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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU/Minor AESB2210)
Thursday July 5th 2018, 13:30-16:30

1. (a) Consider the test equation $y' = \lambda y$, then it follows that

$$w_{n+1} = w_n + (1-\theta)\lambda\Delta t w_n + \theta\lambda\Delta t w_{n+1}.$$

Solving for w_{n+1} gives

$$w_{n+1} = \frac{1+(1-\theta)\lambda\Delta t}{1-\theta\lambda\Delta t} w_n.$$

Hence the amplification factor is given by

$$Q(\lambda\Delta t) = \frac{1+(1-\theta)\lambda\Delta t}{1-\theta\lambda\Delta t}.$$

(b) The local truncation error for the test equation $y' = \lambda y$ is given by

$$\tau_{n+1}(\Delta t) = \frac{e^{\lambda\Delta t} - Q(\lambda\Delta t)}{\Delta t} y_n.$$

The Taylor Series around 0 for $e^{\lambda\Delta t}$ is:

$$e^{\lambda\Delta t} = 1 + \lambda\Delta t + \frac{1}{2}(\lambda\Delta t)^2 + \mathcal{O}(\Delta t^3).$$

The Taylor Series around 0 for $Q(\lambda\Delta t)$ is:

$$\begin{aligned}
Q(\lambda\Delta t) &= (1+(1-\theta)\lambda\Delta t)\frac{1}{1-\theta\lambda\Delta t}\\
&= (1+(1-\theta)\lambda\Delta t)\left(1+\theta\lambda\Delta t + \theta^2(\lambda\Delta t)^2 + \mathcal{O}\left(\Delta t^3\right)\right)\\
&= 1 + \lambda\Delta t + \theta(\lambda\Delta t)^2 + \mathcal{O}\left(\Delta t^3\right).
\end{aligned}$$

Hence, this gives

$$e^{\lambda\Delta t} - Q(\lambda\Delta t) = \left(\frac{1}{2}-\theta\right)\left(\lambda\Delta t^2\right) + \mathcal{O}\left(\Delta t^3\right),$$

and hence

$$\begin{aligned}
\tau_{n+1}(\Delta t) &= \frac{\left(\frac{1}{2}-\theta\right)\left(\lambda\Delta t^2\right) + \mathcal{O}\left(\Delta t^3\right)}{\Delta t} y_n\\
&= \left(\frac{1}{2}-\theta\right)(\lambda\Delta t)\, y_n + \mathcal{O}\left(\Delta t^2\right) = \mathcal{O}(\Delta t).
\end{aligned}$$

Furthermore, $\tau_{n+1} = \mathcal{O}(\Delta t^2)$ if and only if $\theta = \frac{1}{2}$.

(c) To this extent, we determine the eigenvalues of the matrix. Subsequently, these eigenvalues are substituted into the amplification factor. The eigenvalues of the matrix are given by $-1 \pm 3i$.

Using $\Delta t = 1$, $\theta = \frac{1}{2}$ and taking $\lambda = -1 - 3i$ (alternatively, $\lambda = -1 + 3i$), it follows that

$$\begin{aligned} Q(\lambda \Delta t) &= \frac{1 + \frac{1}{2}(-1 - 3i)}{1 - \frac{1}{2}(-1 - 3i)} \\ &= \frac{\frac{1}{2} - \frac{3}{2}i}{\frac{3}{2} + \frac{3}{2}i}. \end{aligned}$$

Herewith, it follows that $|Q(\lambda \Delta t)|^2 = \frac{5}{9} \leq 1$. (Different methods to show this are possible.)

As the two eigenvalues are each others complex conjugate, only one eigenvalue has to be considered during the stability analysis. (Also correct: Repeating the above calculations for the other eigenvalue.)

Hence for $\Delta t = 1$ and $\theta = \frac{1}{2}$ it follows that the method applied to the given system is stable.

