Solution of the vector wave equation using a Krylov solver with an algebraic multigrid approximated preconditioner

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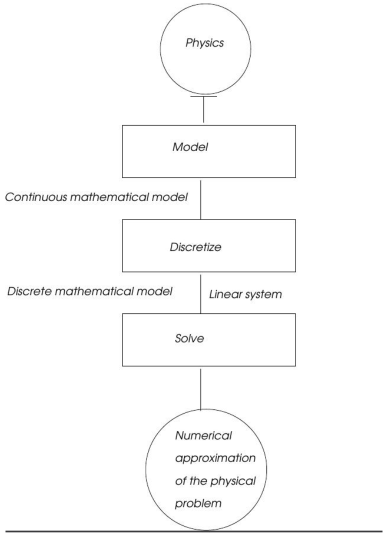
Outline

Introduction numerical analysis



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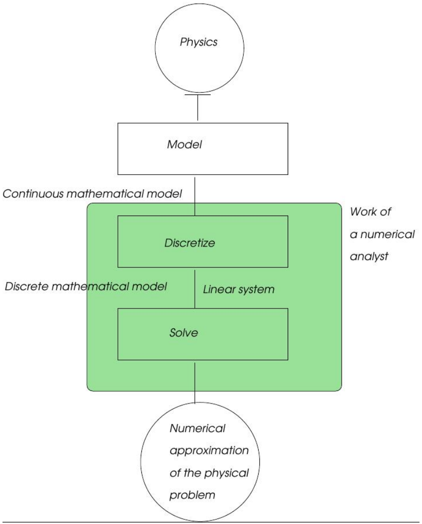

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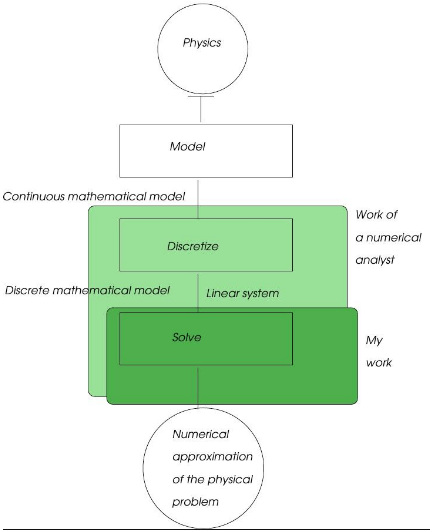
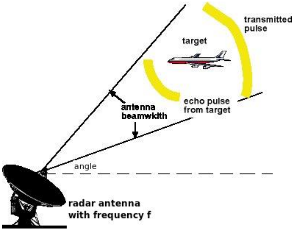


RADAR and RCS

- RADAR (RAdio Detection and Ranging): technology to detect military platforms (e.g. aircraft, ship or tank) by using electromagnetic waves.
- Goal: identify the range, altitude, direction, or speed of both moving and fixed objects.

- Measure of detectability: radar cross section (RCS) (depends on observation angle, frequency, polarization).
- A platform is detected when the *received* signal-to-noise ratio exceeds a certain threshold.

Problem description



- When the radar signature of a military platform cannot be determined experimentally, *numerical prediction* techniques are used.
- In this thesis a *faster* and more *memory efficient* prediction method is proposed.

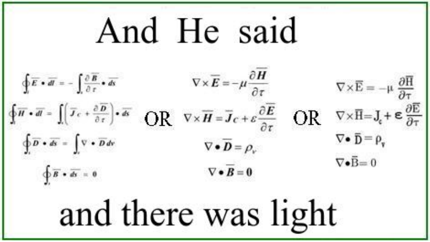
Importance of inlet cavity scattering

Radar excited from the front: jet engine air *intake* (of a modern fighter aircraft) accounts for the major part of the RCS for a large range of observation angles.

Figure: Jet engine air intake (left) closed by jet engine compressor fan (right) – together forms a large and deep open-ended cavity with varying cross section – *Dimensions:* $d \approx 30\lambda$ *and* $L \approx 200\lambda$ *for X-band excitation (10 GHz)*.

