Discontinuous Galerkin Method for Numerical Weather Prediction

Discontinuous Galerkin in a large-eddy simulation

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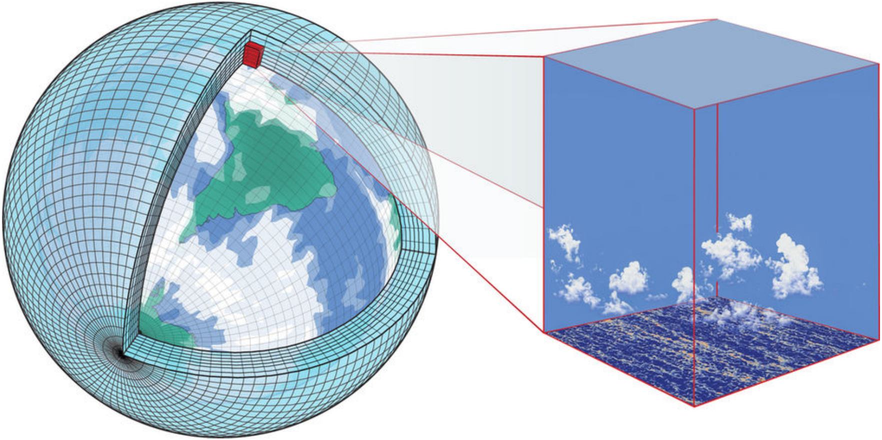
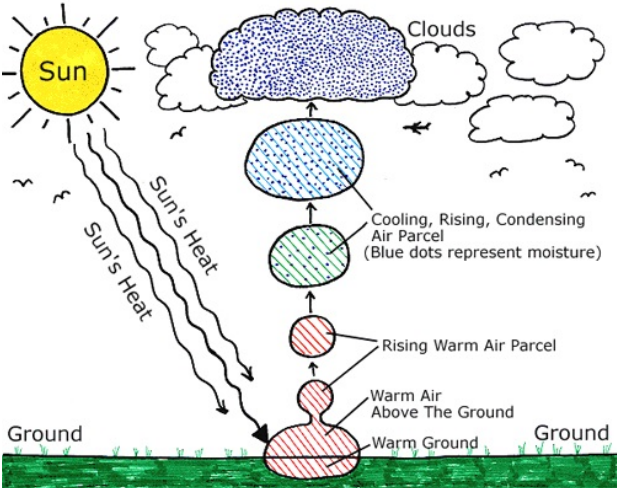
Numerical weather prediction models



Atmospheric processes



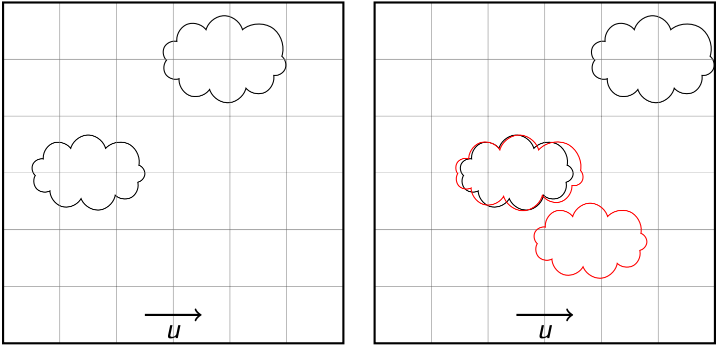
Numerical methods

Conserved variables:

$$\frac{d}{dt}\varphi(x,t) = 0$$

1D advection equation:

$$\frac{\partial \varphi}{\partial t} + u\frac{\partial \varphi}{\partial x} = 0$$



Requirements advection scheme

- High numerical accuracy
- Fast

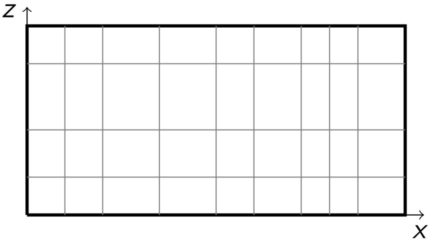
Proposed solution

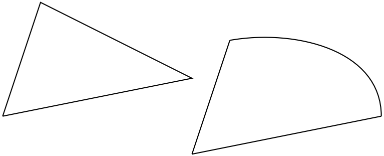
Discontinuous Galerkin method

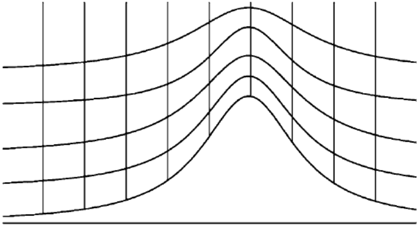
Advantages:

- superconvergence $\mathcal{O}(h^{p+1})$,
- high scalability,
- dynamic h-p refinements,
- unstructured grids,
- conservation of mass.