(d) The given method, applied to the system $\underline{x}' = A\underline{x}$ as given in the question and taking $\theta = \frac{1}{2}$, gives

$$\underline{w}_{n+1} = \underline{w}_n + \frac{1}{2}\Delta t A \underline{w}_n + \frac{1}{2}\Delta t A \underline{w}_{n+1}.$$

Rearranging gives the linear system

$$\left(I - \frac{1}{2}\Delta t A\right) \underline{w}_{n+1} = \left(I + \frac{1}{2}\Delta t A\right) \underline{w}_n.$$

With $\Delta t = 1$ and the initial condition, $\underline{w}_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$, this gives

$$\begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix} \underline{w}_1 = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}.$$

Solving for $\underline{w}_1$ gives

$$\underline{w}_1 = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}.$$

2. (a) The three relevant Lagrange basis polynomials are with $n = 2$ given by

$$\begin{aligned}
L_{02}(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \\
&= \frac{(x-3)(x-4)}{(1-3)(1-4)} \\
&= \frac{1}{6}x^2 - \frac{7}{6}x + 2, \\
L_{12}(x) &= \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \\
&= \frac{(x-1)(x-4)}{(3-1)(3-4)} \\
&= -\frac{1}{2}x^2 + \frac{5}{2}x - 2, \\
L_{22}(x) &= \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \\
&= \frac{(x-1)(x-3)}{(4-1)(4-3)} \\
&= \frac{1}{3}x^2 - \frac{4}{3}x + 1.
\end{aligned}$$

The resulting perturbed interpolating polynomial is then

$$\begin{aligned}
\hat{L}_2(x) &= \hat{f}(x_0)L_{02}(x) + \hat{f}(x_1)L_{12}(x) + \hat{f}(x_2)L_{22}(x) \\
&= 3\left(\frac{1}{6}x^2 - \frac{7}{6}x + 2\right) + 6\left(-\frac{1}{2}x^2 + \frac{5}{2}x - 2\right) + 5\left(\frac{1}{3}x^2 - \frac{4}{3}x + 1\right) \\
&= -\frac{5}{6}x^2 + \frac{29}{6}x - 1.
\end{aligned}$$

Evaluation in $x = 2$ finally gives

$$\hat{L}_2(2) = \frac{16}{3}.$$

Any alternative, but correct, route to the above answer gives the same amount of (total) points.

(b) The unperturbed error $|f(2) - L_2(2)|$ can be bounded from above by the following steps:

$$\begin{aligned}
|f(2) - L_2(2)| &\leq \left| \frac{(2-1)(2-3)(2-4)}{3!} f'''(\zeta(x)) \right| \\
&= \frac{1}{3} \left| f'''(\zeta(x)) \right| \\
&\leq \frac{\delta}{3}.
\end{aligned}$$

The perturbation error $\left|L_2(2) - \hat{L}_2(2)\right|$ can be bounded from above by the following steps:

$$\begin{aligned}\left|L_2(2) - \hat{L}_2(2)\right| &= \left|(f(x_0) - \hat{f}(x_0))L_{02}(2) + (f(x_1) - \hat{f}(x_1))L_{12}(2) + (f(x_2) - \hat{f}(x_2))L_{22}(2)\right| \\ &\leq \frac{1}{3}\left|f(x_0) - \hat{f}(x_0)\right| + \left|f(x_1) - \hat{f}(x_1)\right| + \frac{1}{3}\left|f(x_2) - \hat{f}(x_2)\right| \\ &\leq \frac{5\epsilon}{3}.\end{aligned}$$

Combining these upper bounds gives for the total error

$$\begin{aligned}\left|f(2) - \hat{L}_2(2)\right| &= \left|f(2) - L_2(2) + L_2(2) - \hat{L}_2(2)\right| \\ &\leq |f(2) - L_2(2)| + \left|L_2(2) - \hat{L}_2(2)\right| \\ &\leq \frac{\delta + 5\epsilon}{3}.\end{aligned}$$

3. (a) A fixed point p satisfies the equation $p = g(p)$. Substitution gives: $p = \frac{p^3}{6} + \frac{23}{48}$. Rewriting this expression gives:

