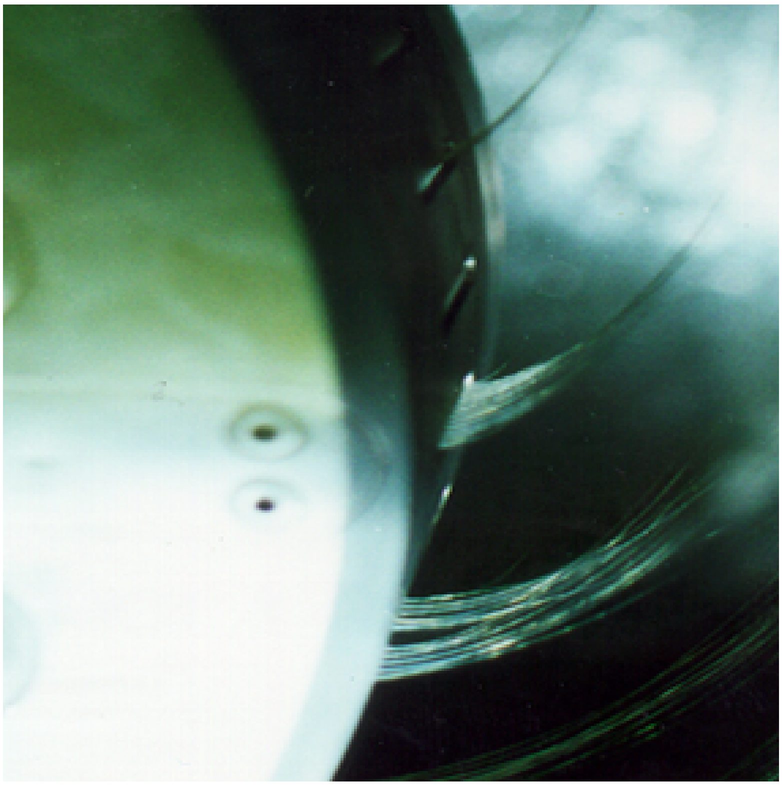
Mathematical Models for a Rotor Spinning Process
Interim report

Everdien Kolk

May 2005



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ir. P. den Decker (Teijin Twaron)

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Preface

In this report some models are described for a rotor spinning process for Twaron®. Twaron is a para-aramid polymer made by Teijin Twaron. Some problems to control this process remain, which is why Teijin Twaron is interested in a mathematical model of the process.
In 2004 a study week called "Mathematics with Industry" took place. In this meeting the problem described in Section 2 was presented to several persons who are active in the mathematical world. This week was too short to answer all questions, so a brief report was the final result [2]. After this week, one of the participants, Bas van 't Hof, wrote a more extensive report about one of the suggested approaches [4]. The problem also became a master thesis project in cooperation with Teijin Twaron.
This report describes the interim research of this master project for the degree Master of Science, done at Delft University of Science at the faculty Electrical Engineering, Mathematics and Computer Science, at Delft Institute of Applied Mathematics.

The first section of this report starts with an introduction of Teijin and Teijin Twaron. Next the problem is defined in Section 2. In Section 3 a derivation of a stationary model with curvilinear coordinate s in a rotating coordinate system is given. This derivation is a detailed version of the derivation given in [2].
In Section 4 a model for the instationary case in a fixed coordinate system is derived. Instead of coordinate s, polar coordinates are introduced. This section is an extensive version of the report of van 't Hof [4]. Van 't Hof did not mention boundary conditions in his report, but they are included here.
Section 5 treats also the instationary case, but this time in a rotating coordinate system. The part where the system of equations is derived, is a detailed discussion of van 't Hof [4]. The same applies to the first part of Section 6. In this section the stationary case in a rotating coordinate system is derived. This is done by ignoring the time derivatives in the system of equations from Section 5. In both sections the boundary conditions are mentioned, in contrast to [4]. The last subsection of Section 6 compares the stationary case in a rotating coordinate system with coordinate s and the case with polar coordinates.
In Section 7 the system from Section 6 is made dimensionless. Also a symbol for one of the components of the momentum equation is introduced. This symbol k plays an important part in numerical analysis.
In Section 8 some numerical methods for differential equations are introduced. Also some error definitions are given. Another way to solve differential equations is by using perturbation theory. An introduction to this theory is given in Section 9.
In the final section, Section 10, of this report an overview of topics for further research is given. A distinction is made between four subjects for further research: the model, boundary conditions, solving the system and model extensions.

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1 Teijin and Teijin Twaron

1.1 The Teijin Group

Teijin Twaron is part of the Industrial Fibers Business Group of Teijin Limited. Teijin is a global technology-driven company based in Osaka, Japan that operates in six main business segments: fibers and textiles; films and plastics; pharmaceuticals and home health care; machinery and engineering, wholesale and retail, and IT and new products. World wide there are approximately 22,000 employees working for Teijin.

