Conservation properties of a new unstructured staggered scheme

I. Wenneker*, A. Segal and P. Wesseling

J.M. Burgers Center and Department of Applied Mathematical Analysis, Faculty of Information Technology and Systems, Delft University of Technology, Mekelweg 4, 2628 CD, Delft, Netherlands

E-mail: Ivo Wenneker (i.wenneker@its.tudelft.nl)
Fax: 0031 - 15 - 278 7209
Telephone: 0031 - 15 - 278 1692

Abstract

In [14] an unstructured staggered scheme for the two-dimensional Euler equations is discussed. Such a scheme cannot be in classic conservation form for momentum. The aim of the present paper is to formulate a generalized conservation form for momentum on unstructured staggered grids, and to demonstrate by numerical experiments that the scheme satisfies the Rankine-Hugoniot conditions. Numerical results for one-dimensional Riemann problems on two-dimensional unstructured grids confirm that the scheme converges to the entropy solution. In addition, transonic flow around an airfoil is computed.

Keywords: Euler equations; compressible flows; conservative schemes; unstructured grids; staggered schemes

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1 Introduction

Most established methods for compressible gas dynamics use colocated schemes. For incompressible flows, a straightforward discretization on a colocated grid results in odd-even oscillations of the pressure. To remedy this, artificial stabilizing measures, like the pressure-weighted interpolation method of Rhie and Chow [11], have to be taken. This problem of spurious pressure oscillations does not occur with staggered schemes. The difficulty also does not arise in the compressible case. Because on non-orthogonal grids colocated discretization is more straightforward than staggered discretization, colocated schemes are prevalent for compressible flows, and have reached a certain degree of maturity.

The classic incompressible staggered scheme of Harlow and Welch [6] can be applied to compressible flows as well, as shown already in [4, 5]. In addition, staggered schemes can be devised that are accurate on highly non-orthogonal structured grids, see [16, 15] for the incompressible flow case and [2, 13] for compressible flows. Because an extension of an incompressible scheme is involved, a unified method for compressible and incompressible flows can be obtained, with Mach-uniform accuracy and efficiency. To achieve this with colocated schemes, special measures, such as preconditioning, have to be taken, leading to additional computing costs, particularly in the nonstationary case. For a recent discussion of difficulties and remedies concerning the zero Mach limit of compressible colocated schemes, see [3].

The above remarks pertain to structured schemes. It has become generally recognized that structured grid generation in complicated domains cannot be automated to a satisfactory extent. Therefore unstructured grids are now receiving widespread attention.

Unstructured staggered schemes for compressible flows have, apart from [14], apparently not yet been considered. In this paper we focus on the conservation properties of the spatial discretization that is introduced in [14]. Generalization to a Mach-uniform compressible-incompressible method has been done also and will be discussed in a forthcoming paper. Here we restrict ourselves to the fully compressible case, because this is least explored.

Weak solutions of hyperbolic conservation laws are called genuine if they satisfy the jump conditions; these are called the Rankine-Hugoniot conditions in the case of the Euler equations of gasdynamics. The well-known Lax-Wendroff theorem [7] states that convergent numerical schemes in conservation form converge to genuine weak solutions. Colocated finite volume schemes for conservation laws, i.e. schemes in which all primary unknowns reside in the same nodes, are in conservation form. The proof of the Lax-Wendroff theorem assumes colocated schemes. Because the theorem suggests and experience confirms that conservation is important for satisfying the jump conditions, our staggered scheme is in conservation form for mass and energy, and on Cartesian grids also for momentum. However, on non-Cartesian grids, with normal momentum components at the cell faces, there is no conservation form, because momentum and pressure balance only in a fixed direction. Therefore we use a generalized concept of conservation form for momentum and rely on numerical experiment to show that the jump conditions are satisfied.

