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FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS
(WI3097TU WI3097Minor WI3197Minor AESB2210 AESB2210-18 CTB2400)
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1. (a) *Remark: Using the test equation $y' = \lambda y$ results in no points for question (a).*
The local truncation error is given by

$$\tau_{n+1}(\Delta t) = \frac{y_{n+1} - z_{n+1}}{\Delta t}. \tag{1}$$

with $y_{n+1} = y(t_{n+1})$ the exact solution and z_{n+1} the result of applying the method with starting value $y_n = y(t_n)$.
A Taylor expansion of y_{n+1} around t_n yields

$$y_{n+1} = y_n + \Delta t y_n' + \frac{1}{2}\Delta t^2 y_n'' + \frac{1}{6}\Delta t^3 y_n''' + \mathcal{O}(\Delta t^4). \tag{2}$$

For the given system $y' = g(y)$ holds

$$z_{n+1} = y_n + \frac{1}{2}\Delta t \left(g(y_n) + g(y_n + \Delta t g(y_n))\right).$$

A Taylor expansion of $g(y_n + \Delta t g(y_n))$ around y_n yields

$$g(y_n + \Delta t g(y_n)) = g(y_n) + \Delta t g(y_n) g'(y_n) + \frac{1}{2}(\Delta t g(y_n))^2 g''(y_n) + \mathcal{O}(\Delta t^3). \tag{3}$$

Usage of $y' = g(y)$ and the given hint gives

$$g(y_n + \Delta t g(y_n)) = y_n' + \Delta t y_n'' + \frac{1}{2}(\Delta t g(y_n))^2 g''(y_n) + \mathcal{O}(\Delta t^3).$$

Substitution of the above in z_{n+1} gives

$$z_{n+1} = y_n + \Delta t y_n' + \frac{1}{2}\Delta t^2 y_n'' + \frac{1}{4}\Delta t^3 g(y_n) g''(y_n) + \mathcal{O}(\Delta t^4). \tag{4}$$

Equations (2) and (4) are substituted into relation (1) to obtain

$$\tau_{n+1} = \left(\frac{1}{6}y_n''' - \frac{1}{4}g(y_n)g''(y_n)\right)\Delta t^2 + \mathcal{O}(\Delta t^3),$$

hence

$$P = \frac{1}{6}y_n''' - \frac{1}{4}g(y_n)^2 g''(y_n).$$

Alternative correct formulae for P:

$$P = \frac{1}{6}y_n''' - \frac{1}{4}(y_n')^2 g''(y_n),$$
$$P = -\frac{1}{12}g''(y_n)(y_n')^2 + \frac{1}{6}g'(y_n)y_n'',$$
$$P = -\frac{1}{12}g''(y_n)g(y_n)^2 + \frac{1}{6}g'(y_n)y_n'',$$
$$P = -\frac{1}{12}g''(y_n)(y_n')^2 + \frac{1}{6}(g'(y_n))^2 y_n',$$
$$P = -\frac{1}{12}g''(y_n)g(y_n)^2 + \frac{1}{6}(g'(y_n))^2 g(y_n),$$

(b) The test equation is given by

$$y' = \lambda y.$$

Application of the method to the test equation gives

$$w_{n+1} = w_n + \frac{1}{2}\Delta t\left(\lambda w_n + \lambda\left(w_n + \Delta t \lambda w_n\right)\right).$$

This is equivalent to

$$w_{n+1} = \left(1 + \lambda\Delta t + \frac{1}{2}(\lambda\Delta t)^2\right) w_n.$$

