh$, with h as stepsize. Let $x_n = nh = 1$.

(a) Give a *finite differences scheme* (including a derivation) for which the local truncation error is of order $O(h^2)$. (2pt.)

(b) Use a virtual gridnode for the boundary condition at $x = 1$ and give a *finite differences scheme* (including a derivation) for which the local truncation error is of order $O(h^2)$. (1pt.)

(c) Give a *linear system* $A\mathbf{w} = \mathbf{b}$ with *symmetric matrix* A which can be obtained from applying the finite difference discretisation with three (after processing the virtual gridnode) unknowns ($h = 1/3$). (2pt.)

3. We approximate the integral $\int_a^b f(x)dx$ by a numerical method.

(a) Now we derive a new integration method. Let $P_1(x)$ be the Taylor polynomial of degree one of $f(x)$ around the point b. Show that the *integration method* I_P based on $P_1(x)$ for $\int_a^b f(x)dx$ is given by

$$(b-a)f(b) - \frac{1}{2}(a-b)^2 f'(b),$$

and give an upper bound for the corresponding *truncation error*

$$\int_a^b f(x)dx - I_P.$$

(2pt.)

(b) Derive the *composite rule* $I(h)$ for the new integration method I_P. Let the gridnodes be given by $x_j = a + jh$, with h as stepsize and $x_n = a + nh = b$. *Approximate the integral* for $f(x) = x^3$, $a = 0$ and $b = 1$ with this method using $h = \frac{1}{2}$ and determine the *difference with the exact answer.* (1.5pt.)

(c) Which method do you *prefer*: the method given in 3b or the composite Trapezoidal rule. Give arguments that support your preference.
Hint: you may use that the truncation error of the composite Trapezoidal rule is bounded by: $\frac{(b-a)h^2}{12}\max_{\xi\in[a,b]}|f''(\xi)|$. (1.5pt.)

For the answers of this test we refer to:
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html