Discontinuous Galerkin



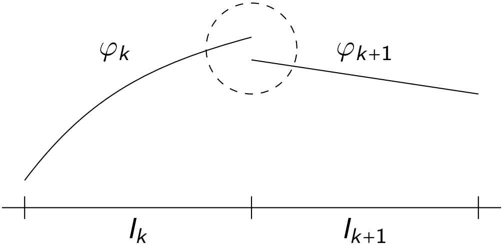




Discontinuous Galerkin

Prescribe the unknown function per element by:

$$\varphi^k(x,t) = \sum_{i=0}^{p} a_i^k(t)\ell_i(x)$$



Basis functions

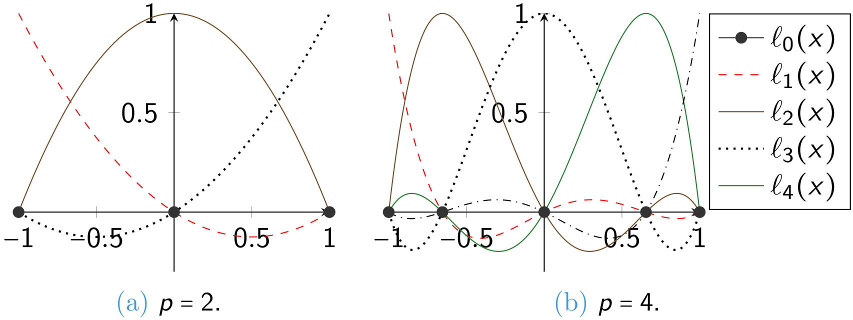


Figure: Lagrange polynomials $\ell_j(x)$ using $p+1$ Legendre-Gauss-Lobatto nodes.

Discontinuous Galerkin

For each element:

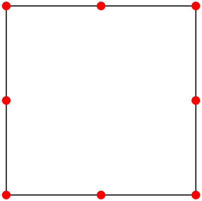
$$\varphi^k(x,t) = \sum_{i=0}^{p} a_i^k(t)\ell_i(x)$$

Advection equation:

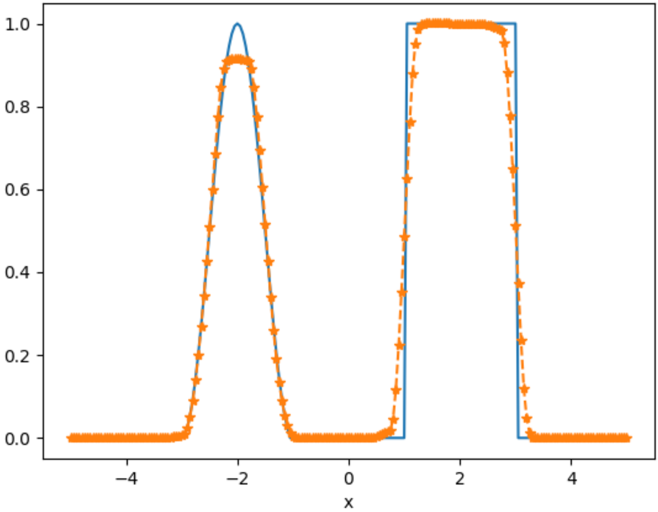
$$\frac{\partial}{\partial t}\varphi + \frac{\partial}{\partial x}(u\varphi) = 0$$

Linear system of ODEs:

$$M^k \frac{d}{dt}\boldsymbol{a}^k = S\boldsymbol{a}^k + \boldsymbol{f}^k$$



Discontinuous Galerkin



DALES

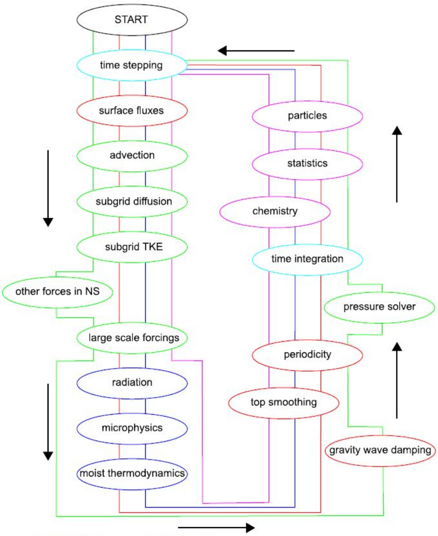
$$\frac{\partial}{\partial t}\tilde{\varphi} = -\frac{1}{\rho_0(z)}\nabla\cdot\left(\rho_0(z)\tilde{\varphi}\tilde{\boldsymbol{u}}\right) \quad \frac{\partial}{\partial t}\tilde{\varphi} = -\frac{1}{\rho_0(z)}\nabla\cdot\left(\rho_0(z)\tilde{\varphi}\tilde{\boldsymbol{u}}\right) \quad + \frac{1}{\rho_0(z)}\nabla\cdot(\rho_0($$