Conservation of kinetic energy, circulation and momentum is demonstrated for two different unstructured staggered schemes for incompressible flows by Perot in [10]. Perot needs a dual mesh, formed by the associated Voronoi tessellation of the triangular mesh, in his conservation

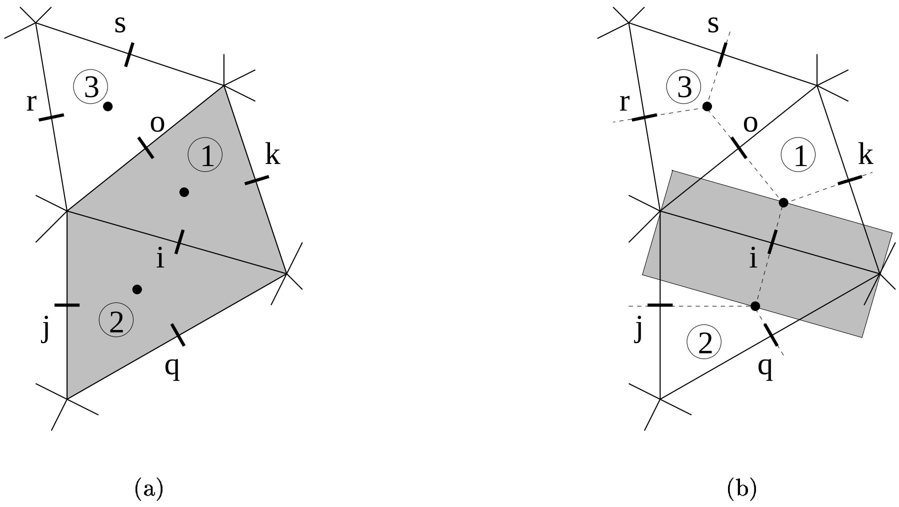


Figure 1: Part of a staggered grid as used by the present authors (a) and Perot (b). The control volume in both approaches is shaded.

proofs. Hence, a Delaunay triangulation has to be employed. The unstructured staggered scheme described in [14] does not need a dual mesh, and is therefore not restricted to Delaunay grids. This is advantageous, since in some cases Delaunay triangulations are unacceptable [8] due to, for example, violation of boundary integrity or the presence of obtuse angles. Since our scheme is applied to compressible flows, it is highly desirable that it satisfies the jump conditions. It is the aim of the present paper to show, by demonstrating conservation of momentum and showing the results of several numerical experiments, that this is indeed the case.

In Section 2 the staggered grid arrangement and some remarks concerning the sequential update procedure will be given. In Section 3 a generalized conservation form for momentum on staggered grids is formulated. Numerical experiments to confirm that the scheme converges to genuine weak solutions are presented in Section 4. It is found that the scheme also satisfies the entropy condition. Section 5 contains conclusions.

2 Solution procedure

In our scheme, the normal momentum components are located at the cell faces and the scalar variables are positioned at the cell centroids, see Figure 1a. In Perot's discretization, the pressure is stored in the circumcenters, which are the vertices of the associated Voronoi tessellation, while the normal velocity components are located at the cell faces, as indicated in Figure 1b.

At a control volume face, 'left' and 'right' state vectors (containing all primitive variables) form the starting point for the familiar flux splitting and flux difference schemes for the Euler equations. In iterative solution methods, the elements of the state vector in a cell are usually updated collectively. However, definition and collective updates of such state vectors are not naturally given on a staggered grid. On the other hand, discretization by a simple finite difference or finite volume scheme for each primary variable separately is natural on a staggered grid. It is also natural to update the primary variables sequentially in a time-stepping or iterative procedure. For the primary variables we take the momentum $\mathbf{m} = \rho\mathbf{u}$, the density ρ and the density times total enthalpy ρH as energy variable. Other variables follow from algebraic relations like the equation of state. Upwind or central interpolations for the convection term in each of the governing equations result in a very simple scheme. Such a staggered and decoupled approach is common in incompressible CFD and for the shallow-water equations, but not in compressible CFD.

3 Conservative discretization of the momentum equation

The scalar equations, i.e. the continuity and energy equation, are integrated over the triangles. The resulting discretization, discussed in more detail in [14], is trivially shown to be conservative.

