DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU/Minor AESB2210)
Tuesday August 14th 2018, 13:30-16:30

1. (a) The local truncation error is given by

$$\tau_{n+1}(\Delta t) = \frac{y_{n+1} - z_{n+1}}{\Delta t}, \quad (1)$$

in which y_{n+1} the solution at time step $n+1$ and z_{n+1} is the numerical approximation obtained by applying the numerical method with starting point y_n, the solution at time step n.
A Taylor series around t_n of y_{n+1} is given by

$$\begin{aligned} y_{n+1} &= y(t_{n+1}) \\ &= y(t_n + \Delta t) \\ &= y(t_n) + \Delta t y'(t_n) + \mathcal{O}(\Delta t^2). \end{aligned} \quad (2)$$

The numerical solution z_{n+1} is given by

$$\begin{aligned} z_{n+1} &= y_n + \Delta t f(t_n, y_n) \\ &= y(t_n) + \Delta t y'(t_n). \end{aligned} \quad (3)$$

Substraction of (3) from (2) gives

$$y_{n+1} - z_{n+1} = \mathcal{O}(\Delta t^2).$$

Substitution of the above in (1) gives

$$\tau_{n+1} = \mathcal{O}(\Delta t),$$

as was requested to show.

(b) Consider the test equation $y' = \lambda y$, then it follows that

$$\begin{aligned} w_{n+1} &= w_n + \lambda \Delta t w_n \\ &= (1 + \lambda \Delta t) w_n. \end{aligned}$$

Hence the amplification factor is given by

$$Q(\lambda \Delta t) = 1 + \lambda \Delta t.$$

(c) For stability must hold $|Q(\lambda\Delta t)| \leq 1$, or equivalently $|Q(\lambda\Delta t)|^2 \leq 1$, for each eigenvalue λ of the given matrix.

The eigenvalues of the given matrix are given by $-3 \pm 2i$.

As the two eigenvalues are each others complex conjugate, only one eigenvalue has to be considered during the stability analysis. (Also correct: Repeating the below calculations for the other eigenvalue.)

Taking $\lambda = -3 - 2i$ (alternatively, $\lambda = -3 + 2i$), it follows that

$$\begin{aligned} Q(\lambda\Delta t) &= 1 + (-3 - 2i)\Delta t \\ &= (1 - 3\Delta t) + (-2\Delta t)i. \end{aligned}$$

Herewith, it follows that

$$\begin{aligned} |Q(\lambda\Delta t)|^2 &= (1 - 3\Delta t)^2 + (-2\Delta t)^2 \\ &= 1 - 6\Delta t + 13\Delta t^2. \end{aligned}$$

Substitution of the above into the stability criterion $|Q(\lambda\Delta t)|^2 \leq 1$ and solving for Δt gives that the given method is stable for the given initial value problem if

$$\Delta t \leq 6/13.$$

(d) The given method, applied to the system $\underline{x}' = A\underline{x}$ as given in the question and taking $\Delta t = 1/5$, gives

$$\begin{aligned} \underline{w}_0 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \underline{w}_1 &= \underline{w}_0 + \Delta t A \underline{w}_0 \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} -3 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 4/5 \end{bmatrix} \\ \underline{w}_2 &= \underline{w}_1 + \Delta t A \underline{w}_1 \\ &= \begin{bmatrix} 0 \\ 4/5 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} -3 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 4/5 \end{bmatrix} \\ &= \begin{bmatrix} -8/25 \\ 8/25 \end{bmatrix}. \end{aligned}$$

2. (a) The solution and its first and second derivative are given by

$$\begin{aligned} u(x) &= x - \frac{1-e^x}{1-e}, \\ u'(x) &= 1 + \frac{e^x}{1-e}, \\ u''(x) &= \frac{e^x}{1-e}. \end{aligned}$$

The function $u(x)$ has a critical point at x^*, with x^* such that $u'(x^*) = 0$. This gives the only critical point $x^* = \ln\left(\frac{1}{e-1}\right)$.

The second derivative in this critical point equals

$$u''(x^*) = \frac{-1}{(e-1)^2} < 0,$$

so $u(x)$ has a (global) maximum at x^*, with the maximum value

$$u(x^*) = x - \frac{1-e^{x^*}}{1-e} = \ln(-1+e) - \frac{2-e}{1-e} \approx 0.1233.$$

Furthermore, $u(0) = 0 = u(1)$, so $u(x)$ is monotonically increasing on $[0, x^*]$ and monotonically decreasing on $[x^*, 1]$. Therefore $u(x)$ does not oscillate. (Other physical/mathematical arguments why $u(x)$ does not oscillate give equal points.)

Since the numerical solution should have the same characteristics as the exact solution, oscillatory solutions should be considered as not reflecting the analytic solution.

(b) The given formulas show that central difference approximations are used so we expect a local truncation error of second order, $\mathcal{O}(\Delta x^2)$.