Advection equation

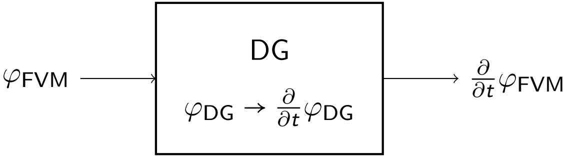
Diffusion

DALES

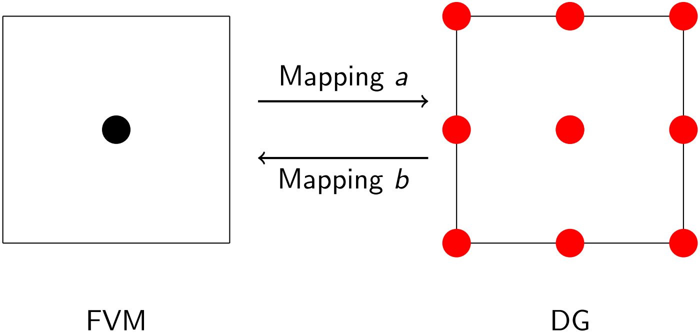
$$\frac{\partial}{\partial t}\tilde{\varphi} = -\frac{1}{\rho_0(z)}\nabla\cdot\left(\rho_0(z)\tilde{\varphi}\tilde{\boldsymbol{u}}\right) + \frac{1}{\rho_0(z)}\nabla\cdot\left(\rho_0(z)K_h\nabla\tilde{\varphi}\right) + S_\varphi$$



Advection scheme



Mappings



Mappings

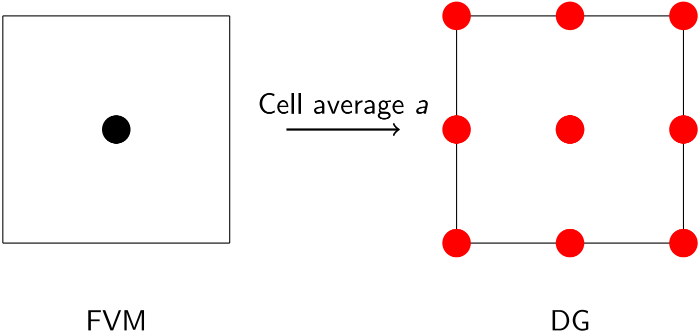
Mappings a (FVM to DG):

- Cell average a
- L_2-projection

Mappings b (DG to FVM):

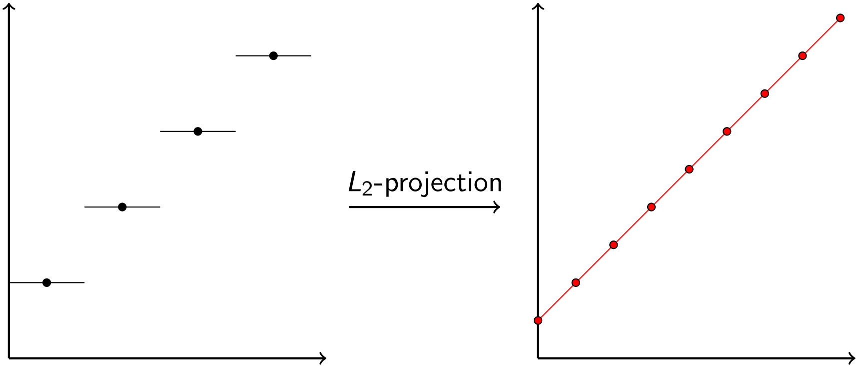
- Cell average of tendency $\frac{\partial \varphi}{\partial t}$
- Cell average b

Cell average *a*



- Simple, computational efficient
- Discontinuities

L_2-projection



- mass conservation
- no discontinuities

Mappings b

From DG values to FVM values:

$$g_{\text{FVM}} = \frac{1}{\Delta x \Delta y \Delta z} \int_{\Omega_k} g(\boldsymbol{x})\ d\Omega_k,$$

Cell average of tendency:

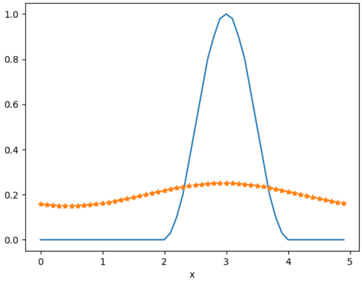
$$\frac{\partial}{\partial t}\varphi_{\text{DG}} \rightarrow \frac{\partial}{\partial t}\varphi_{\text{FVM}}$$