Hence the amplification factor is given by

$$Q(\lambda\Delta t) = 1 + \lambda\Delta t + \frac{1}{2}(\lambda\Delta t)^2.$$

(c) *Remark: Wrong eigenvalues causes a deduction of 1 point in question (c).*
Remark: A correct analysis with wrong eigenvalues still gives the allocated points.
With $\mathbf{x} = [x_1, x_2]^T$, the problem can be written as $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$, with

$$A = \begin{bmatrix} -3/2 & -1/2 \\ -1/2 & -3/2 \end{bmatrix},$$

and

$$\mathbf{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The characteristic equation of A is given by

$$\det(A - \lambda I) = 0$$
$$\Rightarrow \quad \begin{vmatrix} -3/2 - \lambda & -1/2 \\ -1/2 & -3/2 - \lambda \end{vmatrix} = 0$$
$$\Rightarrow \quad (-3/2 - \lambda)^2 - (-1/2)^2 = 0$$

The eigenvalues of A are calculated from this as $\lambda_1 = -2$ and $\lambda_2 = -1$.
For $\lambda_1 = -2$ we obtain

$$\begin{aligned} Q(\lambda_1\Delta t) &= Q(-2\Delta t) \\ &= 1 + (-2\Delta t) + \frac{1}{2}(-2\Delta t)^2 \\ &= 1 - 2\Delta t + 2\Delta t^2. \end{aligned}$$

For stability we must have

$$|Q(\lambda_1 \Delta t)| \leq 1,$$

or equivalently

$$-1 \leq 1 - 2\Delta t + 2\Delta t^2 \leq 1, \tag{5}$$

as $Q(-2\Delta t)$ is a real number.
For the left inequality of (5), we obtain:

$$\begin{aligned} & & -1 \leq 1 - 2\Delta t + 2\Delta t^2 \\ \Rightarrow & & 0 \leq 2 - 2\Delta t + 2\Delta t^2 \\ \Rightarrow & & 0 \leq 1 - \Delta t + \Delta t^2 \end{aligned} \tag{6}$$

The right-hand side above evaluates for $\Delta t = 1$ to 1 and its discriminant

$$D = (-1)^2 - 4 \cdot 1 \cdot 1 = -3,$$

is negative. Therefore the right-hand side of (6) has no real roots and (6) is always satisfied for $\Delta t \geq 0$.
For the right inequality of (5), we obtain:

$$\begin{aligned} & & 1 - 2\Delta t + 2\Delta t^2 \leq 1 \\ \Rightarrow & & -2\Delta t + 2\Delta t^2 \leq 0 \\ \Rightarrow & & -\Delta t + \Delta t^2 \leq 0 \\ \Rightarrow & & -1 + \Delta t \leq 0 \\ \Rightarrow & & \Delta t \leq 1 \end{aligned} \tag{7}$$

For $\lambda_2 = -1$ we obtain

$$\begin{aligned} Q(\lambda_2 \Delta t) &= Q(-\Delta t) \\ &= 1 + (-\Delta t) + \frac{1}{2}(-\Delta t)^2 \\ &= 1 - \Delta t + \frac{1}{2}\Delta t^2. \end{aligned}$$

For stability we must have

$$|Q(\lambda_2 \Delta t)| \leq 1,$$

or equivalently

$$-1 \leq 1 - \Delta t + \frac{1}{2}\Delta t^2 \leq 1, \tag{8}$$

as $Q(-\Delta t)$ is a real number.
For the left inequality of (8), we obtain:

$$\begin{aligned} & & -1 \leq 1 - \Delta t + \frac{1}{2}\Delta t^2 \\ \Rightarrow & & 0 \leq 2 - \Delta t + \frac{1}{2}\Delta t^2 \\ \Rightarrow & & 0 \leq 4 - 2\Delta t + \Delta t^2 \end{aligned} \tag{9}$$

The right-hand side above evaluates for $\Delta t = 1$ to 3 and its discriminant

$$D = (-2)^2 - 4 \cdot 1 \cdot 4 = -12,$$

is negative. Therefore the right-hand side of (9) has no real roots and (9) is always satisfied for $\Delta t \geq 0$.