Integration of the inviscid momentum equation over the domain D with boundary ∂D yields:

$$\frac{d}{dt}\int_D \mathbf{m}\,d\mathbf{x} + \oint_{\partial D} [(\mathbf{u}\cdot\mathbf{n})\mathbf{m} + p\mathbf{n}]\,d\Gamma = 0, \tag{1}$$

where $\mathbf{n}$ is the outward unit normal at ∂D. The only way in which the total amount of momentum $\int_D \mathbf{m}\,d\mathbf{x}$ can change is by having a non-zero net flux over ∂D, which is why (1) is called a conservation law. The control volume (CV) for each internal face consists of the union of the two adjacent triangles, whereas the CV for a boundary face consists of the neighboring boundary cell. Note that Perot has opted for another CV, see Figure 1b. Assume for the moment that at each triangle face the momentum vector is located; we will return to this later. Integration over the CV for the momentum in face f leads to a discretized momentum vector equation of the following form:

$$\Omega_f \frac{d\mathbf{m}_f}{dt} + \sum_{e(f)} [(\mathbf{u}_e\cdot\mathbf{n}_e)\mathbf{m}_e + p_e\mathbf{n}_e]\,l_e = 0, \tag{2}$$

where $e(f)$ indicates the set of faces of the CV and Ω_f is the area of the CV. For example, in Figure 1a, we have $e(i) = \{j, k, o, q\}$ and $\Omega_i = \Omega_1 + \Omega_2$, and, assuming q to be a boundary face, $e(q) = \{q, i, j\}$ and $\Omega_q = \Omega_2$. We will show below that summation of (2) over all faces leads to a discrete equivalent of (1), hence equation (2) is in conservation form. After that we return to (2) and the discretized equation for normal momentum components: it is the latter that is actually solved.

Time derivative
Summation over all faces f of the term containing the time-derivative in (2) yields:

$$\frac{d}{dt}\left(\sum_f \Omega_f\mathbf{m}_f\right) = \frac{d}{dt}\left(\sum_c \Omega_c \sum_{f(c)} \mathbf{m}_f\right) = 3\frac{d}{dt}\left(\sum_c \Omega_c\mathbf{m}_c\right), \tag{3}$$

where we approximate the momentum vector in the cell centroid by means of:

$$\mathbf{m}_c = \frac{1}{3}\sum_{f(c)} \mathbf{m}_f, \tag{4}$$

with $f(c)$ the set of (three) faces of cell c.

Inertia term

The inertia flux term $(\mathbf{u}_i \cdot \mathbf{n}_i)\mathbf{m}_i l_i$ at face i, see Figure 1a, appears in the discretized momentum vector equations for the faces j, k, o and q. The evaluation of this term is done by means of the reconstruction procedure, introduced in [14]. A first order upwind scheme is used. As far as we know no higher order schemes on unstructured staggered grids have been developed yet. The outward unit vector $\mathbf{n}_i$ points in the direction of cell 2 in the equations for faces j and q, and in the opposite direction in the remaining two equations. Hence, summation over all momentum vector equations cancels the inertia flux term at face i. All other internal flux contributions disappear similarly. The inertia term $(\mathbf{u}_q \cdot \mathbf{n}_q)\mathbf{m}_q l_q$ at boundary face q shows up in the momentum vector equations for faces i, j and q. Since this term is not cancelled, we arrive at

$$\sum_f (\mathbf{u}_f \cdot \mathbf{n}_f)\mathbf{m}_f l_f = 3 \sum_b (\mathbf{u}_b \cdot \mathbf{n}_b)\mathbf{m}_b l_b, \tag{5}$$

where the subscript b refers to the boundary faces.

Pressure term

A completely similar reasoning as used for the inertia term leads for the pressure term to

$$\sum_f p_f \mathbf{n}_f l_f = 3 \sum_b p_b \mathbf{n}_b l_b. \tag{6}$$

The path-integral formulation used to approximate the pressure term in [14] cannot be written in this form, and is not conservative. Nevertheless, so far all numerical experiments done with this method have resulted in correct shock speeds.