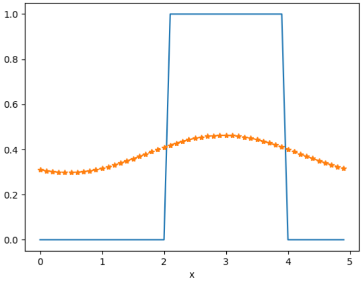
Cell average b:

$$\frac{\partial}{\partial t}\varphi_{\text{DG}} \rightarrow \varphi_{\text{DG}}(t + \beta\Delta t) \rightarrow \varphi_{\text{FVM}}(t + \beta\Delta t) \rightarrow \frac{\partial}{\partial t}\varphi_{\text{FVM}}$$

Numerical Results

Using cell average *a* and the cell average of the tendency:

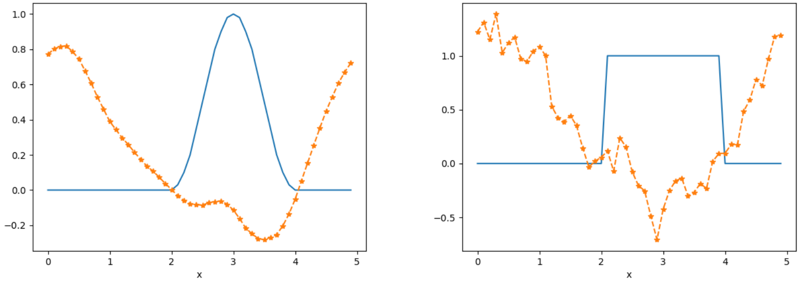




Overdiffusive

Numerical Results

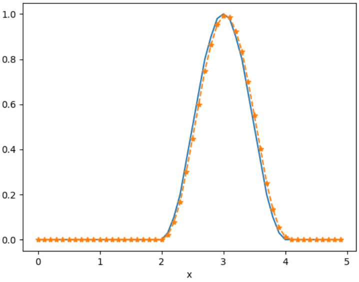
Using the L_2-projection and the cell average of the tendency:

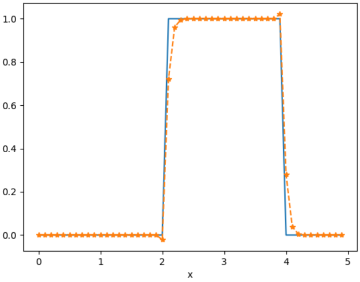


Underdiffusive

Numerical Results

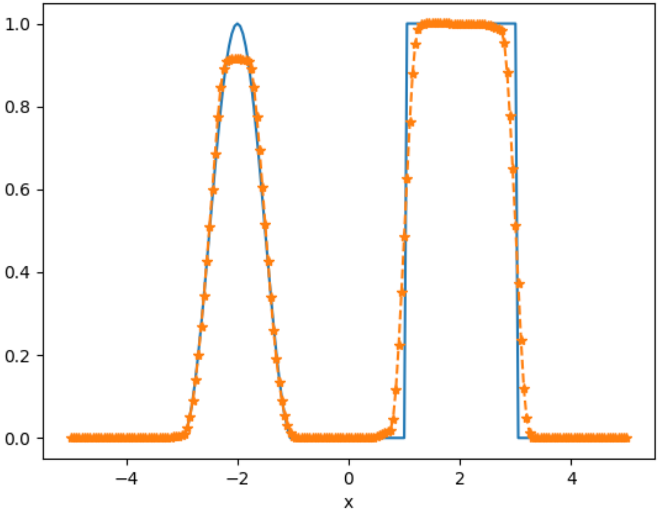
Using the L_2-projection and cell average b:



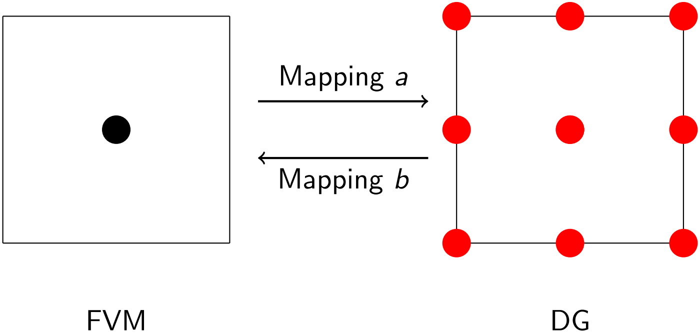


Time delay

Discontinuous Galerkin



Mappings



Conclusions

- DG as advection solver very promising
- in DALES inaccurate due to mappings

A recommendation

Multiple DALES cells as a DG cell:
for example: