Non-linear multigrid methods for Burgers’ equation

The one-dimensional viscous Burgers’ equation – named after the Dutch physicist Johannes Martinus
Burgers (1895–1981) – is a common prototype of a nonlinear partial differential equation (PDE) from
fluid mechanics. It describes the temporal evolution of a conserved quantity \( u \) subject to nonlinear
convective transport as well as viscous effects. The conservative form of the equation reads

\[
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{1}{2} u^2 \right) = \nu \frac{\partial^2 u}{\partial x^2}, \quad t \in (0, T], \quad x \in (a, b)
\]

which is complemented by compatible initial and boundary conditions. Despite its simplicity,
this equation features some of the challenges occurring in complex industrial flow models: the non-
linearity of the convective term, the creation of discontinuous solutions (shock waves) as the viscosity
coefficient \( \nu \to +0 \), etc. A good overview of the theory of scalar conservation laws is given in [LeV92].
The time-dependent one-dimensional Burgers’ equation can be cast into an equivalent space-time station-
ary problem to be solved in the two-dimensional domain \((0, T] \times (a, b)\) with a finite difference scheme.
This approach yields a nonlinear system of equations \( A(u)u = b \) which needs to be solved by a suitable
nonlinear solution algorithm. The main goal of this thesis is to compare the following two approaches:

1. Apply Newton’s method to the nonlinear problem and solve the resulting linear problems

\[
u^{(m+1)} = \nu^{(m)} - J_A^{-1}(\nu^{(m)})A(\nu^{(m)}), \quad m = 0, 1, \ldots
\]

until convergence, where \( J_A \) denotes the Jacobi matrix of the nonlinear operator \( A(u) \).

2. Apply a nonlinear multigrid (FAS) method to the nonlinear problem directly [vE02, Bri00].

Both approaches should be implemented in a MATLAB code and compared with respect to their nonlinear
convergence behavior. A numerical study should be performed to answer the following questions:

- How does the nonlinear convergence behavior depend on the strength of the nonlinearity (magni-
tude of the viscosity parameter)?
- How does the smoothness/discontinuity of the solution profile influence the convergence behavior?
- Which method performs better if only an approximation to the Jacobian matrix is available?

Project details

Advisor: Matthias Möller (m.moller@tudelft.nl)
Prerequisites: Basic knowledge of numerical methods for partial differential equations and/or iterative
solution techniques as well as programming skills (MATLAB/C/C++) is recommended.

References


Center for Appleid Scientific Comuting, Lawrence Livermore National Laboratory, 2002.