What is the Discontinuous Galerkin method? And why should we use it?

DG for a simple one-dimensional advection equation

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Motivation

Two problems of numerical weather prediction and climate models:

1. Mathematically modelling atmospheric processes.
2. Evaluating the models as accurate and efficient as possible.

Motivation

Two problems of numerical weather prediction and climate models:

1. Mathematically modelling atmospheric processes
   - e.g. parametrization of atmospheric processes using DALES.
2. Evaluating the models as accurate and efficient as possible
   - e.g. Improving the advection scheme.

Advection scheme is used for, e.g.:

- the continuity equation,
- important scalars like q_t and θ_l.

But why improve the advection scheme (of DALES)?

Simple 1D Advection Equation

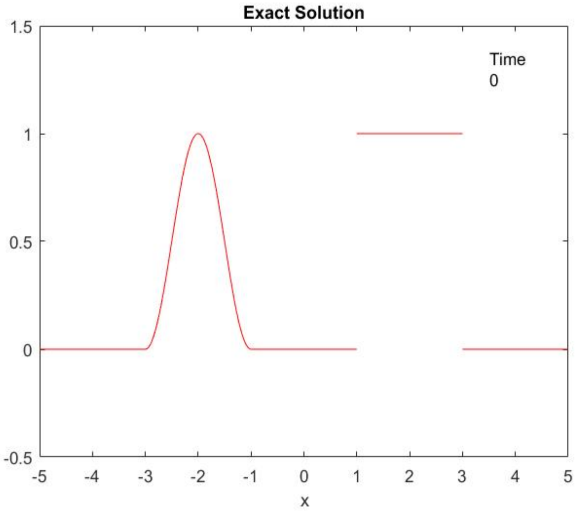
$$\begin{cases} \frac{\partial \varphi}{\partial t} + \frac{\partial f(\varphi)}{\partial x} = 0 & x \in [a, b], t > 0, \\ \varphi(x, 0) = \varphi_0(x) & x \in [a, b], \end{cases}$$

where $f(\varphi) = u\varphi$ with u constant.

Exact solution is given by:

$$\varphi(x, t) = \varphi_0(x - ut)$$

Exact Solution



Tested Finite Difference Methods of DALES

1. First order upwind,
2. Second order central,
3. Fifth order upwind,
4. WENO method.

First Order Upwind at $t = 10$

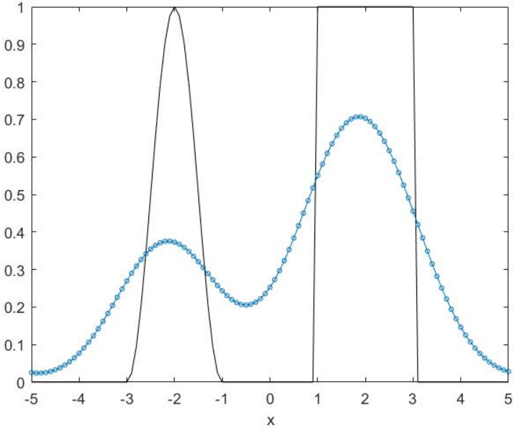


Figure: First order upwind at $t = 10$ with $\Delta x = 0.1$ and $\Delta t = 0.3\frac{u}{\Delta x}$.

First Order Upwind at $t = 50$

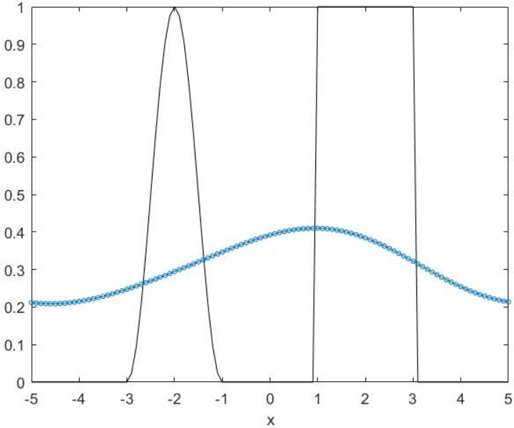


Figure: First order upwind at $t = 50$ with $\Delta x = 0.1$ and $\Delta t = 0.3\frac{u}{\Delta x}$.

