DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

Examiner responsible: C. Vuik
Examination reviewer: D. den Ouden-van der Horst

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS
(CTB2400)
Thursday June 29 2023, 13:30-16:30

Number of questions: This is an exam with 14 open questions, subdivided in 3 main questions.

Answers All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will not give points.

Tools Only a non-graphical, non-programmable calculator is permitted. All other electronic tools are not permitted.

Assessment In total 20 points can be earned. The final not-rounded grade is given by $P/2$, where P is the number of points earned.

1. a The fourth order Runge-Kutta method (RK_4) for the differential equation $y' = f(t, y)$ is given by the following formulas:

$$\begin{aligned} k_1 &= \Delta t f(t_n, w_n) \\ k_2 &= \Delta t f(t_n + \tfrac{1}{2}\Delta t, w_n + \tfrac{1}{2}k_1) \\ k_3 &= \Delta t f(t_n + \tfrac{1}{2}\Delta t, w_n + \tfrac{1}{2}k_2) \\ k_4 &= \Delta t f(t_n + \Delta t, w_n + k_3) \end{aligned}$$

$$w_{n+1} = w_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

Derive the amplification factor $Q(\lambda\Delta t)$ of RK_4. (2pt.)

b Use the fact that $y(t_{n+1}) = e^{\lambda\Delta t}y(t_n)$ holds for the exact solution of $y' = \lambda y$ to show that RK_4 solves the homogeneous test equation with a local truncation error of $\mathrm{O}(\Delta t^4)$. (Hint: $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...$) (2pt.)

In the following you may assume without proof that the last result also holds for systems and that, as a consequence, the global error of RK_4 applied to (1) is $\mathrm{O}(\Delta t^4)$.

We consider the following second order initial value problem:

$$y'' + py' + qy = \sin t, \quad y(0) = y_0, \; y'(0) = y_0'. \tag{1}$$

c Write (1) as a system of first order differential equations of type $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}(t)$. Give $\mathbf{A}$ and $\mathbf{g}$, and and show that the eigenvalues of $\mathbf{A}$ are given by: $\lambda_{1,2} = \frac{-p\pm\sqrt{p^2-4q}}{2}$. (2pt.)

(d) Let us consider p=1000 and q=249999. Give an approximate stability condition for this case. Hint: use that RK_4 is stable as $\Delta t \leq \frac{2.8}{-\lambda}$ for $\lambda < 0$. (2pt.)

(e) Suppose you have the choice between the Trapezoidal rule and RK_4 to integrate the problem given in (d). Motivate your choice as good as possible, taking into account the stability condition, the order of magnitude of the global error, and the amount of work. (2pt.)

2. We have approximated a function f satisfying $f(-1) = 0$, $f(0) = 2$ and $f(1) = 1$ with a natural cubic spline s given by

$$s(x) = \begin{cases} -\dfrac{3}{4}x^3 - \dfrac{9}{4}x^2 + \dfrac{1}{2}x + 2 & \text{if} \quad x \in [-1, 0), \\ \\ \dfrac{3}{4}x^3 - \dfrac{9}{4}x^2 + \dfrac{1}{2}x + 2 & \text{if} \quad x \in [0, 1]. \end{cases} \tag{2}$$

In the next exercises you will prove that s is indeed the natural cubic spline based on f. Then you will use s to approximate $f\left(\frac{1}{2}\right)$.

(a) *Show* that s is a piecewise function consisting of polynomials of degree 3 or lower. ($\frac{1}{2}$ pt.)

(b) *Show* that $s(x)$ equals $f(x)$ in the nodes. (1 pt.)

(c) *Show* that s, s' and s'' are continuous on the interval $[-1, 1]$. (2 pt.)

(d) *Show* that $s''(x)$ equals zero in the end points. ($\frac{1}{2}$ pt.)

(e) *Approximate* $f\left(\frac{1}{2}\right)$ with the use of (2). (1 pt.)

3. We are interested in the numerical integration of the integral

$$\int_0^{2\pi} y(x)dx,$$

with $y(x) = 1 + \cos(x)$.

(a) *Approximate* the above integral with the *right composite Rectangle rule* using $h = \pi/2$. ($1^1/2$ pt.)

(b) *Approximate* the above integral with the *composite Trapezoidal rule* using $h = \pi/2$. (1 pt.)

(c) The magnitude of the errors ε_R and ε_T of the approximations are bounded by

$$\varepsilon_R \leq \pi h \max_{x \in [0, 2\pi]} |y'(x)|,$$

for the right composite Rectangle rule and by

$$\varepsilon_T \leq \frac{\pi}{6} h^2 \max_{x \in [0, 2\pi]} |y''(x)|,$$

for the composite Trapezoidal rule. *Give* explicit upper bounds for both rules for general values of h. ($1^1/2$ pt.)

(d) *Select and motivate* which method you prefer if h becomes small. (1 pt.)

For the answers of this test we refer to:
https://diamhomes.ewi.tudelft.nl/~kvuik/wi3097/tentamen.html