Multigrid-based preconditioners for the heterogeneous Helmholtz equation

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Outline of the Talk

Objective of the talk:

A fast, robust iterative solver for the 2D Helmholtz equation in inhomogeneous media with high wave numbers (of typically 300–600 wave numbers in a unit square domain)

Outline:

- Problem definition: Helmholtz equation, discretization
- Krylov subspace method and multigrid
- Multigrid based preconditioner
- Numerical examples
- Conclusion

Time-harmonic (Helmholtz) wave equation

Helmholtz problem: Given $k = k(x,y)$, the wave number, in a domain $\Omega \subset \mathbb{R}^2$. Find $u \in \mathbb{C}$ such that the following PDE is satisfied:

$$\begin{aligned} \mathcal{L}u \equiv \left(-\partial_{xx} - \partial_{yy} - (1 - i\alpha)k^2\right) u &= f \quad \text{in } \Omega \\ \frac{\partial u}{\partial \nu} - iku - \frac{i}{2k}\frac{\partial^2 u}{\partial \tau^2} &= 0 \quad \text{on } \Gamma = \partial\Omega \end{aligned}$$

with:

- ν the outward normal direction to Γ,
- τ the tangential direction to Γ
- k the wavenumber with $k = 2\pi f/c$. f and c are the frequency and the speed of sound, respectively.
- $0 \leq \alpha \ll 1$

Example of applications: aeroacoustics (sound produced by an engine), electromagnetics (lithophotography), geophysics (subsurface mapping).

Typical in geophysical applications:

- High wave number
- Extreme constrast of k

Discretization

Few discretization methods:

- Finite difference
- Finite element

In our application, finite difference is used. This is common practice in geophysics in which simple domain is usually considered (e.g. rectangular).

- compact, 5 (or 9)-point finite difference stencil [*Harari and Turkel, 1995*]
- boundary: one-sided finite difference, $\mathcal{O}(h)$ (or central difference, $\mathcal{O}(h^2)$)

This leads to the linear system

$$Ax = b, \;\; A \in \mathbb{C}^{N \times N}, \;\; b, x \in \mathbb{C}^N,$$

where A is a sparse, large, highly indefinite symmetric matrix.

For high resolution solutions, very fine grid is required.

Numerical methods for solving $Ax = b$

We solve the linear system iteratively using a Krylov subspace method.

Iterative methods: Krylov subspace methods

Definition

$$\mathcal{K}^j(A, r_0) = \text{span}\{r_0, Ar_0, A^2 r_0, \cdots, A^{j-1} r_0\}$$

where $\mathcal{K}^j(A, r_0)$ is the j-th Krylov subspace and $r_0 = b - Ax_0$ the initial residual related to the starting vector x_0.

- requires only matrix/vector multiplications (of order $\mathcal{O}(N)$) (exploiting sparsity of A)

Computational constraint:

- The Krylov method is chosen such that the iteration can be performed within limited computer storage
- The Krylov method with constant amount of work per iteration is the method of choice
- With respect to the preconditioner, the method should be less stringent to the symmetry property of the preconditioning matrix

Currently we choose BiCGSTAB. [However, if $M^{-1}A$ is symmetric, COCG or SQMR may be the best choice]

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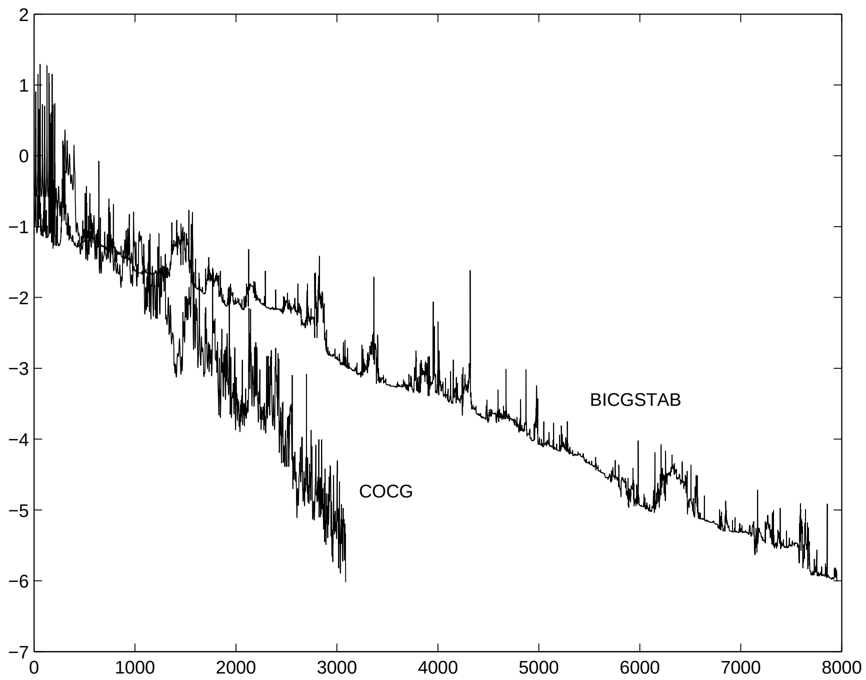
Numerical methods for solving $Ax = b$: illustration