Discussion

Using (3), (5) and (6), summation of (2) over all faces results in:

$$\sum_c \Omega_c \frac{d\mathbf{m}_c}{dt} + \sum_b \left[(\mathbf{u}_b \cdot \mathbf{n}_b)\mathbf{m}_b + p_b \mathbf{n}_b\right] l_b = 0, \tag{7}$$

which is in conservation form. This expression is equivalent to conservation of momentum for the (colocated) cell centered scheme. However, as stated before, the staggered scheme uses only the normal momentum components at the faces, hence it uses the projection of (2) on the corresponding normal vectors, and therefore conservation of momentum is not obvious. The reason to use only the normal components lies in the desire to avoid unwanted pressure oscillations in the incompressible case. Since the normal vectors can have any direction, summation over all projected momentum equations does in general not lead to cancellation of the internal flux contributions. But the set of the momentum equations solved consists of linear combinations of components of (7), and is therefore a subset of the conservative system (7). When this is the case, we say that the system is in generalized conservation (GC) form. Below we show by numerical experiments that the jump conditions are satisfied if the discrete momentum equations are in GC form.

4 Numerical tests

The Lax-Wendroff theorem [7] does not apply to schemes in GC form. However, it suggests that conservation properties are important, which is why we have introduced the concept of generalized conservation form. In addition, satisfaction of the entropy condition has, for the case of the Euler equations, only been proven for the Osher scheme [9]. We verify numerically whether the entropy solution is satisfied, since also for our scheme no such proof is available.

Riemann problems

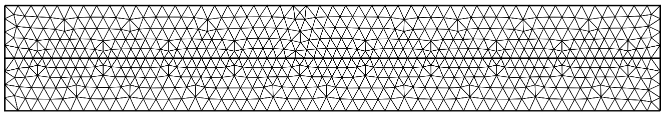
On the grid shown in Figure 2a, numerical solutions for the test case of Lax and the supersonic flow problem posed by Arora and Roe [1] are computed. We have included in Figures 2b,c results from the Roe scheme and the AUSM scheme obtained on an equidistant 1D grid with the same number of gridpoints (70) as on the centerline in the 2D grid. For the Lax problem, all three methods are of comparable accuracy. For the Mach 3 case, the Roe scheme violates the entropy condition, and Harten's sonic entropy fix had to be employed. Again, the accuracy for the three methods is almost similar, with our scheme yielding a slightly more smeared contact discontinuity. Note that the staggered scheme, because of the arbitrary direction of the normal vectors, considers these 1D Riemann problems as 2D flows. On a similar grid, refined to 550 gridpoints at the centerline, we computed the solution to the modified Sod problem [12]. We observe satisfaction of the shock relations and convergence to the genuine weak solution, also on a fine grid.

Transonic flow around the NACA0012 airfoil

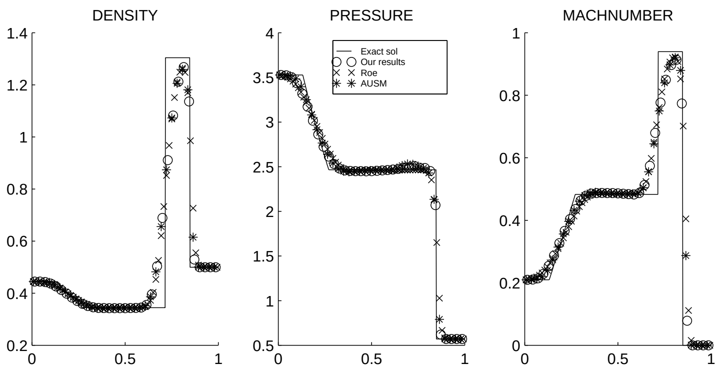
The transonic flow around the NACA0012 airfoil with freestream Mach number $M_\infty = 0.8$ and angle of incidence 1.25^o is computed, with 480 nodes located at the airfoil. In Figure 3 the computed isobars are depicted, and the Mach number at the airfoil is compared to the AGARD reference result [17]. The shock at the upper surface is captured without oscillations at almost the correct location, and the weak shock at the lower surface is absent. The quality of the result is similar to what is generally obtained with first order colocated schemes.

5 Conclusions

We have formulated the property of generalized conservation for momentum in unstructured staggered schemes. Conservation of mass and energy for the scheme is obtained in the same manner as for colocated schemes. Numerical experiments have been conducted using the generalized conservative scheme, showing that this scheme satisfies the Rankine-Hugoniot shock conditions and leads to the genuine weak solutions. The accuracy is comparable to that of state-of-the-art approximate Riemann solvers. Higher order upwind schemes are currently under development.