1.2 Teijin Twaron

Teijin Twaron is a part of the Teijin Group established in The Netherlands. Teijin Twaron supplies customers throughout the world with para-aramid polymer, yarn, fiber and pulp under the name Twaron®. Twaron, the synthetic fiber made from aramid polymer, can be found in a comprehensive range of products including protective clothing, ballistic vests and helmets, tires and optical fiber cables. The company is a worldwide leader in the field of aramid fibers. Teijin Twaron aims to become the leading and preferred aramid supplier. The company has a longterm vision, continuous work is done on further improving existing products and developing new products.

Teijin Twaron has four establishments, three in The Netherlands: Arnhem, Delfzijl and Emmen, and one in Germany: Wuppertal. All the departments in The Netherlands have a production site. The raw material is made in Delfzijl, the pulp and yarn is produced in Emmen and in Arnhem the head office and the research division are located and pulp is produced on this location. There are working approximately 1,100 employees for Teijin Twaron all over the world.

As part of the Teijin group, Teijin Twaron shares the corporate philosophy: 'Human Chemistry - Human Solutions'. The term 'Human Chemistry' means that Teijin Twaron promises that it will develop chemical technologies that are friendly for both people and the global environment. With 'Human Solutions' the company says that it will make optimal use of the technologies, products and services resulting from the first promise.

1.3 Twaron, an aramid

Twaron, made of an aramid polymer, is extremely strong and very lightweight. The former owner, Akzo, developed Twaron in the early seventies. In 1976 the first pilot plant for Twaron was built and by 1985 five factories where operational on two sites. After a slow start, the market of para-aramid distinctly grows at the end of the nineties. At the end of 2000 Twaron was taken over by the Teijin Group. Aramid fibers are a type of nylon of which the molecular structure are comprised of linked benzene rings and amide bonds. Aramid fibers differ greatly from conventional fibers in both their properties and applications.

Twaron fibers are made by a wet spinning process. The unique characteristics of Twaron are derived from the ability of the aramid molecules to orient themselves along the line of flow during the spinning process producing the fiber, forming straight strands that resemble uncooked spaghetti.
There are two spinning processes. In the first one, the liquid polymer drops down from an outlet and goes through several washing devices to become a yarn. The second spinning process is called advanced or rotor spinning. The liquid polymer leaves a rotating disc which is located inside a cylinder horizontally. The polymer hits the cylinder and drains away with water that is falling down the cylinder wall. The end product of this type of spinning will be worked up to get pulp.

2 Problem Definition

At Teijin Twaron a rotor spinning process is used, to produce pulp. Some problems to control this process remain, which is why Teijin Twaron is interested in a mathematical model of the process. With the model, they hope to gain insight into the process.

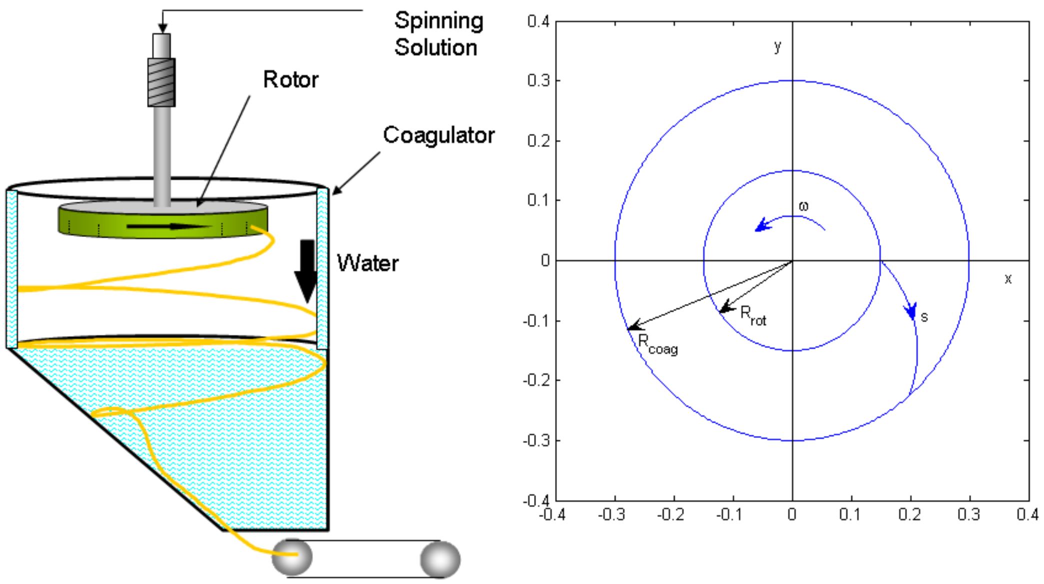

Figure 1: The rotor spinner and a two-dimensional view

The rotor spinning machine consists of a rotor and a coagulator. The coagulator is a cylinder with radius R_{coag} and water along the inside wall. In this cylinder, a disc with radius R_{rot}, the rotor, is rotating anti-clockwise with angular velocity ω. Between the rotor and the coagulator there is an air gap. In this gap the extruded polymer moves from an orifice in the rotor to the coagulator.