Second Order Central at $t = 10$

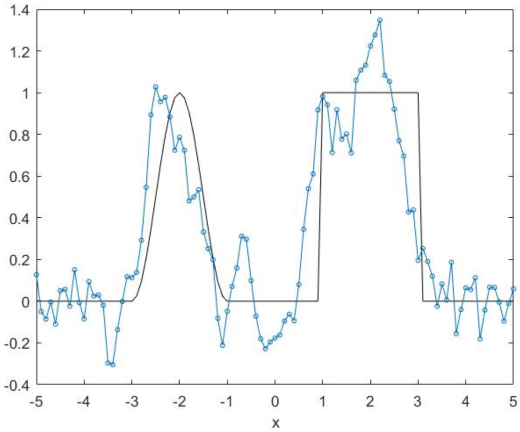


Figure: Second order central at $t = 10$ with $\Delta x = 0.1$ and $\Delta t = 0.3\frac{u}{\Delta x}$.

Second Order Central at $t = 50$

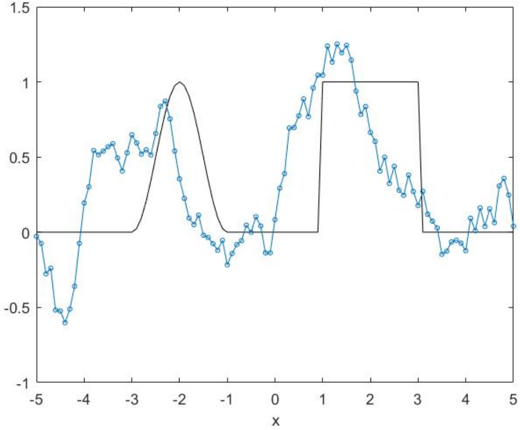


Figure: Second order central at $t = 50$ with $\Delta x = 0.1$ and $\Delta t = 0.3\frac{u}{\Delta x}$.

Fifth Order Upwind at $t = 10$

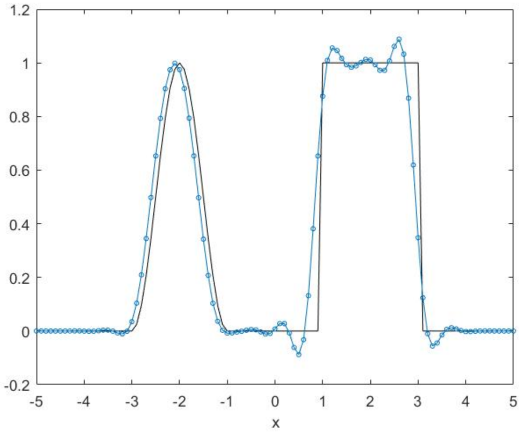


Figure: Fifth order upwind at $t = 10$ with $\Delta x = 0.1$ and $\Delta t = 0.3 \frac{u}{\Delta x}$.

Fifth Order Upwind at $t = 50$

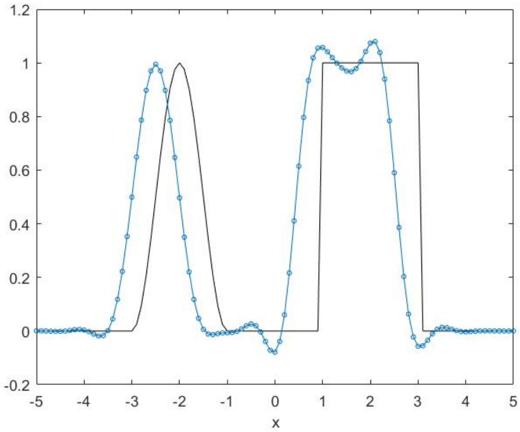


Figure: Fifth order upwind at $t = 50$ with $\Delta x = 0.1$ and $\Delta t = 0.3 \frac{u}{\Delta x}$.

WENO method at $t = 10$

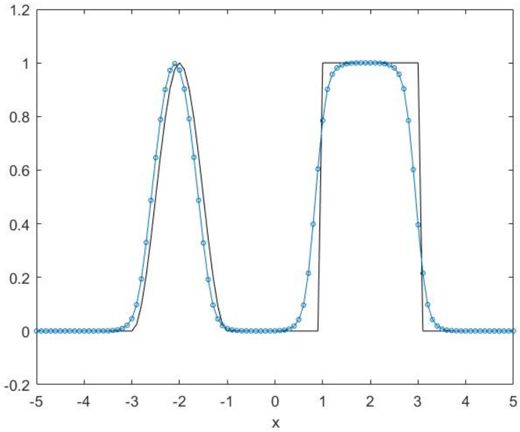


Figure: WENO at $t = 10$ with $\Delta x = 0.1$ and $\Delta t = 0.3\frac{u}{\Delta x}$.