General Maxwell equations – Assumptions

- Start with general Maxwell equations:
  describe properties of electric $(\mathbf{E})$ and magnetic $(\mathbf{H})$ fields and relate them to their source: an electric current density distribution.
- Assume that the field quantities are harmonic oscillating functions with an angular frequency ω^* (*time-harmonic functions*).



The vector wave equation

Use the *constitutive relations*[1], to derive the dimensionless *vector wave equation* for the electric field:

$$\nabla \times \nabla \times \mathbf{E} - k_0^2 \varepsilon_r \mathbf{E} = -jk_0 Z_0 \mathbf{J}. \quad (1)$$

(same type of equation for the magnetic field **H**.)

Boundary conditions to obtain a well-posed problem:

- Impose *global* absorbing boundary conditions on the *aperture*: they lead to more accurate numerical solutions.
- Impose Dirichlet boundary conditions on the cavity surface.

[1] Used to describe the material properties of the medium of interest.

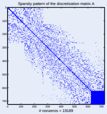
Linear system

FEM discretization

Equation (1) is discretized by a higher order edge based finite element discretization method, resulting in a large linear system:

$$Ax = b.$$

Properties system matrix A:



- Consists of a sparse part and a fully populated part,
- ‘Nearly’ symmetric but not Hermitian,
- Ill-conditioned,
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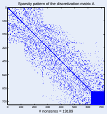

- Consists of a sparse part and a fully populated part,
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- Use a preconditioner matrix M.
- Solve $M^{-1} A x = M^{-1} b$.

Helmholtz equation – Application of multigrid

Based on the work of Erlangga[2]:

- Bi-CGSTAB method to solve the preconditioned system: `bicgstab(A,b,ITER-MAX,TOL,M).`
- Apply *shifted Laplace preconditioner* and solve preconditioner system with multigrid.
- Helmholtz and Maxwell's equation have similar properties.

  ⇒ Expected that the shifted Laplace preconditioner and multigrid will also be very effective to incorporate in the current application.

[2] See my report for the complete reference.

Analogy Helmholtz and vector wave equation

Homogeneous Helmholtz equation

- $-\triangle u - k_0^2 u = 0.$
- Shifted Laplace operator: $\mathcal{M}_{(\beta_1,\beta_2)} = -\triangle - (\beta_1 + \iota\beta_2)k_0^2.$

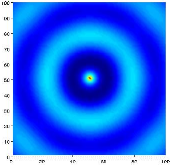
Vector wave equation

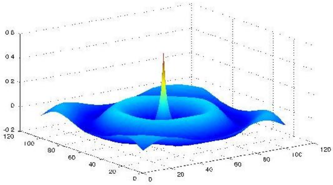
- Vector form of the Helmholtz equation: $-\triangle \mathbf{E} - k_0^2 \mathbf{E} = 0$, with $\mathbf{E} = (E_x, E_y, E_z)$.
- Shifted Laplace operator in *vector* form: $\mathcal{W}_{(\beta_1,\beta_2)} = -\triangle - (\beta_1 + \iota\beta_2)k_0^2.$

Solve the Helmholtz equation using Bi-CGSTAB–AMG

Multigrid type

- Erlangga used *geometric* multigrid to perform the preconditioner solve: Bi-CGSTAB–*MG* algorithm.
- In this experiment *algebraic* multigrid is used with one V-cycle, leading to the same solution as Erlangga.





Surface plot of the real part of the solution of the *two dimensional* Helmholtz equation with local absorbing boundary conditions.

Present implementation

- Dimension of system matrix $A : N \approx 1 \cdot 10^7$ (total number of unknowns).
- Iterative Krylov subspace method: *Generalized Conjugate Residual* (GCR) method.
- Ill-conditionedness and unfavourable spectrum of A $\Rightarrow$ improve convergence GCR by *shifted Laplace preconditioner*.
- GCR long recurrence method $\Rightarrow$ difficult to satisfy memory requirements for storing eigenvectors of Krylov basis.