The purpose of the mathematical modeling is to describe the curve of the spinning line in the air gap. So the spinning line is described from the point where it leaves the rotor until it hits the coagulator. Initially we ignore the gravity, so the problem is reduced to a two-dimensional problem.
We need boundary conditions to describe the curve of the spinning line correctly. These conditions are not that obvious. That is why finding correct boundary conditions is an other purpose of this research.

The radius of the rotor is 0.15 m and the radius of the coagulator is 0.30 m. The angular velocity ω of the rotor is 262 rad/s what corresponds with 2500 rpm. The polymer leaves the orifice with a velocity of 1.0 m/s.

3 The stationary case with rotating s

This section starts with the derivation of a model for the rotor spinning process. This part was derived during the week "Mathematics with Industry" in 2004. In [2], this derivation is written down briefly, here (through Section 3.3) it is more detailed.

We consider a stationary process in a rotating coordinate system. So we omit the time-derivatives in this case. We will look at the movement of the polymer along the trajectory of the polymer, the so called spinning line. Then, the dependent variables are the position of the spinning line (x, y), the velocity v of the polymer along the spinning line and the viscous force F. The arc-length s along the spinning line is the independent variable.

The first equation the model has to satisfy follows from Pythagoras:

$$\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 = 1. \tag{3.1}$$

Now we want to find the momentum balances. Therefore, we need the mass flux, which is constant int his case:

$$\Phi = \rho A v, \tag{3.2}$$

where ρ the mass density of the polymer and A the area of the cross-section of the spinning line (see Appendix A for a list of symbols). To find the momentum balance, we take a small part of the spinning line, $[s, s + \Delta s]$. The forces acting on this part Δs are drawn in Figure 2. When we rotate all forces to the first quadrant we find the forces

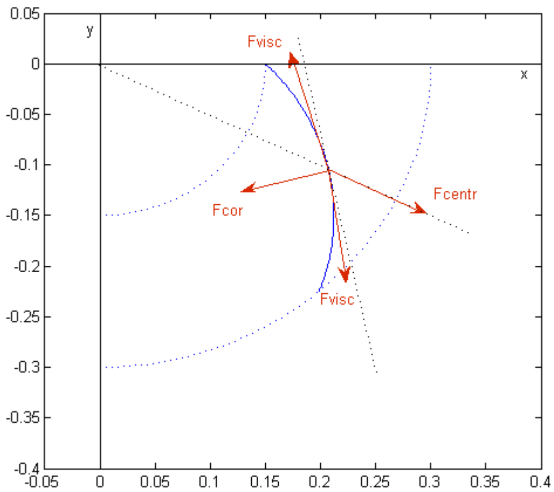

Figure 2: The forces acting on Δs.

and corresponding unit vectors as shown in Figure 3. Now we can derive the several

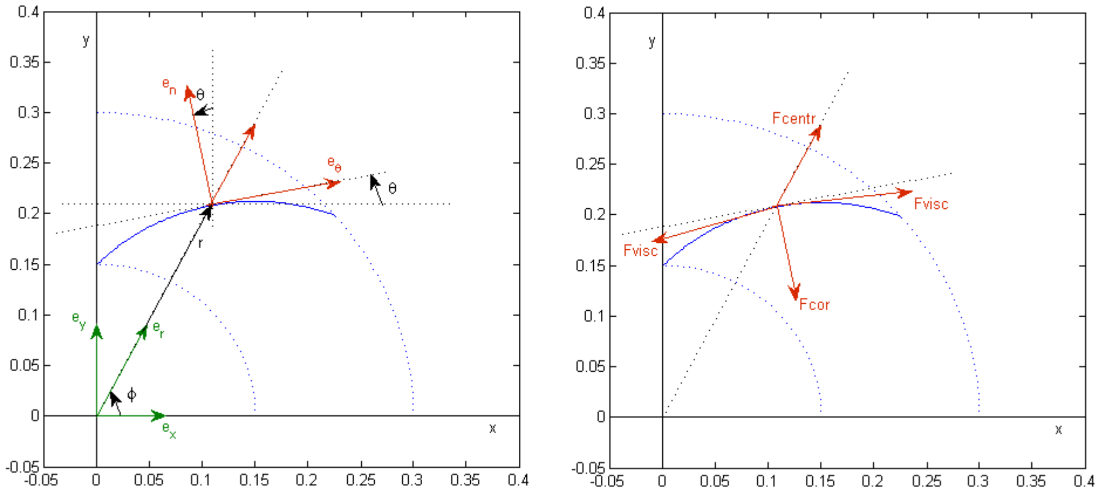

Figure 3: The forces acting on Δs.

forces. The centrifugal force $\mathbf{F}_{centr}$ [1] is given by:

$$\mathbf{F}_{centr} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = \Delta s \rho A \omega^2 \begin{pmatrix} x \\ y \end{pmatrix}, \tag{3.3}$$

and the Coriolis force $\mathbf{F}_{cor}$ [1], directed perpendicular on the spinning line, is given by:

$$\mathbf{F}_{cor} = -2m\boldsymbol{\omega} \times \mathbf{v} = 2\Delta s \rho A \omega v \begin{pmatrix} \frac{\mathrm{d}y}{\mathrm{d}s} \\ -\frac{\mathrm{d}x}{\mathrm{d}s} \end{pmatrix}, \tag{3.4}$$

where $\times$ denotes the cross product, $\mathbf{r} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ and $\boldsymbol{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$.

The viscous force F_{visc} in the x direction can be denoted by

$$\mathrm{F}_{visc} = \Big(F\frac{\mathrm{d}x}{\mathrm{d}s}\Big)_{s+\Delta s} - \Big(F\frac{\mathrm{d}x}{\mathrm{d}s}\Big)_{s}. \tag{3.5}$$

Using Taylor series gives

$$\begin{aligned} \Big(F\frac{\mathrm{d}x}{\mathrm{d}s}\Big)_{s+\Delta s} - \Big(F\frac{\mathrm{d}x}{\mathrm{d}s}\Big)_{s} &= \Big(F\frac{\mathrm{d}x}{\mathrm{d}s}\Big)_{s} + \Delta s\frac{\mathrm{d}}{\mathrm{d}s}\Big(F\frac{\mathrm{d}x}{\mathrm{d}s}\Big)_{s} - \Big(F\frac{\mathrm{d}x}{\mathrm{d}s}\Big)_{s} \\ &= \Delta s \frac{\mathrm{d}}{\mathrm{d}s}\Big(F\frac{\mathrm{d}x}{\mathrm{d}s}\Big). \end{aligned} \tag{3.6}$$

In the same way you can find the viscous force F_{visc} in the y direction. Then the viscous force satisfies:

$$\mathbf{F}_{visc} = \Delta s \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}s}\Big(F\frac{\mathrm{d}x}{\mathrm{d}s}\Big) \\ \frac{\mathrm{d}}{\mathrm{d}s}\Big(F\frac{\mathrm{d}y}{\mathrm{d}s}\Big) \end{bmatrix}. \tag{3.7}$$

In this equation for the viscous force F denotes the norm of the viscous force vector at s. It is assumed that the polymer is Newtonian. From this assumption it follows that the viscous force F satisfies:

$$F = \eta A \frac{\mathrm{d}v}{\mathrm{d}s}, \tag{3.8}$$

so

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{\rho v}{\eta} \frac{F}{\Phi}. \tag{3.9}$$

3.1 The momentum balance

Now we can derive the momentum balances in the x and y direction for the steady state process. The general momentum balance reads:

$$\mathbf{I}_{in} - \mathbf{I}_{out} + \mathbf{F}_{centr} + \mathbf{F}_{cor} + \mathbf{F}_{visc} = 0. \tag{3.10}$$

To write the momentum balance in the x and y direction, we need the tangential unit vector $\mathbf{e}_\theta$:

$$\mathbf{e}_\theta = \mathbf{e}_x cos\theta + \mathbf{e}_y sin\theta = \mathbf{e}_x \frac{\mathrm{d}x}{\mathrm{d}s} + \mathbf{e}_y \frac{\mathrm{d}y}{\mathrm{d}s}. \tag{3.11}$$

The entering momentum flux $\mathbf{I}_{in}$ and the leaving momentum flux $\mathbf{I}_{out}$ are given by:

$$\mathbf{I}_{in} = \rho A v^2 \mid_s, \tag{3.12}$$

$$\mathbf{I}_{out} = \rho A v^2 \mid_{s+\Delta s} . \tag{3.13}$$

These fluxes are both directed along the tangent $\mathbf{e}_\theta$, so with $\Phi = \rho A v = constant$ and equation (3.11) follows:

$$\begin{aligned} \mathbf{I}_{in} - \mathbf{I}_{out} = \rho A v^2 \mathbf{e}_\theta \mid_s - \rho A v^2 \mathbf{e}_\theta \mid_{s+\Delta s} = \Phi v \mathbf{e}_\theta \mid_s - \Phi v \mathbf{e}_\theta \mid_{s+\Delta s} = \\ - \Delta s \frac{\mathrm{d}}{\mathrm{d}s}\Big(\Phi v \mathbf{e}_\theta\Big) = -\Delta s \Phi \Big(\mathbf{e}_x \frac{\mathrm{d}}{\mathrm{d}s}\Big(v\frac{\mathrm{d}x}{\mathrm{d}s}\Big) + \mathbf{e}_y \frac{\mathrm{d}}{\mathrm{d}s}\Big(v\frac{\mathrm{d}y}{\mathrm{d}s}\Big)\Big). \end{aligned} \tag{3.14}$$