WENO method at $t = 50$

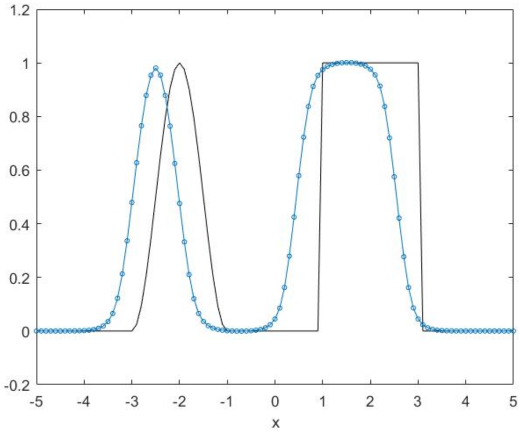


Figure: WENO at $t = 50$ with $\Delta x = 0.1$ and $\Delta t = 0.3\frac{u}{\Delta x}$.

Moment Limited DG at $t = 10$

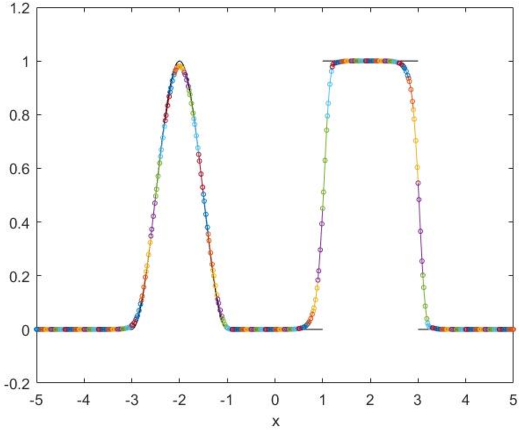


Figure: Moment limited DG at $t = 10$ with $N = 4$, $\Delta x = 0.1$ and $\Delta t = 0.95\text{CFL}_2 \frac{u}{\Delta x}$

Moment Limited DG at $t = 50$

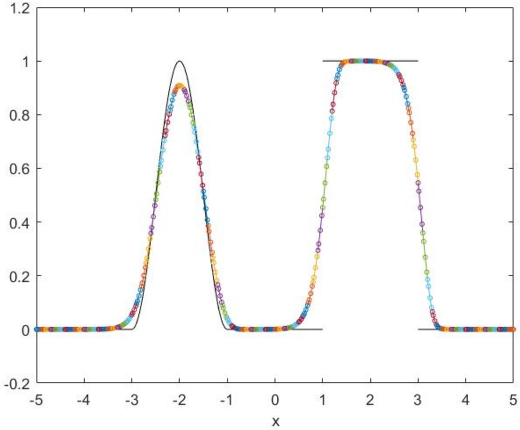


Figure: Moment limited DG at $t = 50$ with $N = 4$, $\Delta x = 0.1$ and $\Delta t = 0.95 \mathrm{CFL}_2 \frac{u}{\Delta x}$

Differences between FDM, FVM and FEM

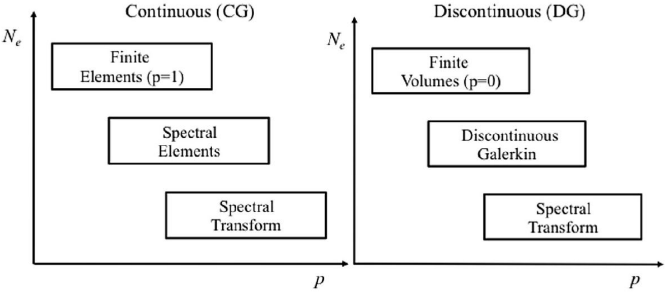
FDM - Finite Difference Method
FVM - Finite Volume Method
FEM - Finite Element Method

	FDM	FVM	FEM
solves	direct	integral	weak
discontinuities	✗	✓	✗
values	nodal	cell average	nodal
unstructured grids	✗	✓	✓
conservation of mass	✗	✓	✗

DG in comparison with FDM, FVM and FEM

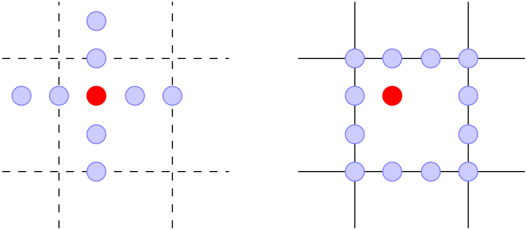
	FDM	FVM	FEM	DG
solves	direct	integral	weak	weak
discontinuities	✗	✓	✗	✓
values	nodal	cell average	nodal	nodal
unstructured grids	✗	✓	✓	✓
conservation of mass	✗	✓	✗	✓