For the right inequality of (8), we obtain:

$$\begin{aligned} & & 1 - \Delta t + \frac{1}{2}\Delta t^2 &\leq 1 \\ &\Rightarrow & -\Delta t + \frac{1}{2}\Delta t^2 &\leq 0 \\ &\Rightarrow & -2\Delta t + \Delta t^2 &\leq 0 \\ &\Rightarrow & -2 + \Delta t &\leq 0 \\ &\Rightarrow & \Delta t &\leq 2 \end{aligned} \tag{10}$$

So for stability we must have from (6), (7), (9) and (10)

$$\Delta t \leq 1,$$

and therefore

$$\Delta t_{\max} = 1.$$

Alternative (c) *With $\mathbf{x} = [x_1, x_2]^T$, the problem can be written as $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$, with*

$$A = \begin{bmatrix} -3/2 & -1/2 \\ -1/2 & -3/2 \end{bmatrix},$$

and

$$\mathbf{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The characteristic equation of A is given by

$$\begin{aligned} & & \det(A - \lambda I) &= 0 \\ &\Rightarrow & \begin{vmatrix} -3/2 - \lambda & -1/2 \\ -1/2 & -3/2 - \lambda \end{vmatrix} &= 0 \\ &\Rightarrow & (-3/2 - \lambda)^2 - (-1/2)^2 &= 0 \end{aligned}$$

The eigenvalues of A are calculated from this as $\lambda_1 = -2$ and $\lambda_2 = -1$.

The given method is the Modified Euler method.

From the stability region of this method can be deduced:

$$\Delta t \leq \frac{2}{|\lambda|},$$

for all eigenvalues of A, if these eigenvalues are all real.

In this case we obtain:

$$\Delta t \leq \frac{2}{|\lambda_1|} = 1,$$

and

$$\Delta t \leq \frac{2}{|\lambda_2|} = 2.$$

So for stability we must have

$$\Delta t \leq 1,$$

and therefore

$$\Delta t_{\max} = 1.$$

(d) *Remark: No points will be given if a different method is used or a different system of differential equations is solved.*

Remark: Small miscalculations cost $\frac{1}{4}$ point per miscalculation.

First note:

$$\mathbf{w}_0 = \mathbf{0}.$$

Therefore

$$A\mathbf{w}_0 = \mathbf{0}.$$

Applying the given method with $\Delta t = 1$ gives

$$\begin{aligned}
\mathbf{w}_1 &= \mathbf{w}_0 + \frac{1}{2}\left(A\mathbf{w}_0 + \mathbf{b} + A(\mathbf{w}_0 + A\mathbf{w}_0 + \mathbf{b}) + \mathbf{b}\right) \\
&= \mathbf{0} + \frac{1}{2}\left(\mathbf{0} + \mathbf{b} + A(\mathbf{0} + \mathbf{0} + \mathbf{b}) + \mathbf{b}\right) \\
&= \frac{1}{2}\left(\mathbf{b} + A\mathbf{b} + \mathbf{b}\right) \\
&= \mathbf{b} + \frac{1}{2}A\mathbf{b} \\
&= \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -3/2 & -1/2 \\ -1/2 & -3/2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}.
\end{aligned}$$

2. (a) *Remark: A correct analysis of the order with the wrong choice for $U(\Delta x)$ is at most rewarded with 1 point.*

Remark: A choice for $U(\Delta x)$ without a correct argument gives a subtraction of $\frac{1}{2}$ points.

For the given differential equation the convection speed is -3. As this is negative, the forward difference

$$U(\Delta x) = \frac{y(x+\Delta x) - y(x)}{\Delta x}$$

should be taken.

