DELFT UNIVERSITY OF TECHNOLOGY
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TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS
(WI3097TU WI3097Minor WI3197Minor AESB2210 AESB2210-18 CTB2400)
Thursday July 4th 2019, 13:30-16:30

Number of questions: This is an exam with 12 open questions, subdivided in 3 main questions.

Answers All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will not give points.

Tools Only a non-graphical, non-programmable calculator is permitted. All other electronic tools are not permitted.

Assessment In total 20 points can be earned. The final not-rounded grade is given by $P/2$, where P is the number of points earned.

1. For the initial value problem $y' = f(t,y)$, $y(t_0) = y_0$, we use the following integration method:

$$\begin{cases} w_{n+1}^* = w_n + \Delta t f(t_n, w_n) \\ w_{n+1} = w_n + \dfrac{\Delta t}{2}\left(f(t_n, w_n) + f(t_{n+1}, w_{n+1}^*)\right). \end{cases} \tag{1}$$

Here Δt denotes the timestep and w_n represents the numerical approximation at time t_n.

(a) Show that the local truncation error of the integration method is of the order $\mathcal{O}(\Delta t^2)$. (You are not allowed to use the test equation here.) (3pt.)

Consider the following initial value problem

$$\begin{cases} \dfrac{d^2y}{dt^2} + 4y = \cos t, \\ y(0) = 1, \quad \dfrac{dy}{dt}(0) = 0. \end{cases} \tag{2}$$

(b) Show that the above initial value problem can be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos t \end{pmatrix}. \tag{3}$$

Give the initial conditions for $x_1(0)$ and $x_2(0)$ as well. (1pt.)

(c) Calculate one step with integration method (1), in which $\Delta t = 0.1$ and $t_0 = 0$, applied to (3) and use the given initial conditions. (2pt.)

(d) Derive the amplification factor for the integration method. (2pt.)

(e) Examine for which stepsizes Δt, the integration method (1), applied to the initial value problem (3), is stable. (2pt.)

2. We want to find an approximation of the zero p of a function f, i.e. we want to find p such that $f(p) = 0$. However, we do not known the function f, but only some values of f in some points x are known, which are given in the table to the right.

x	$f(x)$
1	-1
$4/3$	$-2/9$
$7/5$	$-1/25$
$10/7$	$2/49$
2	2

Therefore we consider using the Secant method:

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{K_{n-1}}, \tag{4}$$

with K_{n-1} an approximation to $f'(p_{n-1})$, and in which p_{n-2}, p_{n-1} and p_n are three consecutive approximations of the zero p.
Equation (4) is based on formulating the linear interpolating polynomial L of f based on p_0 and p_1:

$$L(x) = f(p_0) + \frac{f(p_1) - f(p_0)}{p_1 - p_0}(x - p_0),$$

after which p_2 is found by solving $L(p_2) = 0$ for p_2.

(a) *Show* that, for $n = 2$, K_1 is given by

$$K_1 = \frac{f(p_1) - f(p_0)}{p_1 - p_0},$$

by solving $L(p_2) = 0$ for p_2. (2 pt.)

(b) Take $p_0 = 1$ and $p_1 = 2$. *Approximate* the zero p of f by calculating p_2. You may round p_2 to a value of x as given in the table. (1 pt.)

(c) Repeat the above steps with $n = 3$ by *stating* the formula for K_2 and *calculating* p_3. You may round p_3 to a value of x as given in the table. (2 pt.)

3. We are interested in the numerical integration of the integral

$$\int_0^{2\pi} y(x)dx,$$

with $y(x) = 1 + \sin(x)$.

(a) *Approximate* the above integral with the *right composite Rectangle rule* using $h = \pi/2$. (1½ pt.)

(b) *Approximate* the above integral with the *composite Trapezoidal rule* using $h = \pi/2$. (1 pt.)

(c) The magnitude of the errors ε_R and ε_T of the approximations are bounded by

$$\varepsilon_R \le \pi h \max_{x\in[0,2\pi]} |y'(x)|,$$

for the right composite Rectangle rule and by

$$\varepsilon_T \le \frac{\pi}{6} h^2 \max_{x\in[0,2\pi]} |y''(x)|,$$

for the composite Trapezoidal rule. *Give* explicit upper bounds for both rules for general values of h. (1½ pt.)

(d) *Select and motivate* which method you prefer if h becomes small. (1 pt.)

For the answers of this test we refer to:
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html