Using $a_1 \neq 0$ we obtain

$$A^k \underline{q}_0 = a_1 \lambda_1^k \left(\underline{v}_1 + \sum_{j=2}^{n} \frac{a_j}{a_1} \left(\frac{\lambda_j}{\lambda_1} \right)^k \underline{v}_j \right) \quad (8.1)$$

This can be used to prove:

$$|\lambda_1 - \lambda^{(k)}| = O\left(\left| \frac{\lambda_2}{\lambda_1} \right|^k \right) \quad (8.2)$$

Proof: For simplicity we assume $\underline{q}_0 \in \mathbb{R}^n$

We know that $\| \underline{q}_{k-1} \|_2 = 1$.

Since $|\lambda_2| \geq |\lambda_3| \ldots |\lambda_n|$ (8.1) implies:

$$\left\| \frac{A^k \underline{q}_0}{a_1 \lambda_1^k} - \underline{v}_1 \right\| \leq \sum_{j=2}^{n} \frac{|a_j|}{|a_1|} \left| \frac{\lambda_j}{\lambda_1} \right|^k \underbrace{\| \underline{v}_j \|_2}_{=1}$$

$$\leq C \left| \frac{\lambda_2}{\lambda_1} \right|^k = O\left(\left| \frac{\lambda_2}{\lambda_1} \right|^k \right) \quad (*)$$

To simplify notation we write (8.1) as:

$$\underline{q}_{k-1} = \frac{A^k \underline{q}_0}{\| A^k \underline{q}_0 \|_2} = \gamma \left(\underline{v}_1 + \underline{w} \right) \quad (**)$$

$$\gamma = \frac{a_1 \lambda_1^k}{\| A^k \underline{q}_0 \|_2}$$ and $\underline{w}$ contains the remaining vectors but $\| \underline{w} \|_2 = O\left(\left| \frac{\lambda_2}{\lambda_1} \right|^k \right)$

see (*)

Since $\lambda^{(k)} = \underline{q}_{k-1}^T \underline{z}_k = \underline{q}_{k-1}^T A \underline{q}_{k-1}$
we obtain (see (**))

$$\lambda^{(k)} = \gamma(\underline{v}_1 + \underline{w})^T A \gamma(\underline{v}_1 + \underline{w})$$

$$= (\gamma\underline{v}_1 + \gamma\underline{w})^T \gamma(\lambda_1 \underline{v}_1 + A\underline{w})$$

$$= \lambda_1 (\gamma\underline{v}_1)^T (\gamma\underline{v}_1) + O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) \quad \text{due to } (*)$$

$$= \lambda_1 \|\gamma\underline{v}_1\|_2^2 + O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right) \qquad (***)$$

From (**) it follows that

$$\gamma\underline{v}_1 = \underline{q}_{k-1} - \underline{w}$$

thus
$$\|\gamma\underline{v}_1\|_2 = \|\underline{q}_{k-1} - \underline{w}\|_2$$
$$= \|\underline{q}_{k-1}\|_2 + O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)$$
$$= 1 + O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)$$

Combining this with (***) show that

$$\lambda^{(k)} = \lambda_1 + O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)$$

Q.E.D.

which shows that $|\lambda^{(k)} - \lambda_1| = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)$