(a)



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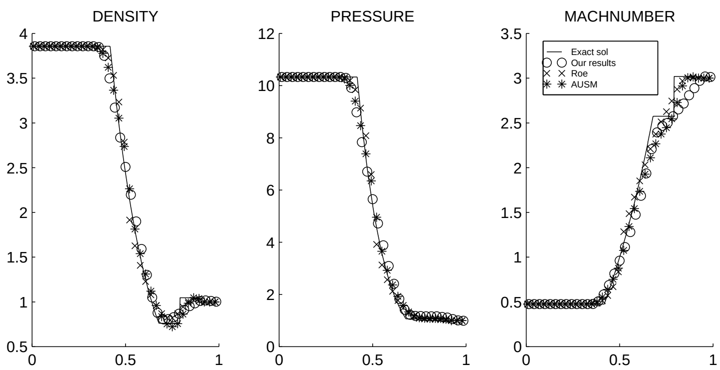

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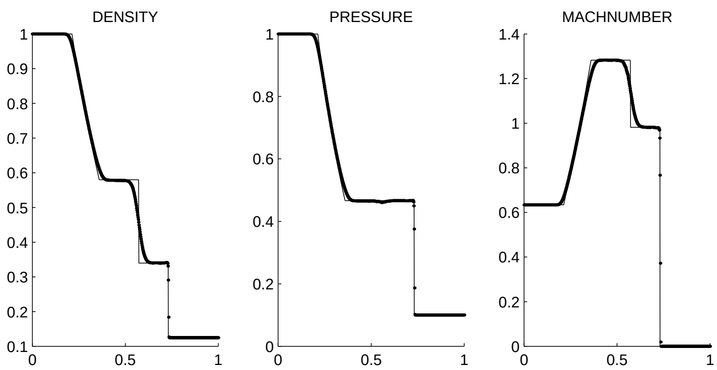
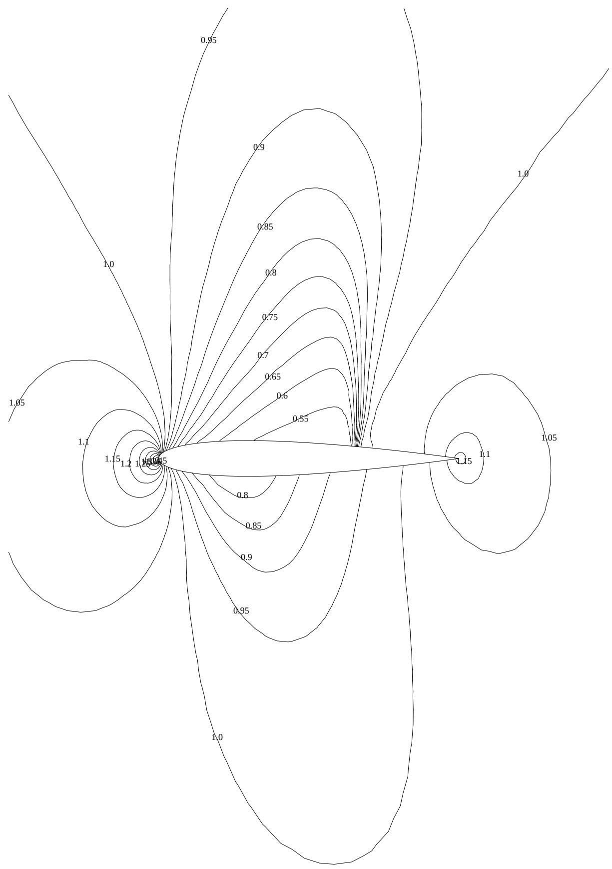


(d)

Figure 2: Grid (a) used for Riemann problems. Solutions for the Lax (b), Arora and Roe (c) and modified Sod problem (d) at the centerline. For clarity, in (b) and (c) only half of the solution points are displayed.