Summation of equation (3.14), (3.3), (3.4) and (3.7) results in the momentum balances (3.10) in the x and y direction:

$$-\Phi \frac{\mathrm{d}}{\mathrm{d}s}\Big(v\frac{\mathrm{d}x}{\mathrm{d}s}\Big) + \rho A \omega^2 x + 2\rho A \omega v \frac{\mathrm{d}y}{\mathrm{d}s} + \frac{\mathrm{d}}{\mathrm{d}s}\Big(F\frac{\mathrm{d}x}{\mathrm{d}s}\Big) = 0, \tag{3.15}$$

$$-\Phi \frac{\mathrm{d}}{\mathrm{d}s}\Big(v\frac{\mathrm{d}y}{\mathrm{d}s}\Big) + \rho A \omega^2 y - 2\rho A \omega v \frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\mathrm{d}}{\mathrm{d}s}\Big(F\frac{\mathrm{d}y}{\mathrm{d}s}\Big) = 0. \tag{3.16}$$

We can rewrite these equations by using the product rule of differentiation and $\Phi = \rho A v$.

$$\Big(F - \Phi v\Big)\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} = -\frac{\Phi\omega^2}{v} x - 2\Phi\omega \frac{\mathrm{d}y}{\mathrm{d}s} - \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}s}\Big(F - \Phi v\Big), \tag{3.17}$$

$$\Big(F - \Phi v\Big)\frac{\mathrm{d}^2 y}{\mathrm{d}s^2} = -\frac{\Phi\omega^2}{v} y + 2\Phi\omega \frac{\mathrm{d}x}{\mathrm{d}s} - \frac{\mathrm{d}y}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}s}\Big(F - \Phi v\Big). \tag{3.18}$$

What will happen if $F - \Phi v = 0$ will be discussed later on.

3.2 Replacing Pythagoras

Equation (3.1) results in a square root when you use it to find an expression for the derivatives of x and y with respect to s. To avoid this, we can we can replace this equation by a differential equation. Therefore, we have to take the inner product of the vectorial momentum equations (3.17-3.18) and the vector $\left(\frac{\mathrm{d}x}{\mathrm{d}s}, \frac{\mathrm{d}y}{\mathrm{d}s}\right)^T$:

$$\begin{aligned}\Big(F - \Phi v\Big)\Big(\frac{\mathrm{d}^2 x}{\mathrm{d}s^2}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\mathrm{d}^2 y}{\mathrm{d}s^2}\frac{\mathrm{d}y}{\mathrm{d}s}\Big) &= \\ &= -\frac{\Phi\omega^2}{v}\Big(x\frac{\mathrm{d}x}{\mathrm{d}s} + y\frac{\mathrm{d}y}{\mathrm{d}s}\Big) - \Big(\Big(\frac{\mathrm{d}x}{\mathrm{d}s}\Big)^2 + \Big(\frac{\mathrm{d}y}{\mathrm{d}s}\Big)^2\Big)\frac{\mathrm{d}}{\mathrm{d}s}\Big(F - \Phi v\Big).\end{aligned} \tag{3.19}$$

The derivative of equation (3.1) is zero. So substituting this derivative and equation (3.1) gives:

$$\frac{\mathrm{d}F}{\mathrm{d}s} = \Phi\frac{\mathrm{d}v}{\mathrm{d}s} - \frac{\Phi\omega^2}{v}\Big(x\frac{\mathrm{d}x}{\mathrm{d}s} + y\frac{\mathrm{d}y}{\mathrm{d}s}\Big). \tag{3.20}$$

Instead of equation (3.1), this equation does not results in a square root

3.3 The stationary system with rotating s

The set of equations for the stationary case in a rotating coordinate system with coordinate s we have found is:

$$\Big(F - \Phi v\Big)\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} = -\frac{\Phi\omega^2}{v}x - 2\Phi\omega\frac{\mathrm{d}y}{\mathrm{d}s} - \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}s}\Big(F - \Phi v\Big), \tag{3.21}$$

$$\Big(F - \Phi v\Big)\frac{\mathrm{d}^2 y}{\mathrm{d}s^2} = -\frac{\Phi\omega^2}{v}y + 2\Phi\omega\frac{\mathrm{d}x}{\mathrm{d}s} - \frac{\mathrm{d}y}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}s}\Big(F - \Phi v\Big), \tag{3.22}$$