Illustrative problem: The Helmholtz problem in homogeneous medium. $\Omega = (0,1)^2$, $k = 50$, $N = 250^2$, with Sommerfeld's condition at $x = 0, 1$ and $y = 1$, and Dirichlet condition at $y = 0$. Without residual smoother.



x-axis: number of iteration, y-axis: $\log_{10}(\|r\|_2/\|b\|_2)$

For unpreconditioned case, COCG outperforms BiCGSTAB. GMRES is found to be less effective as compared to Bi-CGSTAB

Multigrid

Multigrid is known as a good and efficient method for elliptic PDE. For Poisson equation, e.g., the complexity can be up to order $O(N)$.

Multigrid principles:

- Error smoothing $\longrightarrow$ some iterative methods may have smoothing property of the error
- Coarse grid correction $\longrightarrow$ a smooth error can be well approximated on the coarse grid

Multigrid for the Helmholtz equation: the problem is indefinite,

- standard iteration (e.g. Jacobi) does not converge. Multigrid only converges if the coarsest grid is fine enough to represent smooth frequencies $\longrightarrow$ the convergence is limited by the coarsest grid. Therefore, the method is no longer $O(N)$.
- eigenvalues close to the origin may change signs $\longrightarrow$ convergence degradation

Example: Standard multigrid, constant k, $N = 256^2$, 0.8-JAC, V(1,1)

k	1	2.5	5
Iter	17	14	div
ρ	0.535	0.44	$>$1.0

Multigrid based preconditioner

The idea: use multigrid as the preconditioner to accelerate the Krylov subspace iteration.

Meaning: solve

$$AC^{-1}\tilde{x} = b, \qquad \tilde{x} = Cx,$$

where $C \in \mathbb{C}^{N \times N}$ is the preconditioner and C^{-1} is approximated by multigrid.

Example: $C = A$. C^{-1} is approximated with **one** V(1,1) multigrid iteration

k	1	2.5	5	20
GMRES	8	9	14	93
Bi-CGSTAB	5	5	11	$>$100

Choosing $C = A$ does not result in a good method. As the problem becomes more indefinite (more eigenvalues change signs) for larger k, the method diverges

Multigrid based preconditioner

Find C such that:

- AC^{-1} better conditioned than A
- C^{-1} easy to solve using multigrid

Complex shift of the Laplace operator as the preconditioner:

$$\mathcal{M} := -\partial_{xx} - \partial_{yy} - (\beta_1 - i\beta_2)k^2, \qquad \beta_1, \beta_2 \in \mathbb{R}, \qquad i = \sqrt{-1}.$$

C is built based on discretization of $\mathcal{M}$.

For operator based preconditioner:

- the convergence of CG-like iteration can be made h-independent
- for AC^{-1}, h-independent convergence is obtained if the same boundary condition is used in $\mathcal{L}$ and $\mathcal{M}$.

Multigrid based preconditioner

Case 1: $\mathcal{M}$ is a complex, symmetric positive definite operator (CSPD). In this case $\beta_1 \leq 0$.