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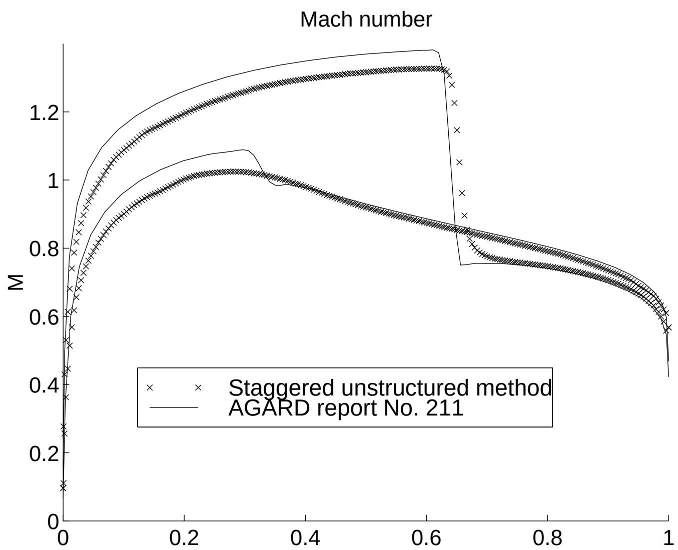

(b)

Figure 3: Isobars (a) and Mach number (b) for the transonic profile flow.

References

[1] M. Arora and P.L Roe. A well-behaved TVD limiter for high-resolution calculations of unsteady flow. *J. Comp. Phys.*, 132:3–11, 1997.

[2] H. Bijl and P. Wesseling. A unified method for computing incompressible and compressible flows in boundary-fitted coordinates. *J. Comp. Phys.*, 141:153–173, 1998.

[3] H. Guillard and C. Viozat. On the behavior of upwind schemes in the low Mach number limit. *Computers and Fluids*, 28:63–86, 1999.

[4] F.H. Harlow and A.A. Amsden. Numerical calculation of almost incompressible flows. *J. Comp. Phys.*, 3:80–93, 1968.

[5] F.H. Harlow and A.A. Amsden. A numerical fluid dynamics calculation method for all flow speeds. *J. Comp. Phys.*, 8:197–213, 1971.

[6] F.H. Harlow and J.E. Welch. Numerical calculation of time-dependent viscous incompressible flow of fluid with a free surface. *The Physics of Fluids*, 8:2182–2189, 1965.

[7] P. Lax and B. Wendroff. Systems of conservation laws. *Comm. Pure and Appl. Math.*, 13:217–237, 1960.

[8] D.J. Mavriplis. Unstructured grid techniques. *Annu. Review Fluid Mechanics*, 29:473–514, 1997.

[9] S. Osher and S. Chakravarthy. Upwind schemes and boundary conditions with applications to Euler equations in general geometries. *J. Comp. Phys.*, 50:447–481, 1983.

[10] J.B. Perot. Conservation properties of unstructured staggered mesh schemes. *J. Comp. Phys.*, 159:58–89, 2000.

[11] C.M. Rhie and W.L. Chow. Numerical study of the turbulent flow past an airfoil with trailing edge separation. *AIAA J.*, 21:1525–1532, 1983.

[12] E.F. Toro. *Riemann Solvers and Numerical Methods for Fluid Dynamics.* Springer, Berlin, 2nd edition, 1999.

[13] D.R. van der Heul, C. Vuik, and P. Wesseling. A staggered scheme for hyperbolic conservation laws applied to unsteady sheet cavitation. *Computing and Visualization in Science*, 2:63–68, 1999.

[14] I. Wenneker, A. Segal, and P. Wesseling. Computation of compressible flows on unstructured staggered grids. In E. Oñate, G. Bugeda, and B. Suárez, editors, *Proceedings European Congress on Computational Methods in Applied Sciences and Engineering, ECCOMAS 2000, Barcelona, September 11-14, 2000*, Barcelona, 2000. CIMNE. http://ta.twi.tudelft.nl/users/wesseling/pub.html.

[15] P. Wesseling, A. Segal, and C.G.M. Kassels. Computing flows on general three-dimensional nonsmooth staggered grids. *J. Comp. Phys.*, 149:333–362, 1999.

[16] P. Wesseling, A. Segal, C.G.M. Kassels, and H. Bijl. Computing flows on general two-dimensional nonsmooth staggered grids. *J. Eng. Math.*, 34:21–44, 1998.

[17] H. Yoshihara and P. Sacher. Test cases for inviscid flow field methods. AGARDograph No. 211, AGARD, Neuilly-sur-Seine, France, 1985.