DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

The final grade of the test: (total number of points)/5

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TEST SCIENTIFIC COMPUTING (wi4201 / wi4201COSSE)
Wednesday January 25 2023, 13:30-16:30

1. Below 5 statements are given. If the statement is true give a short proof. If the statement is wrong give a counter example or an explanation.

   (a) For $A \in \mathbb{R}^{m \times n}$ the matrix norm $\|A\|_{max}$ is defined as $\|A\|_{max} = \max_{1 \le i \le m, 1 \le j \le n} |a_{i,j}|$. This norm has the multiplicative property. (2 pt.)

   (b) For $R \in \mathbb{R}^{m \times n}$ the following expression holds: $\|R\|_1 = \max_{1 \le i \le m} \sum_{j=1}^{n} |r_{ij}|$ (maximum absolute row sum) (2 pt.)

   (c) For any invertible matrix A the following inequality holds: $\kappa_p(A) \ge 1$. (2 pt.)

   (d) $A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$ The spectral radius of A is larger than 4. (2 pt.)

   (e) Let $\rho(A)$ be the spectral radius of A.

   $$\rho(A) < 1 \Rightarrow (I - A) \text{ is non-singular, and } \sum_{k=0}^{\infty} A^k = (I - A)^{-1}$$

   (2 pt.)

2. We consider the following boundary value problem:

   $$-(1 + x)\frac{d^2u(x)}{dx^2} + 4u(x) = x^2 \text{ for } 0 < x < 1\,, \tag{1}$$

   with boundary conditions

   $$u(0) = 0 \text{ and } u(1) = 0\,. \tag{2}$$

   A finite difference method is used on a uniform mesh with N intervals and mesh width $h = 1/N$.

   (a) Give the finite difference stencil for internal grid points and show that it is $O(h^2)$. (4 pt.)

(b) We use elimination of the boundary conditions. The numerical approximation $\mathbf{u}$ satisfies the linear system $A\mathbf{u} = \mathbf{f}$. Give matrix A and vector $\mathbf{f}$ for $N = 4$. (3 pt.)

(c) Give a lower bound of the real part of the eigenvalues of A for any value of N. (3 pt.)

3. Consider the linear system $A\mathbf{u} = \mathbf{f}$, where $A \in \mathbb{R}^{n \times n}$ is a nonsingular matrix.

   (a) Given that a non-singular matrix M exists we can split A as follows: $A = M - N$. Show that $\mathbf{u}^{k+1} = \mathbf{u}^k + M^{-1}\mathbf{r}^k$. (2 pt.)

   (b) Derive the Jacobi iteration matrix B_{Jac}. (hint $\mathbf{e}^{k+1} = B_{Jac}\mathbf{e}^k$) (2 pt.)

   (c) Consider a 2D Poisson equation with Dirichlet boundary conditions on a square domain, discretized by a 5-point stencil. Give the stencil notation of the Jacobi iteration matrix for an interior grid point. (2 pt.)

   (d) Derive the damped Jacobi method and give the damped Jacobi iteration matrix. (2 pt.)

   (e) Do 1 iteration with the backward Gauss-Seidel method to the following linear system, where we start with the zero vector.

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$

   (2 pt.)

4. In this exercise we have to solve a linear system $A\mathbf{u} = \mathbf{f}$, where A is an $n \times n$ non-singular matrix.

   (a) Take $\mathbf{u}_1 = \alpha\mathbf{f}$. Derive an expression for α such that $\|\mathbf{u} - \mathbf{u}_1\|_{A^TA}$ is minimal. (2 pt.)

   (b) The CGNR method is the CG method applied to $A^TA\mathbf{u} = A^T\mathbf{f}$. Show that the 2-norm of the residuals is monotone decreasing. (2 pt.)

   (c) Give a 4×4 non-diagonal matrix such that CGNR converges in one iteration for every right-hand side vector $\mathbf{f}$. (2 pt.)