DG is a combination of FEM and FVM



Advantages of DG

- Unstructured grids, discontinuities and conservation of mass,
- Dynamic *h*-*p* refinements,
- Compact stencil,



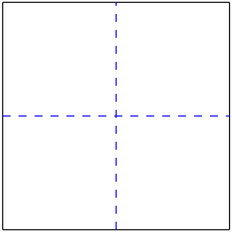
- High scalability.

Discontinuous Galerkin Method

1. Split domain into non-overlapping *elements*.
2. Find *weak form* of the partial differential equations.
3. Fill in the *approximation* in the weak form.
4. Find *element matrices*.
5. *Solve* $M_k \mathbf{a}_k' = S_k \mathbf{a}_k$ for each element.

Steps 1 and 2

1. Split domain into non-overlapping elements.



Find weak form of the partial differential equations:

$$\begin{cases} \int_{I_k} \left[\frac{\partial \varphi}{\partial t} + \frac{\partial}{\partial x} f(\varphi) \right] \eta \; dx = 0, \\ \int_{I_k} \varphi(x,0) \eta \; dx = \int_{I_k} \varphi_0(x) \eta \; dx. \end{cases}$$

Step 3 (1/2)

3. Fill in the approximation in the weak form.

$$\varphi_h^k(x,t) = \sum_{j=0}^{N} a_h^k(x_j^k,t)\ell_j^k(x), \ \ \forall x \in I_k$$

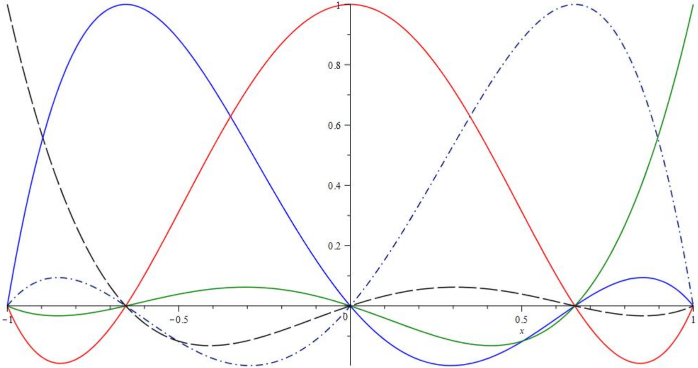


Figure: Lagrangian polynomials with LGL nodes.

Steps 3 (2/2), 4 and 5

3. Fill in the approximation in the weak form:

$$\int_{I_k} \sum_{j=0}^{N} \frac{\partial}{\partial t} a^k(x_j^k, t)\ell_j(\xi(x))\ell_i(\xi(x))\ dx - \int_{I_k} f(\varphi_h)\frac{\partial \ell_i(\xi(x))}{\partial x}\ dx + \left[f(\varphi_h)\ell_i(\xi(x))\right]_{x_{k-1/2}}^{x_{k+1/2}} = 0,$$

$$\int_{I_k} \sum_{j=0}^{N} a^k(x_j^k, t)\ell_j(\xi(x))\ell_i(\xi(x))\ dx = \int_{I_k} \varphi_0(x)\ell_i(\xi(x))\ dx.$$

4. Find element matrices.

$$M_k \mathbf{a}_k' = S_k \mathbf{a}_k,$$
$$M_k \mathbf{a}_k(0) = \tilde{\varphi}_{\mathbf{0}}$$

5. Solve $M_k \mathbf{a}_k' = S_k \mathbf{a}_k$ for each element.

DG with $N = 4$ at $t = 10$

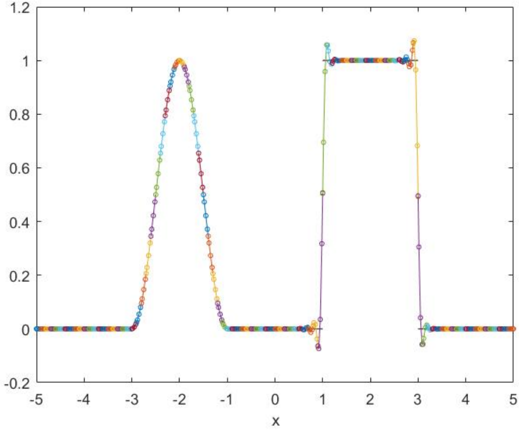


Figure: DG with $N = 4$ at $t = 10$.