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{\rho v}{\eta}\frac{F}{\Phi}, \tag{3.23}$$

$$\frac{\mathrm{d}F}{\mathrm{d}s} = \Phi\frac{\mathrm{d}v}{\mathrm{d}s} - \frac{\Phi\omega^2}{v}\Big(x\frac{\mathrm{d}x}{\mathrm{d}s} + y\frac{\mathrm{d}y}{\mathrm{d}s}\Big), \tag{3.24}$$

with $\Phi = \rho A v$ constant. There are four equations and four unknowns: F, x, y and v. The momentum equation in the x direction (3.21) and the momentum equation in the y direction (3.22) are second order equations. The other equations are of first order. Therefore, to solve the whole system six boundary or initial conditions are needed. Some of the initial conditions of the system (3.21-3.24) are obvious:

$$x(0) = R_{rot}, \qquad y(0) = 0, \qquad v(0) = v_0, \qquad F(0) = F_0. \tag{3.25}$$

The viscous force F_0 is unknown and should follow from conditions imposed on the spinning line at the coagulator. When the spinning line leaves the orifice perpendicular to the rotor the initial conditions are:

$$\frac{\mathrm{d}x}{\mathrm{d}s} = 1, \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = 0, \qquad \text{for } s = 0. \tag{3.26}$$

Against all expectations, the spinning line seems to leave the orifice not perpendicular to the rotor. This subject will be discussed later. Denote the arc length of the spinning line at the coagulator by L. Then we need two boundary conditions:

$$x(L)^2 + y(L)^2 = R_{coag}^2, \qquad v(L) = v_e. \tag{3.27}$$

A problem is that we do not know the length L when the spinning line hits the coagulator.

3.4 Balances perpendicular to the tangent and the radius vector

The balance perpendicular to the tangent and the balance perpendicular to the radius vector are also useful. These balances will be used in Section 7.

The balance perpendicular to the tangent appears when equation (3.21) is multiplied by $\frac{\mathrm{d}y}{\mathrm{d}s}$ and added to equation (3.22) multiplied by $-\frac{\mathrm{d}x}{\mathrm{d}s}$:

$$(F - \Phi v)\left(\frac{\mathrm{d}^2x}{\mathrm{d}s^2}\frac{\mathrm{d}y}{\mathrm{d}s} - \frac{\mathrm{d}^2y}{\mathrm{d}s^2}\frac{\mathrm{d}x}{\mathrm{d}s}\right) = -\frac{\Phi\omega^2}{v}\left(x\frac{\mathrm{d}y}{\mathrm{d}s} - y\frac{\mathrm{d}x}{\mathrm{d}s}\right) - 2\Phi\omega. \tag{3.28}$$

Assume $F - \Phi v = 0$, then

$$x\frac{\mathrm{d}y}{\mathrm{d}s} - y\frac{\mathrm{d}x}{\mathrm{d}s} = -\frac{2v}{\omega}. \tag{3.29}$$

In the same way, the balance perpendicular to the radius vector can be found. Multiply equation (3.21) by y and equation (3.22) by $-x$, adding those two equations gives:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left((F - \Phi v)\left(y\frac{\mathrm{d}x}{\mathrm{d}s} - x\frac{\mathrm{d}y}{\mathrm{d}s}\right)\right) = -\Phi\omega\frac{\mathrm{d}}{\mathrm{d}s}\left(y^2 + x^2\right). \tag{3.30}$$

Integration of this equation leads to

$$\left((F - \Phi v)\left(y\frac{\mathrm{d}x}{\mathrm{d}s} - x\frac{\mathrm{d}y}{\mathrm{d}s}\right)\right) = -\Phi\omega\left(y^2 + x^2\right) + C_1, \tag{3.31}$$

where C_1 an integration constant.

3.5 The case $F - \Phi v \equiv 0$

When $F - \Phi v \equiv 0$, so $F \equiv \Phi v$, the set of equations (3.21-3.24) can be reduced to the following system

$$0 = -\frac{\omega}{v}x - 2\frac{\mathrm{d}y}{\mathrm{d}s} \tag{3.32}$$

$$0 = -\frac{\omega}{v}y + 2\frac{\mathrm{d}x}{\mathrm{d}s} \tag{3.33}$$

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{\rho}{\eta}v^2 \tag{3.34}$$

$$x\frac{\mathrm{d}x}{\mathrm{d}s} + y\frac{\mathrm{d}y}{\mathrm{d}s} = 0 \tag{3.35}$$

If the first two equations (3.32-3.33) hold, the last one (3.35) automatically also holds, so this one cancels. The velocity can be solved from equation (3.34), when we use the initial condition $v(0) = 1$ the velocity corresponds with

$$v(s) = \frac{1}{1 - \frac{\rho}{\eta}s} \tag{3.36}$$

When we substitute this in equation (3.32) and (3.33) the remaining system is

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}s} &= \frac{1}{2}\omega y + \frac{1}{2}\frac{\rho\omega}{\eta}sy \\ \frac{\mathrm{d}y}{\mathrm{d}s} &= -\frac{1}{2}\omega x + \frac{1}{2}\frac{\rho\omega}{\eta}sx \\ x(0) &= R_{rot}, \qquad y(0) = 0\end{aligned} \tag{3.37}$$

Solving this gives a spinning line which is following the rotor exactly. So when $F - \Phi v \equiv 0$, the spinning line will stick to the rotor.