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Proposed solution:

Modify existing algorithm by using a *multigrid solution method* for the shifted Laplace preconditioner system and use a *short recurrence method* for the preconditioned system.

IDR(4)–ML-AMG algorithm

Description of the methods

- IDR(4) short recurrence Krylov method from Van Gijzen and Sonneveld.
- ML: Sandia's laboratories main multigrid preconditioning package[3].

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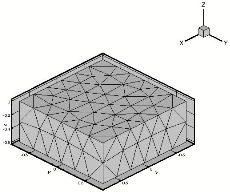
Limitations using ML

- ML can only perform computations in real valued arithmetic.
- Use optimal real shift for the shifted Laplace preconditioner.
- Only applications with non-absorbing materials (permittivity and permeability) can be considered.

[3] See my report for the complete references.

IDR(4)–ML-AMG algorithm for a small cavity scattering model with dimensions $1.5\lambda \times 1.5\lambda \times 0.6\lambda$



p	h	N	$(-1, 0)$	$(1, -0.5)$	ML-solve $(-1, 0)$
0	0.25	1402	245	150	308
0	0.20	2796	301	226	374
1	0.35	2914	343	270	394
1	0.30	4344	342	427	417
1	0.25	7960	413	459	530
2	0.35	5316	390	477	546
2	0.30	8730	473	494	833

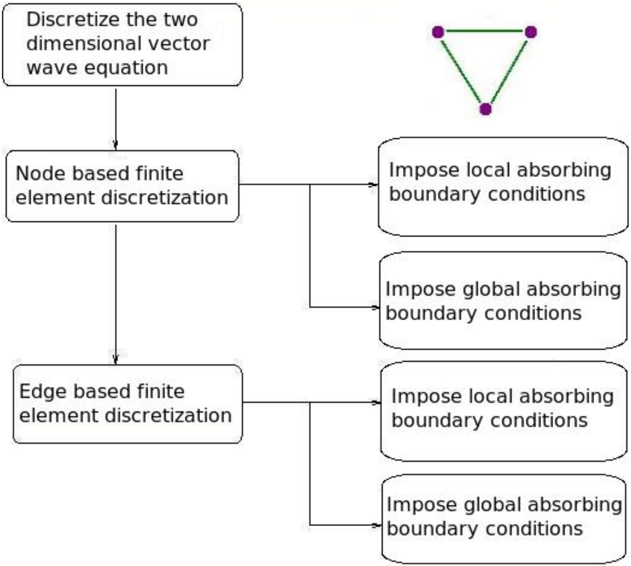
Two dimensional vector wave equation

To improve the performance of the IDR(4)−ML-AMG algorithm, the two dimensional vector wave equation is considered ⇒ two dimensional Maxwell solver[4].

- Relatively small number of unknowns for high wavenumbers.
- Extend results to three dimensional case.

[4] Made by Duncan van der Heul and Shiraz Abdoel.

Flowchart two dimensional vector wave equation



Two dimensional vector wave equation

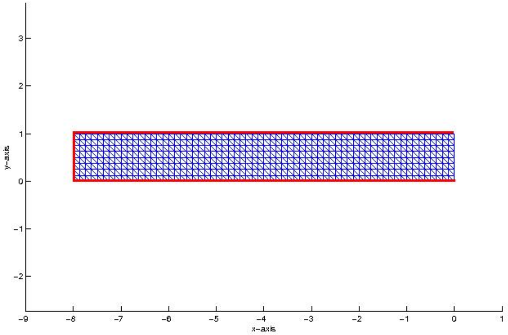


Figure: Mesh of a two dimensional cavity with dimensions 8×1.

Node based implementation: E_z solution

The equation considered here:

$$-\triangle E_z - k_0^2 \varepsilon_r E_z = 0$$

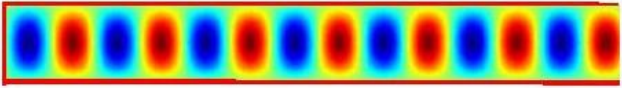


Figure: Contour plot of the real part of E_z. It can be seen how the wave travels through the inlet and is reflected on the bottom.