We find for AC^{-1} that [E, Vuik, Oosterlee, 2004]:

- $|\lambda|_{\text{max}} \rightarrow 1$
- $|\lambda|_{\text{min}} \sim \mathcal{O}(\epsilon/k^2)$
- the condition number κ is minimal if $(\beta_1, \beta_2) = (0, \pm 1)$

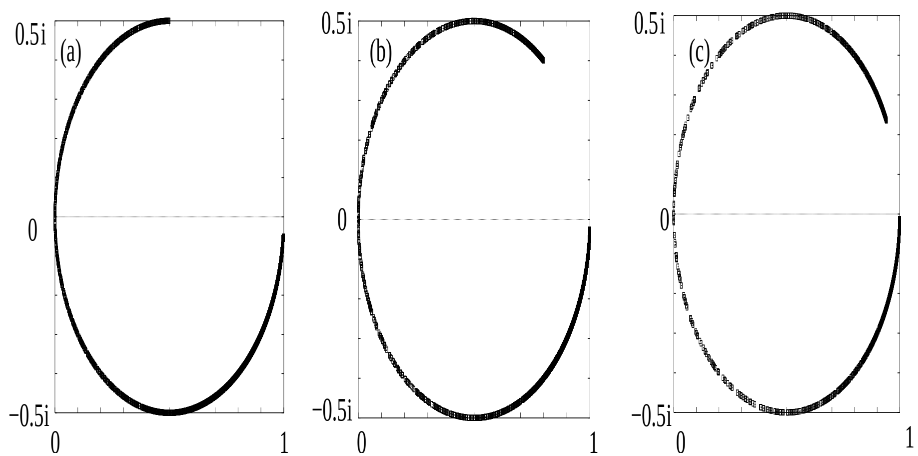
Example: Multigrid as solver for $C_{(0,1)}$. Constant k, $N = 256^2$, 0.8-JAC, V(1,1)

k	1	2.5	5	20
Iter	10	8	8	8
ρ	0.310	0.269	0.201	0.201

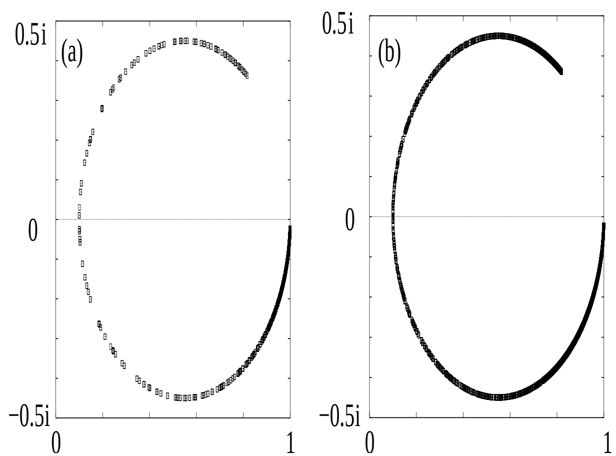
Multigrid based preconditioner

Case 2: $\mathcal{M}$ is a more general operator, but is still suitable for multigrid convergence. For example: $\beta_1 = 1$ and $\beta_2 \in \mathbb{R}$.

We investigate the following pairs: $(\beta_1, \beta_2) = (1, 1), (1, 0.5)$.



0% damp, $k = 100$: (a) $\beta = (1{,}1)$ (b) $\beta = (1{,}0.5)$ (c) $\beta = (1{,}0.3)$



5% damp, $\beta = (1{,}0.5)$: (a) $k = 40$ h $= 1/64$ (b) $k = 100$, h $= 1/160$

Numerical example

Problem 1: Homogeneous medium. $\Omega = (0,1)^2$, $k = 50$, $N = 250^2$.

Convergence with respect to k: Bi-CSGTAB, F(1,1) multigrid cycle. Second order absorbing condition.

Multigrid components:

- for $(\beta_1, \beta_2) = (0, 1)$: 0.8-JAC
- for $(\beta_1, \beta_2) = (1, 1)$: 0.7-JAC
- for $(\beta_1, \beta_2) = (1, 0.5)$: 0.5-JAC
- Coarse grid operator: Galerkin
- Prolongator: bi-linear interpolation/matrix dependent
- Restrictor: full weighting

	k :							
(β_1, β_2)	40	50	80	100	150	200	500	600
(0,1)	57 (0.44)	73 (0.92)	112 (4.3)	126 (7.7)	188 (28.5)	–	–	–
(1,1)	36 (0.30)	39 (0.51)	54 (2.2)	74 (4.5)	90 (13.9)	114 (30.8)	291 (515)	352 (890)
(1,0.5)	26 (0.21)	31 (0.40)	44 (1.8)	52 (3.3)	73 (10.8)	92(25.4)	250 (425)	298 (726)

Numerical Example

Problem 2: Wedge problem. $\Omega = (0, 600) \times (0, 1000)$ m2. Bi-CSSTAB. Second order absorbing condition.