3.6 Solving the set of equations representing the stationary case in a rotating coordinate system with zero viscosity

If the viscosity is zero, the model reduces to the problem of bullets fired from a rotating disc. After eliminating the viscous force, the system (3.21-3.24) becomes

$$\frac{\mathrm{d}^2x}{\mathrm{d}s^2} = \frac{1}{v}\left(\frac{\omega^2}{v}x + 2\omega\frac{\mathrm{d}y}{\mathrm{d}s} - \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\mathrm{d}v}{\mathrm{d}s}\right), \tag{3.38}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}s^2} = \frac{1}{v}\left(\frac{\omega^2}{v}y - 2\omega\frac{\mathrm{d}x}{\mathrm{d}s} - \frac{\mathrm{d}y}{\mathrm{d}s}\frac{\mathrm{d}v}{\mathrm{d}s}\right), \tag{3.39}$$

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{\omega^2}{v}\left(x\frac{\mathrm{d}x}{\mathrm{d}s} + y\frac{\mathrm{d}y}{\mathrm{d}s}\right). \tag{3.40}$$

Equation (3.23) is disappeared because of eliminating viscous force F. Using $v = \frac{\mathrm{d}s}{\mathrm{d}t}$ the system can be written with respect to t instead of s. Then the system becomes

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \omega^2 x + 2\omega\frac{\mathrm{d}y}{\mathrm{d}t}, \tag{3.41}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \omega^2 y - 2\omega\frac{\mathrm{d}x}{\mathrm{d}t}, \tag{3.42}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\omega^2}{v}\left(x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t}\right). \tag{3.43}$$

The solution of this system is given by

$$x(t) = (R_{rot} + v_0 t)\cos(\omega t) + \omega R_{rot} t \sin(\omega t), \tag{3.44}$$

$$y(t) = -(R_{rot} + v_0 t)\sin(\omega t) + \omega R_{rot} t \cos(\omega t), \tag{3.45}$$

$$v(t) = \sqrt{(v_0 + \omega^2 R_{rot} t)^2 + (\omega v_0 t)^2}. \tag{3.46}$$

In the viscous case, realistic values for the physical parameters are:

$$\rho = 1700\ kg/m^3, \qquad \eta = 1200\ Pa \cdot s, \qquad v_0 = 1\ m/s, \tag{3.47}$$

$$R_{rot} = 0.15\ m, \qquad R_{coag} = 0.3\ m, \qquad A(0) = \pi/64 * 10^{-6}\ m^2 \tag{3.48}$$

4 The instationary case with fixed r

We know the differential equations, describing the rotor spinning process, for the instationary case in a rotating coordinate system with coordinate s, where s is the distance along the spinning line from the rotor to a point on the spinning line. Instead of this coordinate s we can use the radial distance r as a coordinate [4]. In the new coordinate, the domain of the problem is known a priori, while in the coordinate system with s it was not because the spinning line length L is unknown. A disadvantage of the choice of coordinate r, is that spinning lines which curve back toward the rotor cannot be represented, because the solution becomes multi-valued in terms of coordinate r. But in the rotor spinning process this will not happen in practice. Again the vertical velocity is neglected, so the spinning line will be modeled as a one-dimensional curve in a two-dimensional space. Section 4.1 until Section 4.4 is derived by Bas van 't Hof [4], in these sections the derivation is written down in more detail.

4.1 Parametrization of the spinning line

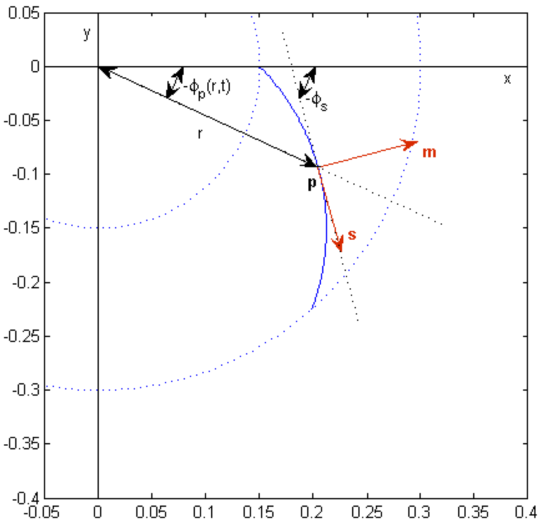

Figure 4: Parametrization of the spinning line.