To prove this we use the following central differences approximation at x_j, for $j \in \{1, \ldots, n\}$:

$$\begin{aligned} u'(x_j) &\approx \frac{u(x_{j+1}) - u(x_{j-1})}{2\Delta x}, \\ u''(x_j) &\approx \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{(\Delta x)^2}. \end{aligned}$$

Since we approximate the derivatives at the point x_j, we use Taylor series expansion about x_j, to obtain:

$$\begin{aligned} u(x_{j+1}) &= u(x_j) + \Delta x u'(x_j) + \frac{(\Delta x)^2}{2} u''(x_j) + \frac{(\Delta x)^3}{6} u'''(x_j) + \mathcal{O}\left(\Delta x^4\right), \\ u(x_{j-1}) &= u(x_j) - \Delta x u'(x_j) + \frac{(\Delta x)^2}{2} u''(x_j) - \frac{(\Delta x)^3}{6} u'''(x_j) + O((\Delta x)^4). \end{aligned}$$

This gives

$$
\begin{aligned}
-\frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{(\Delta x)^2} &= -u''(x_j) + \frac{\mathcal{O}\left(\Delta x^4\right)}{\Delta x^2} \\
&= -u''(x_j) + \mathcal{O}\left(\Delta x^2\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{u(x_{j+1}) - u(x_{j-1})}{2\Delta x} &= u'(x_j) + \frac{\mathcal{O}\left(\Delta x^3\right)}{2\Delta x} \\
&= u'(x_j) + \mathcal{O}\left(\Delta x^2\right).
\end{aligned}
$$

Hence the error is second order, that is $\mathcal{O}\left(\Delta x^2\right)$.

Next, we neglect the truncation error, and set $w_j \approx u(x_j)$ to get

$$
-\frac{w_{j+1} - 2w_j + w_{j-1}}{(\Delta x)^2} + \frac{w_{j+1} - w_{j-1}}{2\Delta x} = 1, \text{ for } j \in \{1, \ldots, n\}. \tag{4}
$$

At the boundaries, we see for $j = 1$ and $j = n$, upon substituting $w_0 = 0$ and $w_{n+1} = 0$, respectively:

$$
\begin{aligned}
-\frac{w_2 - 2w_1 + 0}{(\Delta x)^2} + \frac{w_2 - 0}{2\Delta x} &= 1, \\
-\frac{0 - 2w_n + w_{n-1}}{(\Delta x)^2} + \frac{0 - w_{n-1}}{2\Delta x} &= 1.
\end{aligned}
$$

This can be rewritten more neatly as follows:

$$
-\frac{w_2 - 2w_1}{(\Delta x)^2} + \frac{w_2}{2\Delta x} = 1, \tag{5}
$$

$$
\frac{2w_n - w_{n-1}}{(\Delta x)^2} - \frac{w_{n-1}}{2\Delta x} = 1. \tag{6}
$$

(c) Next, we use $\Delta x = 1/4$, then, from Equations (5), (4) and (6), one obtains the following system:

$$
\begin{aligned}
32w_1 - 14w_2 &= 1 \\
-18w_1 + 32w_2 - 14w_3 &= 1 \\
-18w_2 + 32w_3 &= 1
\end{aligned}
$$

This means with $\mathbf{w} = [w_1, w_2, w_3]^T$ that

$$
A = \begin{bmatrix} 32 & -14 & 0 \\ -18 & 32 & -14 \\ 0 & -18 & 32 \end{bmatrix},
$$

and

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

3. (a) The linear Lagrangian interpolation polynomial, with nodes a and b, is given by

$$p_1(x) = \frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b).$$

We approximate $f(x)$ by $p_1(x)$ in the integral $\int_a^b f(x)\mathrm{d}x$, from which follows:

$$\begin{aligned}
\int_a^b f(x)\,\mathrm{d}x &\approx \int_a^b p_1(x)\,\mathrm{d}x \\
&= \int_a^b \left\{ \frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b) \right\} \mathrm{d}x \\
&= \left[\frac{1}{2} \frac{(x-b)^2}{a-b} f(a) \right]_a^b + \left[\frac{1}{2} \frac{(x-a)^2}{b-a} f(b) \right]_a^b \\
&= \frac{1}{2}(b-a)(f(a)+f(b)).
\end{aligned}$$

This is the Trapezoidal rule.

(b) The magnitude of the error of the numerical integration over interval $[a,b]$ is given by

$$\begin{aligned}
\left| \int_a^b f(x)\,\mathrm{d}x - \int_a^b p_1(x)\,\mathrm{d}x \right| &= \left| \int_a^b \left(f(x) - p_1(x)\right)\,\mathrm{d}x \right| \\
&= \left| \int_a^b \frac{1}{2}(x-a)(x-b) f''(\xi(x))\,\mathrm{d}x \right| \\
&\leq \frac{1}{2} \max_{x\in[a,b]} |f''(x)| \int_a^b |(x-a)(x-b)|\,\mathrm{d}x \\
&= \frac{1}{12}(b-a)^3 \max_{x\in[a,b]} |f''(x)|.
\end{aligned}$$

(c) The composite Trapezoidal rule for $\displaystyle\int_0^1 x^2\,\mathrm{d}x$ with $h = 1/4$ is given by

$$\begin{aligned}
\frac{1}{h}\left(\frac{1}{2}x_0^2 + \left(\sum_{j=2}^{3} x_j^2 \right) + \frac{1}{2}x_4^2 \right) &= \frac{1}{4}\left(\frac{1}{2}0^2 + \frac{1}{4}^2 + \frac{1}{2}^2 + \frac{3}{4}^2 + \frac{1}{2}1^2 \right) \\
&= \frac{11}{32} = 0.34375.
\end{aligned}$$

(d) Since $\displaystyle\int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}$ the absolute value of the truncation error is:

$$\left|\frac{1}{3} - \frac{22}{64}\right| = \frac{1}{96} = 0.01041\overline{6}.$$