Multigrid components:

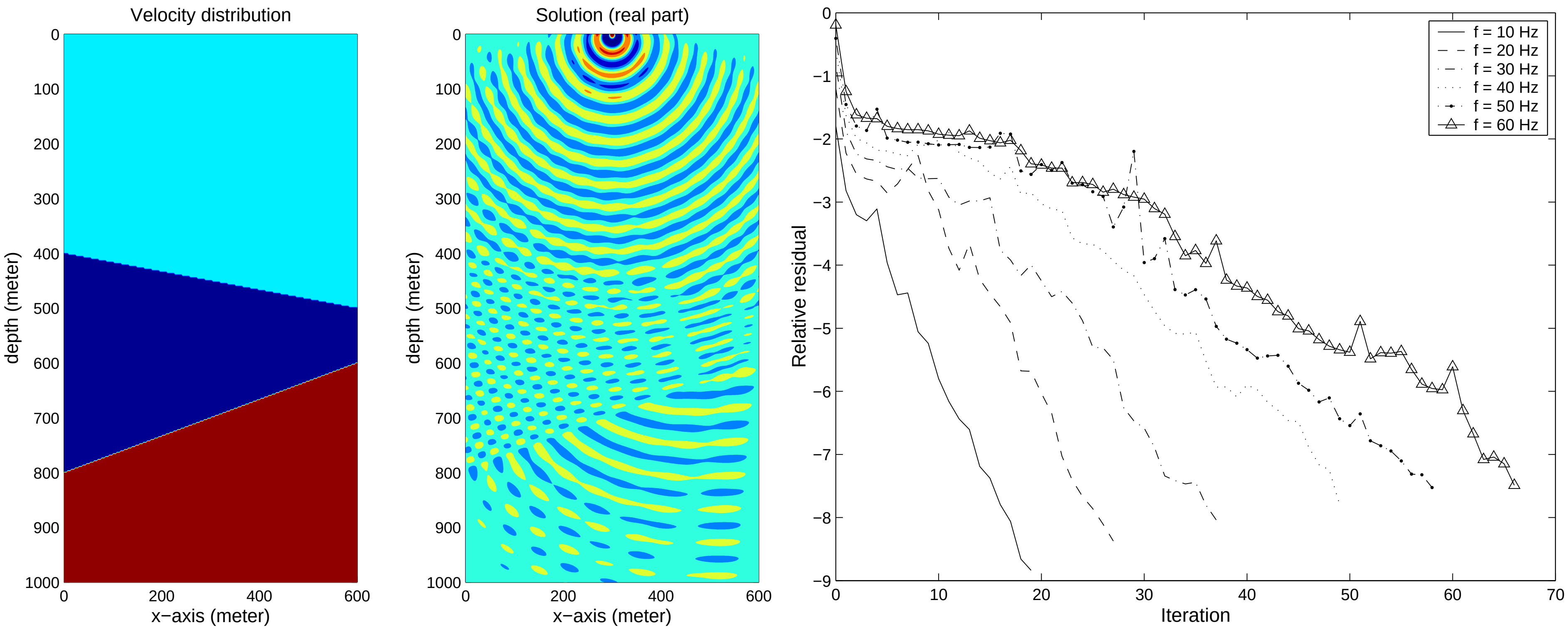
- prolongator : matrix dependent
- restrictor : full weighting

f (Hz)	Grid	(β_1, β_2) (0,1)	(1,1)	(1,0.5)
10	75 × 125	52 (1.2)	30 (0.67)	19 (0.42)
20	149 × 249	91 (8.8)	45 (4.5)	27 (2.8)
40	301 × 501	161 (66.1)	80 (33.5)	49 (20.8)
60	481 × 801	232 (247.3)	118 (127.6)	66 (71.9)

In [Plessix, Mulder, 2004], Separation of Variables based preconditioner does not converge for f = 50 Hz after 2000 iterations.

Numerical Example

Solution of wedge problem, f = 50 Hz.



