DELFT UNIVERSITY OF TECHNOLOGY
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TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (CTB2400) Tuesday July 12 2022, 13:30-16:30

Number of questions: This is an exam with 11 open questions, subdivided in 3 main questions.

Answers All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will not give points.

Tools Only a non-graphical, non-programmable calculator is permitted. All other electronic tools are not permitted.

Assessment In total 20 points can be earned. The final not-rounded grade is given by $P/2$, where P is the number of points earned.

1. We consider the following method

$$w_{n+1} = w_n + \frac{1}{2}\Delta t \left(f(t_n, w_n) + f(t_{n+1}, w_{n+1}) \right) \tag{1}$$

for the integration of the **initial value problem** $y' = f(t, y)$, $y(t_0) = y_0$.

(a) Demonstrate that the *amplification factor* is given by

$$Q(\lambda \Delta t) = \frac{1 + \frac{1}{2}\lambda \Delta t}{1 - \frac{1}{2}\lambda \Delta t}.$$

($1\frac{1}{2}$ pt.)

(b) Show that the *local truncation error* of (1) for the test equation $y' = \lambda y$ takes on the form

$$\tau_{n+1} = T\Delta t^2 + \mathcal{O}(\Delta t^3),$$

and *give* a formula for T. ($3\frac{1}{2}$ pt.)
Hint: $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \mathcal{O}(x^4)$.
Hint: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \mathcal{O}(x^4)$.

(c) We consider the following *system of linear differential equations*: $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$, where:

$$A = \begin{bmatrix} -1 & 2 & -2 \\ 0 & -2 & -2 \\ 0 & 2 & -2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 9 \\ 4 \\ 8 \end{bmatrix} \text{ and } \mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \tag{2}$$

Show that the application of (1) to (2) is stable for $\Delta t = 1$. (3 pt.)

(d) The approximation $\mathbf{w}_1$ of the solution of the system (2) at time $t = 1$ obtained by applying (1) to (2) with $\Delta t = 1$ is calculated by us as

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}.$$

Show that the given value for $\mathbf{w}_1$ is correct. (2 pt.)

please turn over

2. To approximate $\int_a^b f(x)\,dx$ the Trapezoidal rule $\frac{b-a}{2}(f(a)+f(b))$ can be used.

   (a) Give the *linear Lagrange interpolatory polynomial* $p_1(x)$ with nodes a and b and *derive* the Trapezoidal rule by the use of $p_1(x)$. ($1\frac{1}{2}$ pt.)

   (b) The error for linear interpolation over nodes a and b is given by

   $$f(x) - p_1(x) = \frac{1}{2}(x-a)(x-b)f''(\xi(x)), \text{ for some } \xi(x) \in (a,b).$$

   Derive that an *upper bound* of the truncation error of the Trapezoidal rule applied to the interval $[a,b]$ is given by

   $$\frac{1}{12}(b-a)^3 \max_{x\in[a,b]} |f''(x)|,$$

   given that the second-order derivative of f is continuous over $[a,b]$. ($1\frac{1}{2}$ pt.)

   (c) *Approximate* $\int_0^1 x^2\,dx$ with the composite Trapezoidal rule using $h = \frac{1}{4}$. (1 pt.)

   (d) Determine the absolute value of the *truncation error* of the answer given in (c). (1 pt.)

3. We consider the **boundary-value problem**

$$\begin{cases} -y''(x) + (x+1)y(x) = x^3 + x^2 - 2, & 0 < x < 1, \\ y'(0) = 0, \quad y(1) = 1, \end{cases} \tag{3}$$

   where $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$.

   (a) We aim at solving the boundary value problem (3) using finite differences, upon setting $x_j = j\Delta x$, $(n+1)\Delta x = 1$, where Δx denotes the uniform step size.
   Give a discretisation method (+proof) where

   - the truncation error is of order $\mathcal{O}((\Delta x)^2)$;
   - the boundary conditions are taken into account;
   - and the discretisation matrix is symmetric.

   Use a virtual point for the boundary condition at $x = 0$. (2.5 pt.)

   (b) Give the linear system of equations $\mathbf{A}\mathbf{w} = \mathbf{f}$ that results from applying the finite-difference scheme from (a) with three (after processing the virtual points) unknowns (i.e. $\Delta x = 1/3$).
   Remark: You do **not** have to solve this linear system of equations. (1 pt.)

   (c) For another boundary value problem one obtains the $n \times n$ matrix $\mathbf{A}$ with components: $a_{i,i} = \frac{2}{(\Delta x)^2} + 1$ for $i = 1 \ldots n$, $a_{i-1,i} = \frac{-1}{(\Delta x)^2}$ for $i = 2 \ldots n$ and $a_{i,i-1} = \frac{-1}{(\Delta x)^2}$ for $i = 2 \ldots n$. All other components are equal to zero. Use the Gershgorin circle theorem to estimate the smallest eigenvalue $|\lambda|_{\min}$. From that conclude that the finite-difference scheme is stable, that is, $\mathbf{A}^{-1}$ exists and there is a constant C such that $\|\mathbf{A}^{-1}\| \leq C$ for $\Delta x \to 0$. (1.5 pt.)

For the answers of this test we refer to:
http://homepage.tudelft.nl/d2b4e/wi3097/tentamen.html