DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

The final grade of the test: (total number of points)/5

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TEST SCIENTIFIC COMPUTING (wi4201)
Wednesday January 24 2024, 13:30-16:30

1. Below 5 statements are given. If the statement is true give a short proof. If the statement is wrong give a counter example or an explanation.

   (a) $Q \in \mathbb{R}^{n\times n}$ is an orthogonal matrix. For all $A \in \mathbb{R}^{n\times n}$ the equality $\|A\|_2 = \|Q^T A\|_2$ holds. (2 pt.)

   (b) $A \in \mathbb{R}^{n\times n}$ $\rho(A) \leq \|A\|$ for any multiplicative norm $\|.\|$. (2 pt.)

   (c) $A \in \mathbb{R}^{n\times n}$ is SPD, and $\mathbf{v}_1, \mathbf{v}_2$ are eigenvectors of A, where $\lambda_1 = \lambda_2$. Assume that $\mathbf{r} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2$. $\Rightarrow$ The Krylov subspace $K^5(A, \mathbf{r})$ has dimension 3. (2 pt.)

   (d) Suppose that $A \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix. $\Rightarrow$ For all $k = 1, ..., n$ the matrix A_k consisting of the first k rows and columns of A is also a symmetric and positive definite matrix. (2 pt.)

   (e) $I \in \mathbb{R}^{n\times n}$ is the identity matrix. $\Rightarrow$ $K_p(I) = 1$. (2 pt.)

2. For a given function f we consider the following boundary value problem:

$$-\frac{d^2u(x)}{dx^2} - 10u(x) = f(x) \text{ for } 0 < x < 1\,, \tag{1}$$

   with boundary conditions

$$u(0) = 0 \text{ and } u(1) = 0\,. \tag{2}$$

   A finite difference method is used on a uniform mesh with N intervals and mesh width $h = 1/N$.

   (a) Give the finite difference stencil for internal grid points and show that it is $O(h^2)$. (2.5 pt.)

   (b) The eigenvectors $\mathbf{v}^k$ of the resulting coefficient matrix A are given and have components:

$$v_i^k = \sin(k\,\pi\,x_i) = \sin(k\,\pi\,(i-1)\,h) \text{ for } 1 \leq i \leq N+1 \tag{3}$$

derive an expression for the corresponding eigenvalues λ_k as a function of the meshwidth h by computing the action of A^h on these eigenvectors. It suffices here to consider the matrix rows corresponding to internal grid nodes. (Hint: $\sin(\alpha+\beta)+\sin(\alpha-\beta) = 2\sin(\alpha)\cos(\beta)$). (2.5 pt.)

(c) If the boundary conditions are eliminated, is the matrix A symmetric positive definite for $N = 10$? (give a proof or a counter example) (2.5 pt.)

(d) Solving this problem in 2 dimensions one can use a 5 point stencil. We take 4 intervals in x and y-direction ($h = 0.25$). Give the nonzero pattern of the matrix (after elimination of the boundary conditions) if a red-black ordering is used (hint: you can use graph paper). (2.5 pt.)

3. (a) Given the linear system $A\mathbf{u} = \mathbf{f}$ with $A \in \mathbb{R}^{n\times n}$. Consider a splitting of the form $A = M - N$ where M is non-singular. Derive a recursion formula for the BIM (Basic Iterative Method) iterates $\mathbf{u}^k$. Derive a recursion formula for the residual vector $\mathbf{r}^k$. (2 pt.)

   (b) Give the iteration matrix B for this BIM and give a sufficient condition such that the BIM converges. (2 pt.)

   (c) Suppose A is a non-singular upper triangular matrix. Show that the Jacobi method converges for such a matrix. (2 pt.)

   (d) Suppose A is a non-singular upper triangular matrix. Which Gauss Seidel variant is optimal for this matrix (motivate your answer with advantages and/or disadvantages)? (2 pt.)

   (e) Give three different stopping criteria and specify the good and bad properties of these stopping criteria. (2 pt.)

4. In this exercise we solve a linear system $A\mathbf{u} = \mathbf{f}$, where A is an $n \times n$ SPD matrix.

   (a) Take $\mathbf{u}^1 = \alpha\mathbf{f}$. Derive an expression for α such that $\|\mathbf{u}-\mathbf{u}^1\|_2$ is minimal. (2 pt.)

   (b) Show that the value of α as given in part (a) is not easy to calculate. Adapt the minimization property such that α can be computed easily (+ proof). (2 pt.)

   (c) Give the optimality property of the CG method. Motivate why there is a $k \leq n$ such that $\mathbf{u}^k = \mathbf{u}$ (without rounding errors). (2 pt.)

   (d) We consider two different matrices. The extreme eigenvalues of A_1 are given by $\lambda_1 = 1$ and $\lambda_n = 10$. The extreme eigenvalues of A_2 are given by $\lambda_1 = 0.1$ and $\lambda_n = 20$. For which matrix can you expect the CG method to converge faster (+ motivation)? (2 pt.)

   (e) For the preconditioned CG method a preconditioner matrix M is needed. Give three properties that M should satisfy. (2 pt.)

5. (a) Show that the inverse of the Gauss transformation $M_k = I - \alpha^{(k)}\mathbf{e}_k^T$ is the rank-one modification $M_k^{-1} = I + \alpha^{(k)}\mathbf{e}_k^T$. The k-th Gauss-vector $\alpha^{(k)} \in \mathbb{R}^n$ is defined as

$$\alpha^{(k)} = (\underbrace{0, \ldots, 0}_{k}, \underbrace{\mathbf{b}_k / a_{k,k}^{(k-1)}}_{n-k})^T . \tag{4}$$

(2.5 pt.)

(b) Given the linear system $A\mathbf{u} = \mathbf{f}$ with $A \in \mathbb{R}^{n\times n}$ where A is a non-singular matrix and the perturbed system $A(\mathbf{u} + \Delta\mathbf{u}) = \mathbf{f} + \Delta\mathbf{f}$. Derive an upperbound for $\frac{\|\Delta\mathbf{u}\|}{\|\mathbf{u}\|}$ where $\|.\|$ is an arbitrary vector norm, which has the multiplicative property. (2.5 pt.)

(c) Suppose we have a tri-diagonal matrix $A \in \mathbb{R}^{n\times n}$. We know that elements $a(i-1,i)$, $a(i,i)$, and $a(i,i+1)$, are nonzero. Give the outline of the LU-decomposition method (without pivotting) to solve $A\mathbf{u} = \mathbf{f}$, where $A \in \mathbb{R}^{n\times n}$ is a non-singular tri-diagonal matrix and give the amount of flops.

(3 pt.)

(d) Do 1 step of the Gaussian elimination process on the following matrix:

$$\begin{pmatrix} 4 & -1 & 0 & 0 & -1 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ -1 & 0 & 0 & -1 & 4 \end{pmatrix}$$

What is the size of the fill in? (2 pt.)