Analysis two dimensional node based FEM discretizaion

- Test cases with varying wavenumber and varying mesh size: IDR(4)–ML-AMG algorithm performs good.
- Several test cases with different boundary conditions.

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k_0	2π	4π	6π
(A_{loc}, M_{loc})	83	225	440
(A_{gl}, M_{loc})	79	247	379
(A_{gl}, M_{gl})	75	239	427

Table: Total number of matrix vector operations for the IDR(4)−ML-AMG algorithm.

Edge based implementation: vector plot of E_x, E_y

The equation considered here:

$$\nabla \times \nabla \times \mathbf{E} - k_0^2 \varepsilon_r \mathbf{E} = 0, \text{ with } \mathbf{E} = (E_x, E_y, 0)^T.$$

It holds that: $\nabla \times \mathbf{E} = \iota\omega\mathbf{H} = \iota\omega(0, 0, H_z)^T.$

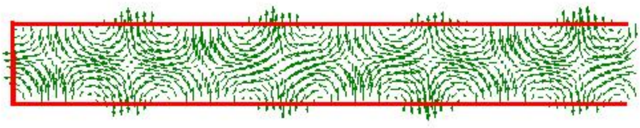


Figure: Vector plot of E_x and E_y.

Edge based implementation: H_z solution

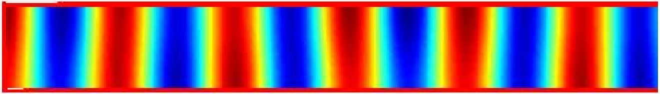


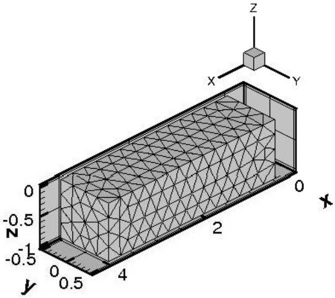
Figure: Contour plot of the real part of H_z.

Edge based implementation with local ABC similar to node based implementation.

Conclusions for two dimensional case

- Work involved in exact solve preconditioner system versus single V-cycle AMG solve similar for *real* shift ⇒ Expect the same for optimal *complex* shift.
- Use preconditioner with operator based on *local* ABC ⇒ more efficiently solved (e.g. with a complex algebraic multigrid method).
- Optimal real shift is a *restriction* compared to optimal complex shift.

Three dimensional cavity



mesh size h	0.10
degrees of freedom N	79,428
wave number k_0	2π
wave length λ	1
preconditioner system	solved using ML for optimal real shift
	exact solve for optimal complex shift

Performance of nested GCR algorithm versus the IDR(4)−ML-AMG algorithm

Nested GCR algorithm

- Optimal shift used: $(\beta_1, \beta_2) = (0.5, 3.0)$.
- Orthogonalisation of all basisvectors ⇒ lot of work.
- # MAT-VEC-OPs: 423,000.

Performance of nested GCR algorithm versus the IDR(4)−ML-AMG algorithm

Nested GCR algorithm

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- Orthogonalisation of all basisvectors $\Rightarrow$ lot of work.
- # MAT-VEC-OPs: 423,000.

Algorithm	$(\beta_1, \beta_2) = (1, -0.5)$	$(\beta_1, \beta_2) = (-1, 0)$
IDR(4)	543	–
IDR(4)−ML-AMG	–	5307
expected factor	800	80

Table: Total number of matrix vector operations for the different algorithms and the expected factor which can be gained compared to the currently used algorithm.

Conclusions

- Restriction to real valued arithmetic leads to a reduction in work by factor 100.
- Optimal real shift restriction on effectiveness which can be gained using optimal complex shift.
- Optimal complex shift leads to a reduction in work by factor 1000.

Recommendations for future research

- Realize a complex algebraic multigrid solver.
- Incorporate IDR(s) method in the existing algorithm.
- Introduce preconditioner based on local absorbing boundary conditions.

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Questions & Discussion