   (d) Given the algorithm

Conjugate Gradient Squared method

$\mathbf{u}^0$ is an initial guess; $\mathbf{r}^0 = \mathbf{f} - A\mathbf{u}^0$;
$\tilde{\mathbf{r}}^0$ is an arbitrary vector, such that
$(\mathbf{r}^0)^T\tilde{\mathbf{r}}^0 \neq 0$,
e.g., $\tilde{\mathbf{r}}^0 = \mathbf{r}^0$; $\rho_0 = (\mathbf{r}^0)^T\tilde{\mathbf{r}}^0$;
$\beta_{-1} = \rho_0$; $\mathbf{p}_{-1} = \mathbf{q}_0 = 0$;
for $i = 0, 1, 2, ...$ do
  $\mathbf{w}^i = \mathbf{r}^i + \beta_{i-1}\mathbf{q}^i$;
  $\mathbf{p}^i = \mathbf{w}^i + \beta_{i-1}(\mathbf{q}^i + \beta_{i-1}\mathbf{p}^{i-1})$;
  $\hat{\mathbf{p}} = M^{-1}\mathbf{p}^i$;
  $\hat{\mathbf{v}} = A\hat{\mathbf{p}}$;
  $\alpha_i = \frac{\rho_i}{(\tilde{\mathbf{r}}^0)^T\hat{\mathbf{v}}}$;
  $\mathbf{q}^{i+1} = \mathbf{w}^i - \alpha_i\hat{\mathbf{v}}$;
  $\hat{\mathbf{w}} = M^{-1}(\mathbf{w}^i + \mathbf{q}^{i+1})$
  $\mathbf{u}^{i+1} = \mathbf{u}^i + \alpha_i\hat{\mathbf{w}}$;
  if $\mathbf{u}^{i+1}$ is accurate enough then quit;
  $\mathbf{r}^{i+1} = \mathbf{r}^i - \alpha_i A\hat{\mathbf{w}}$;
  $\rho_{i+1} = (\tilde{\mathbf{r}}^0)^T\mathbf{r}^{i+1}$;
  if $\rho_{i+1} = 0$ then method fails to converge!;
  $\beta_i = \frac{\rho_{i+1}}{\rho_i}$;
end for

The matrix M in this scheme represents the preconditioning matrix. Determine the minimal amount of memory and flops per iteration. (2 pt.)

(e) Give a comparison of the mathematical properties of the CGNR and CGS method (both without preconditioning). (2 pt.)

5. We consider the following boundary value problem:

$$-\frac{d^2u(x)}{dx^2} = f \text{ for } 0 < x < 1\,, \tag{3}$$

with boundary conditions

$$u(0) = 0 \text{ and } u(1) = 0\,. \tag{4}$$

A finite difference method is used on a uniform mesh with mesh width h. Elimination of boundary conditions is used. This leads to a linear system $A_h\mathbf{u}_h = \mathbf{f}_h$, where $A_h \in \mathbb{R}^{n\times n}$ is a nonsingular matrix. We consider the two grid and multi-grid method to solve this linear system. For the two grid method the intergrid transfer operators I_h^H and I_H^h are used.

(a) For the Coarse Grid Correction operator we use the Galerkin approach to obtain A_H, with $I_H^h = (I_h^H)^T$. Show that A_H is an SPD matrix. (2 pt.)

(b) We now take $h = \frac{1}{6}$. For the integrid operator we use the injection operator so

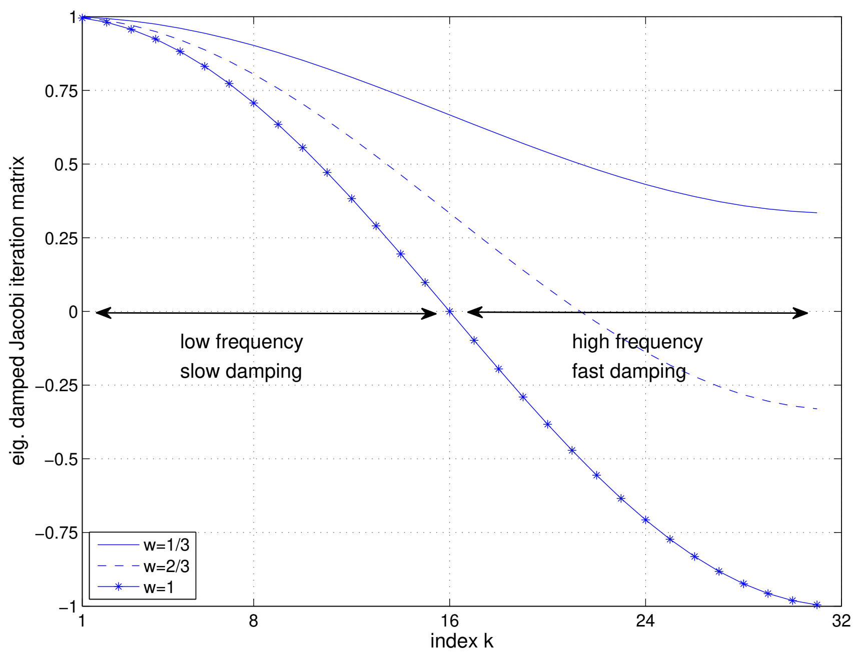
$$I_h^H = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Compute the Coarse Grid Correction matrix A_H. (2 pt.)

(c) The error vector of the two grid method satisfies the following relation: $\mathbf{e}^{k+1} = B_{CGC}\mathbf{e}^k$. Give matrix B_{CGC}.

(2 pt.)

(d) The smoothing behaviour of the damped Jacobi method for $n = 32$ and three values of ω is given in the next figure. Which value of ω leads to the best



smoother. Motivate your answer. (2 pt.)

(e) The solution vector $\mathbf{u}_h$ has length n. Using the multi-grid method, also the coarse grid vectors $\mathbf{u}_{2h}, \mathbf{u}_{4h}, \mathbf{u}_{8h}, \ldots.$ has to be stored. Give the amount of memory needed to store all solution vectors. (2 pt.)