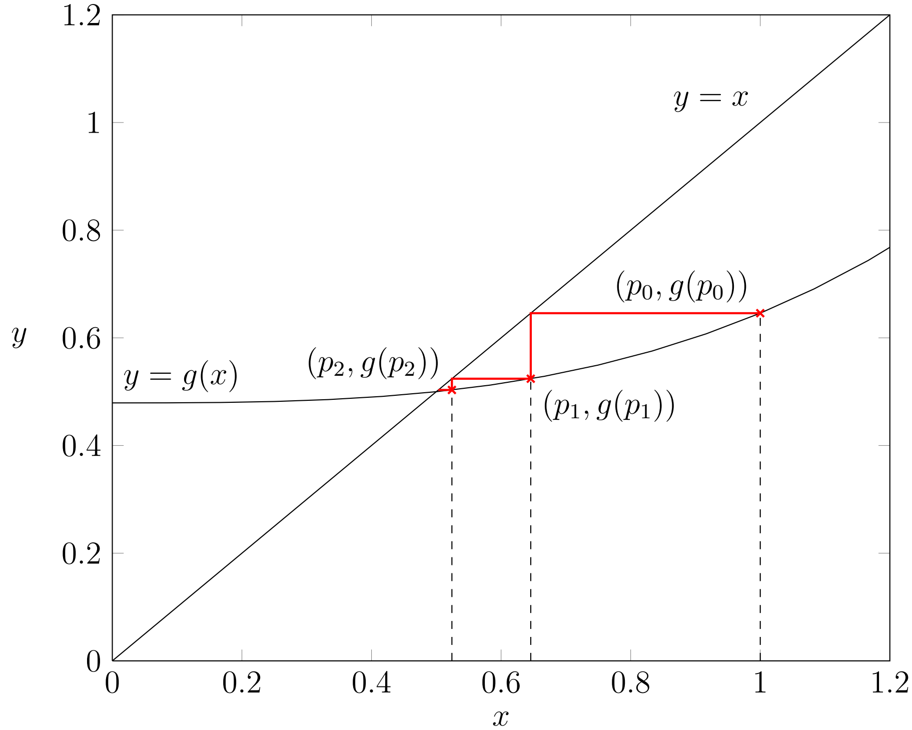
$$\begin{aligned}-\tfrac{p^3}{6} + p - \tfrac{23}{48} &= 0 \\ \Rightarrow \quad -p^3 + 6p - \tfrac{23}{8} &= 0 \\ \Rightarrow \quad f(p) &= 0,\end{aligned}$$

which shows that a fixed point of $g(x)$ also a root of $f(x)$ is.

(b) Starting with $p_0 = 1$ we obtain:

$$\begin{aligned}p_1 &= \tfrac{31}{48} &\approx 0.6458, \\ p_2 &= \tfrac{347743}{663552} &\approx 0.5241, \\ p_3 &= \tfrac{882018880783482655}{1752976676930715648} &\approx 0.5032.\end{aligned}$$

A sketch of this fixed-point iteration is given by



(c) For the convergence three conditions should be satisfied:

- $g \in C[0, 1]$.
- $g(p) \in [0, 1]$ for all $p \in [0, 1]$.
- $|g'(p)| \leq k < 1$ for all $p \in [0, 1]$.

Since $g(p) = \frac{p^3}{6} + \frac{23}{48}$ it follows that g is continuous everywhere, so the first condition holds.

Furthermore, $g'(x) = \frac{x^2}{2}$. Note that $g'(p) \geq 0$ for all $p \in [0, 1]$. This implies that $g(x)$ is increasing on $[0, 1]$. A lower bound for $g(x)$ is given by

$$g(x) \geq g(0) = \frac{23}{48} \geq 0,$$

and an upper bound is given by

$$g(x) \leq g(1) = \frac{31}{48} \leq 1.$$

So $0 \leq g(x) \leq 1$ and the second conditions holds.

For the third condition we note that $|g'(x)| = \frac{x^2}{2} \leq \frac{1}{2} = k < 1$ for all $x \in [0, 1]$, so the third condition is also satisfied.

As all conditions are satisfied, the fixed point iteration converges for all $p_0 \in [0, 1]$ to the fixed point $p \in [0, 1]$.