DG with $N = 4$ at $t = 50$

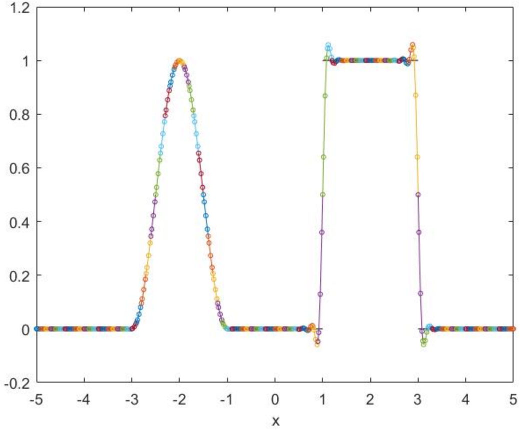


Figure: DG with $N = 4$ at $t = 50$.

Moment Limiter

Krivodonova: $\hat{a}_j^k \approx \frac{\partial^j \varphi_h^k}{\partial x^j}$

Idea:
Compare $\hat{a}_j^k$ with numerical derivatives using forward and backward differences.

Moment Limited DG at $t = 10$

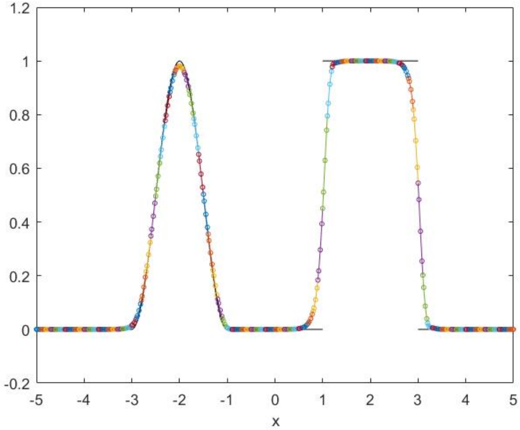


Figure: Moment limited DG at $t = 10$ with $N = 4$, $\Delta x = 0.1$ and $\Delta t = 0.95 \mathrm{CFL}_2 \frac{u}{\Delta x}$

Moment Limited DG at $t = 50$

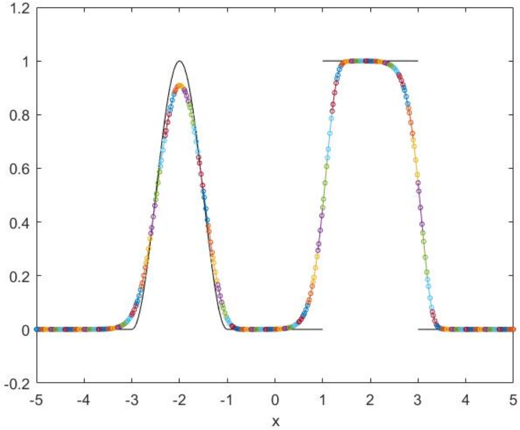


Figure: Moment limited DG at $t = 50$ with $N = 4$, $\Delta x = 0.1$ and $\Delta t = 0.95 \mathrm{CFL}_2 \frac{u}{\Delta x}$

Conclusion

DG is very promising method.

- No time lags,
- Unstructured grids,
- Dynamic *h*-*p* refinements,
- Compact stencil,
- High scalability.

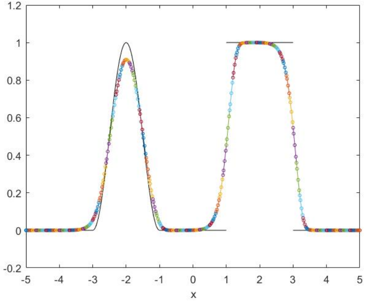


Figure: Moment limited DG at $t = 50$.

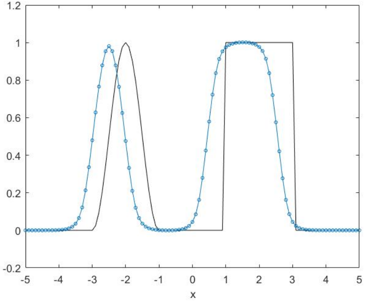


Figure: WENO at $t = 50$.