The order can be determined with the use of Taylor expansions. We need the following:

$$y(x+\Delta x) = y(x) + y'(x)\Delta x + \frac{1}{2}y''(x)\Delta x^2 + \frac{1}{6}y'''(x)\Delta x^3 + \mathcal{O}(\Delta x^4).$$

Substitution of the above in the chosen finite difference $U(\Delta x)$ gives

$$\begin{aligned} U(\Delta x) &= \frac{y'(x)\Delta x + \frac{1}{2}\Delta x^2 y''(x) + \frac{1}{6}\Delta x^3 y'''(x) + \mathcal{O}(\Delta x^4)}{\Delta x} \\ &= y'(x) + \mathcal{O}(\Delta x). \end{aligned}$$

This shows that $U(\Delta x)$ is a $\mathcal{O}(\Delta x)$ approximation of $y'(x)$.

(b) To obtain the correct formula, we use Taylor expansions:

$$\begin{aligned} y(x+\Delta x) &= y(x) + y'(x)\Delta x + \frac{1}{2}y''(x)\Delta x^2 + \frac{1}{6}y'''(x)\Delta x^3 + \mathcal{O}(\Delta x^4), \\ y(x-\Delta x) &= y(x) - y'(x)\Delta x + \frac{1}{2}y''(x)\Delta x^2 - \frac{1}{6}y'''(x)\Delta x^3 + \mathcal{O}(\Delta x^4). \end{aligned}$$

Substitution of the above in the given formula gives

$$\begin{aligned} Q(\Delta x) = &\frac{(\alpha_1 + \alpha_0 + \alpha_{-1})y(x) + (\alpha_1 - \alpha_{-1})\Delta x y'(x)}{\Delta x^2} \\ &+ \frac{\frac{1}{2}(\alpha_1 + \alpha_{-1})\Delta x^2 y''(x) + \frac{1}{6}(\alpha_1 - \alpha_{-1})\Delta x^3 y'''(x) + \mathcal{O}(\Delta x^4)}{\Delta x^2}. \end{aligned}$$

From this we obtain the following system of linear equations:

$$\begin{cases} \alpha_1 + \alpha_0 + \alpha_{-1} = 0, \\ \alpha_1 - \alpha_{-1} = 0, \\ \alpha_1 + \alpha_{-1} = 2. \end{cases}$$

The solution of this system is $\alpha_1 = \alpha_{-1} = 1, \alpha_0 = -2$, which leads to the formula

$$Q(\Delta x) = \frac{y(x+\Delta x) - 2y(x) + y(x-\Delta x)}{\Delta x^2}.$$

(c) *Remark: No points will be deducted for incorrect $U(\Delta x)$ and/or incorrect $Q(\Delta x)$, if applied correctly.*

Evaluation of the ode in $x = x_j$ and replacing $y''(x_j)$ with $Q(\Delta x)$ and $y'(x_j)$ with $U(\Delta x)$ gives

$$-\frac{y(x_{j+1}) - 2y(x_j) + y(x_{j-1})}{\Delta x^2} + \mathcal{O}\left(\Delta x^2\right) - 3\frac{y(x_{j+1}) - y(x_j)}{\Delta x} + \mathcal{O}(\Delta x) = 1.$$

Next, we neglect the truncation errors, and set $w_j \approx y(x_j)$ to obtain

$$-\frac{w_{j+1} - 2w_j + w_{j-1}}{\Delta x^2} - 3\frac{w_{j+1} - w_j}{\Delta x} = 1, \tag{11}$$

the general equation for an internal point x_j.
At the left boundary, $x = 0 = x_0$, substitution of $j = 0$ in (11) gives

$$-\frac{w_1 - 2w_0 + w_{-1}}{\Delta x^2} - 3\frac{w_1 - w_0}{\Delta x} = 1,$$

from which the virtual value w_{-1} must be eliminated.
The boundary condition $y'(0) = 0$ can be transformed (after ignoring errors) to

$$\frac{w_1 - w_{-1}}{2\Delta x} = 0,$$

from which $w_{-1} = w_1$ follows and the final equation for $j = 0$ becomes

$$-\frac{2w_1 - 2w_0}{\Delta x^2} - 3\frac{w_1 - w_0}{\Delta x} = 1.$$

At the right boundary, $x = 1$, we have $w_{n+1} = 1$, which after substitution in (11) for $j = n$ gives

$$-\frac{-2w_n + w_{n-1}}{\Delta x^2} - 3\frac{-w_n}{\Delta x} = 1 + \frac{1}{\Delta x^2} + \frac{3}{\Delta x}.$$

