

Using $a_1 \neq 0$ we obtain

$$A^k \underline{q}_0 = a_1 \lambda_1^k \left(\underline{v}_1 + \sum_{j=2}^n \frac{a_j}{a_1} \left(\frac{\lambda_j}{\lambda_1} \right)^k \underline{v}_j \right) \quad (0.1)$$

This can be used to prove:

$$\left| \lambda_1 - \lambda^{(k)} \right| = O\left(\left| \frac{\lambda_2}{\lambda_1} \right|^k \right) \quad (0.2)$$

Proof: For simplicity we assume $\underline{q}_0 \in \mathbb{R}^n$

We know that $\| \underline{q}_{k-1} \|_2 = 1$.

Since $|\lambda_2| \geq |\lambda_3| \dots, |\lambda_n|$ (0.1) implies:

$$\begin{aligned} \left\| \frac{A^k \underline{q}_0}{a_1 \lambda_1^k} - \underline{v}_1 \right\| &\leq \sum_{j=2}^n \frac{|a_j|}{|a_1|} \left| \frac{\lambda_j}{\lambda_1} \right|^k \|\underline{v}_j\|_2 \\ &\leq C \left| \frac{\lambda_2}{\lambda_1} \right|^k = O\left(\left| \frac{\lambda_2}{\lambda_1} \right|^k \right) \quad (*) \end{aligned}$$

To simplify notation we write (0.1) as:

$$\underline{q}_{k-1} = \frac{A^k \underline{q}_0}{\|A^k \underline{q}_0\|_2} = \gamma (\underline{v}_1 + \underline{w}) \quad (**)$$

$\gamma = \frac{a_1 \lambda_1^k}{\|A^k \underline{q}_0\|_2}$ and \underline{w} contains the remaining vectors but $\|\underline{w}\|_2 = O\left(\left| \frac{\lambda_2}{\lambda_1} \right|^k \right)$

see (*)

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method of ...

$$(1) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

... ..

$$\|A\| = \sqrt{\lambda_{max}}$$

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$$(3) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

... ..

$$(4) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

... ..

$$\|A\| = \sqrt{\lambda_{max}} = \sqrt{1} = 1$$

(a) ...

Since $\lambda^{(k)} = \underline{q}_{k-1}^T \underline{z}_k = \underline{q}_{k-1}^T A \underline{q}_{k-1}$

we obtain (see (**))

$$\lambda^{(k)} = \gamma(\underline{v}_1 + \underline{w})^T A \gamma(\underline{v}_1 + \underline{w})$$

$$= (\gamma \underline{v}_1 + \gamma \underline{w})^T \gamma(\lambda_1 \underline{v}_1 + A \underline{w})$$

$$\Rightarrow \lambda_1 (\gamma \underline{v}_1)^T (\gamma \underline{v}_1) + O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) \text{ due to (*)}$$

$$= \lambda_1 \|\gamma \underline{v}_1\|_2^2 + O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) \text{ (***)}$$

From (***) it follows that

$$\gamma \underline{v}_1 = \underline{q}_{k-1} - \underline{w}$$

$$\begin{aligned} \text{thus } \|\gamma \underline{v}_1\|_2 &= \|\underline{q}_{k-1} - \underline{w}\|_2 \\ &= \|\underline{q}_{k-1}\|_2 + O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) \\ &= 1 + O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) \end{aligned}$$

Combining this with (***) show that,

$$\lambda^{(k)} = \lambda_1 + O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) \quad \text{Q.E.D.}$$

$$\text{which shows that } |\lambda^{(k)} - \lambda_1| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)$$