Numerical Example

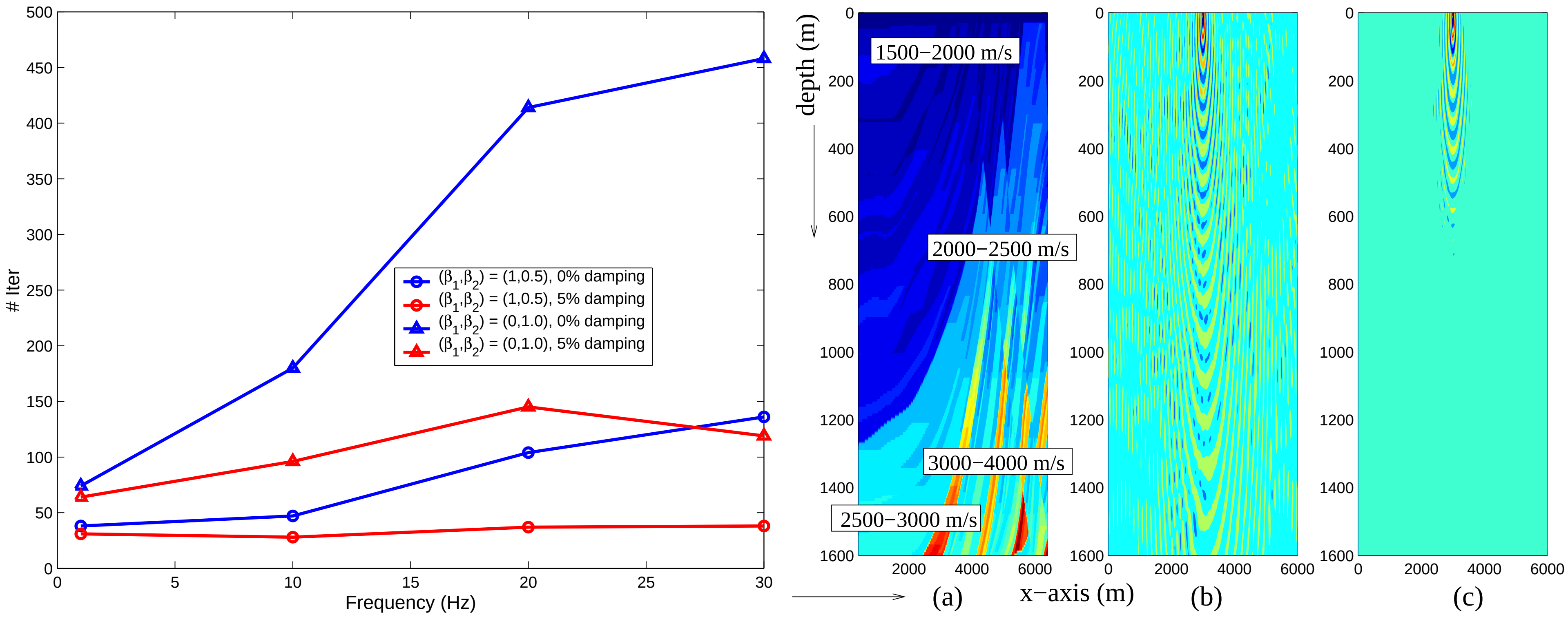
Problem 3: Marmousi. $\Omega = (0, 6000) \times (0, 1600)$ m2. Bi-CGSTAB and F(1,1) multigrid. Second order absorbing condition.

f (Hz)	Grid	damping	(β_1, β_2) (0,1)	(1,1)	(1,0.5)
1	751 × 201	0.0%	74 (37.5)	54 (27.6)	38 (19.7)
		5.0%	64 (32.5)	53 (27.3)	31 (16.5)
10	751 × 201	0.0%	180 (89.2)	84 (42.4)	47 (24.2)
		5.0%	96 (48.3)	48 (24.8)	28 (15.0)
20	1501 × 401	0.0%	414 (832.3)	168 (308.7)	104 (212.1)
		5.0%	145 (294.7)	64 (133.4)	37 (79.3)
30	2001 × 534	0.0%	458 (1724.8)	211 (799.4)	136 (519.4)
		5.0%	119 (455.3)	61 (238.4)	38 (151.9)

In [Plessix, Mulder, 2004], Separation of Variables based preconditioner does not converge for f = 30 Hz after 2000 iterations.

Numerical Example

Solution of Marmousi problem, f = 30 Hz. Figure (b): 0 % damping. (c): 5 % damping.



Conclusion

- A new class of preconditioner for the Helmholtz equation based on Shifted-Laplace operator is proposed and analyzed.
- This class of preconditioners clusters the eigenvalues around the origin, which give some very small eigenvalues
- Numerical tests show the effectiveness of the preconditioners
- Extension to varying k can be done easily