The entire scheme therefore is

$$\begin{aligned} -\frac{2w_1 - 2w_0}{\Delta x^2} - 3\frac{w_1 - w_0}{\Delta x} &= 1, \\ -\frac{w_{j+1} - 2w_j + w_{j-1}}{\Delta x^2} - 3\frac{w_{j+1} - w_j}{\Delta x} &= 1, \qquad \text{for } j = 1, \ldots, n, \\ -\frac{-2w_n + w_{n-1}}{\Delta x^2} - 3\frac{-w_n}{\Delta x} &= 1 + \frac{1}{\Delta x^2} + \frac{3}{\Delta x}. \end{aligned}$$

3. (a) $p = \sqrt{3}$ is a fixed point of the function g if $g(\sqrt{3}) = \sqrt{3}$. We calculate $g(p)$:

$$\begin{aligned} g(p) &= g(\sqrt{3}) \\ &= -\frac{1}{3}(\sqrt{3})^2 + \sqrt{3} + 1 \\ &= -1 + \sqrt{3} + 1 \\ &= \sqrt{3}, \end{aligned}$$

So $p = \sqrt{3}$ is indeed a fixed-point of the function g.

(b) The function g is a polynomial and polynomials are continuous everywhere, so there for g also is continuous on the interval $[1, 2]$.

(c) First note that g is a parabola opening to the bottom and therefore g has a maximum at the point where $g'(x) = 0$. We solve this equation:

$$\begin{aligned} & & g'(x) &= 0 \\ \Rightarrow & & -\frac{2}{3}x + 1 &= 0 \\ \Rightarrow & & x &= \frac{3}{2}, \end{aligned}$$

so the position of the maximum of g is located in the interval $[1, 2]$ and attains the value $g(3/2) = 7/4$. Therefore we conclude

$$g(x) \leq 2 \quad \text{for } x \in [1, 2].$$

The function g attains its minimum on the boundary of the interval $[1, 2]$, so evaluation of g at these points gives

$$\begin{aligned} g(1) &= 5/3, \\ g(2) &= 5/3. \end{aligned}$$

Therefore we conclude

$$g(x) \geq 1 \quad \text{for } x \in [1, 2].$$

Putting everything together, we have found

$$1 \leq g(x) \leq 2, \quad \text{for } x \in [1, 2].$$

as requested.

(d) The derivative of g is given by

$$g'(x) = -\frac{2}{3}x + 1,$$

which is a monotonous decreasing function. Therefore the minimum and maximum value are located on the boundary of the interval, leading to

$$\begin{array}{ccccccc} & g'(2) & \leq & g'(x) & \leq & g'(1) \\ \Rightarrow & -1/3 & \leq & g'(x) & \leq & 1/3 \\ \Rightarrow & & & |g'(x)| & \leq & 1/3 \end{array}$$

So $k = 1/3$.

(e) *Remark: The final value of p_1 should be given in 4 significant digits. Failure to do so results in a deduction of $\frac{1}{4}$ point if $p_1 = 5/3$ is stated.*
Remark: Calculation of p_2 using $p_1 = 5/3$ is incorrect, and causes a deduction of $\frac{1}{4}$ point.
Remark: The final value of p_2 should be given in 4 significant digits. Failure to do so results in a deduction of $\frac{1}{4}$ point if $p_2 = 47/27$ is stated.
Straightforward application of the fixed point iteration gives

$$\begin{aligned} p_1 &= g(p_0) \\ &= g(2.000) \\ &= 1.667, \end{aligned}$$

and

$$\begin{aligned} p_2 &= g(p_1) \\ &= g(1.667) \\ &= 1.741. \end{aligned}$$