DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

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TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS
(WI3097TU WI3197Minor AESB2210-18 CTB2400)
January 31st, 2020, 13:30 - 16:30

Number of questions: This is an exam with 9 open questions, subdivided in 3 main questions.

Answers All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will give less or no points.

Tools Only a non-graphical, non-programmable calculator is permitted. All other electronic tools are not permitted.

Assessment In total 20 points can be earned. The final not-rounded grade is given by $P/2$, where P is the number of points earned.

1. For the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$, we use the Modified Euler method:

$$\begin{cases} w^*_{n+1} = w_n + \Delta t f(t_n, w_n) \\ w_{n+1} = w_n + \dfrac{\Delta t}{2}\left(f(t_n, w_n) + f(t_{n+1}, w^*_{n+1})\right). \end{cases} \tag{1}$$

Here Δt denotes the time step and w_n represents the numerical approximation of $y(t_n)$ after n time steps.

We also consider the following system of differential equations

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin(t) \end{pmatrix}, \tag{2}$$

combined with the initial conditions $x_1(0) = 0$ and $x_2(0) = 1$ into an initial value problem.

(a) Show that the local truncation error of the time integration method is of the order $\mathcal{O}(\Delta t^2)$. ($3\frac{1}{2}$ pt.)
(You are not allowed to use the test equation here.)

(b) Calculate one step with the Modified Euler method (1), in which $\Delta t = 1$ and $t_0 = 0$, applied to (2) and use the given initial conditions. ($1\frac{1}{2}$ pt.)

(c) Determine the amplification factor $Q(\lambda \Delta t)$ of the integration method (1). (1 pt.)

(d) Determine for which time steps $\Delta t > 0$, the integration method (1), applied to the system (2), is stable. (4 pt.)

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2. We consider the following boundary-value problem:

$$\begin{cases} -y''(x) + 4y(x) &= 4e^{2x}, \quad x \in (0,1), \\ y(0) &= \frac{3}{2}, \\ y'(1) &= 0. \end{cases} \tag{3}$$

In this exercise we try to approximate the exact solution with a numerical method.

We solve the boundary value problem (3) using central finite differences with a local truncation error of $\mathcal{O}(\Delta x^2)$, upon setting $x_j = j\Delta x$, $(n+1)\Delta x = 1$, where Δx denotes the uniform step size. After discretization we obtain the following formulas:

$$\begin{aligned} -\frac{w_2 - 2w_1}{\Delta x^2} + 4w_1 &= 4e^{2\Delta x} + \frac{3}{2\Delta x^2}, \\ -\frac{w_{j+1} - 2w_j + w_{j-1}}{\Delta x^2} + 4w_j &= 4e^{2j\Delta x}, \qquad \text{for } j \in \{2, \ldots, n\}, \\ -\frac{-w_{n+1} + w_n}{\Delta x^2} + 2w_{n+1} &= 2e^2. \end{aligned}$$

(a) Give (with arguments) the derivation of this scheme. ($3\frac{1}{2}$ pt.)

(b) Choose $\Delta x = 1/4$ and derive the system of equations resulting from this choice. Furthermore, rewrite this system to the form $A\mathbf{w} = \mathbf{b}$ with $\mathbf{w} = [w_1, \ldots, w_{n+1}]^T$. Explicitly state A and $\mathbf{b}$ in your answer. ($1\frac{1}{2}$ pt.)

3. To approximate $\int_a^b f(x)\,\mathrm{d}x$ Simpson's rule

$$I_S = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

can be used. Simpson's rule is based on the assumption $f(x) \approx L_2(x)$, with $L_2(x)$ the quadratic Lagrange interpolation polynomial with nodes $x_0 = a$, $x_1 = \frac{a+b}{2}$ and $x_2 = b$:

$$L_2(x) = \sum_{k=0}^{2} L_{k2}(x) f(x_k),$$

where $L_{k2}(x)$ is the quadratic Lagrange basis polynomial of node x_k.

We furthermore know the following integrals:

$$\int_{x_0}^{x_2} L_{k2}(x)\mathrm{d}x = \begin{cases} \frac{1}{6}(x_2 - x_0) & \text{if } k \in \{0, 2\}, \\ \frac{2}{3}(x_2 - x_0) & \text{if } k = 1. \end{cases}$$

Given is also that an upper bound for the truncation error of Simpson's rule I_S is

$$\left| \int_a^b f(x)\,\mathrm{d}x - I_S \right| \leq \frac{1}{2880} m_4 (b-a)^5,$$

where $m_4 = \max_{a \leq x \leq b} |f^{(4)}(x)|$.

(a) Give a derivation of Simpson's rule I_S. ($2\frac{1}{2}$ pt.)

(b) Show that Simpson's rule is exact for polynomials of degree 3 and lower. ($1\frac{1}{2}$ pt.)

(c) Approximate $\int_0^\pi \sin(x)\,\mathrm{d}x$ with Simpson's rule and give an upper bound for the absolute value of the truncation error in this approximation. (1 pt.)

For the answers of this test we refer to:
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html