First we need a parameterization of the spinning line (See Figure 4). This is done by means of ϕ_p

$$\mathbf{p}(r,t) = r\begin{pmatrix}\cos(\phi_p(r,t))\\ \sin(\phi_p(r,t))\end{pmatrix}. \tag{4.1}$$

Here, r is the radial distance, so $r = \sqrt{p_1^2 + p_2^2}$. The velocity of the spinning line is parametrized by:

$$\mathbf{v}(r,t) = V\begin{pmatrix}\cos(\phi_v(r,t))\\ \sin(\phi_v(r,t))\end{pmatrix}. \tag{4.2}$$

Here, V is the length of the vector, so $V(r,t) = \sqrt{v_1^2(r,t) + v_2^2(r,t)}$. In this section the angle ϕ_p is negative in all figures. A tangent vector to the spinning line is given by the derivative $\frac{\partial \mathbf{p}}{\partial r}$. The spinning line direction ϕ_s is the argument of the spinning line tangent vector (see [4], Section 4.2.1):

$$\phi_s = \arctan\left(\frac{\sin(\phi_p) + r\cos(\phi_p)\frac{\partial \phi_p}{\partial r}}{\cos(\phi_p) - r\sin(\phi_p)\frac{\partial \phi_p}{\partial r}}\right). \tag{4.3}$$

Rewriting equation (4.3) with simple addition and subtraction formulas leads to

$$\frac{\partial \phi_p}{\partial r} = \frac{-\tan(\phi_p - \phi_s)}{r}. \tag{4.4}$$

By definition, the unit tangent vector $\mathbf{s}$ along the spinning line is given by:

$$\mathbf{s} = \frac{1}{\|\frac{\partial \mathbf{p}}{\partial r}\|}\frac{\partial \mathbf{p}}{\partial r} = \begin{pmatrix}\cos(\phi_s)\\ \sin(\phi_s)\end{pmatrix}, \tag{4.5}$$

and a unit normal vector $\mathbf{m}$ can be represented by

$$\mathbf{m} = \mathbf{J}\mathbf{s}, \tag{4.6}$$

because then $\mathbf{m}\cdot\mathbf{s} = 0$. Here, $\mathbf{J}$ is the rotation operator denoted by $\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In the model, we consider the flow velocity $\mathbf{v}(r,t)$ and the cross-section area of the spinning line $A(r,t)$. Then the dependent variables are $\mathbf{v}$, A, ϕ_s and ϕ_p. The radial distance r and the time are the independent variables.

4.2 Kinematic equation

The kinematic equation is the only equation in the system that is not a conservation law. This equation leads to an expression for the time derivative of the angle ϕ_p. It is derived from the fact that the spinning line moves with flow velocity $\mathbf{v}$. The fluid particle $\mathbf{p}(r,t)$ will be at $\mathbf{p}(r,t) + \Delta t\mathbf{v}(r,t)$ after a small time Δt. So there must be a coordinate $r + \Delta r$ (see Figure 5) such that

$$\mathbf{p}(r+\Delta r, t+\Delta t) = \mathbf{p}(r,t) + \Delta t\mathbf{v}(r,t). \tag{4.7}$$

The new position of the particle after Δt time is equal to the old position on time t plus the covered way in time Δt. Here Δr is dependent of t. We can take the total derivative of $\mathbf{p}$, where $\mathbf{p}$ is denoted by $\mathbf{p}(R(t),t)$. Then the total derivative of $\mathbf{p}$, the velocity $\mathbf{v}$, is denoted by:

$$\frac{\mathrm{D}\mathbf{p}}{\mathrm{D}t} = \frac{\mathrm{D}\mathbf{p}(R(t),t)}{\mathrm{D}t} = \frac{\partial \mathbf{p}}{\partial t} + \frac{\partial \mathbf{p}}{\partial r}\frac{\mathrm{d}R}{\mathrm{d}t} \tag{4.8}$$

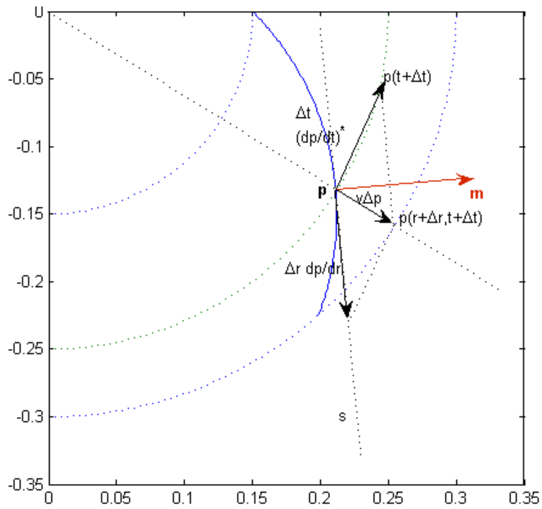

Figure 5: The kinematic equation.

Because $\mathbf{m}$ and $\mathbf{s}$ are perpendicular we get the following kinematic equation, by using (4.5):

$$\mathbf{m} \cdot \mathbf{v} = \mathbf{m} \cdot \frac{\partial \mathbf{p}}{\partial t} + \mathbf{m} \cdot \mathbf{s} \| \frac{\partial \mathbf{p}}{\partial r} \| \frac{\mathrm{d}R}{\mathrm{d}t} = \mathbf{m} \cdot \frac{\partial \mathbf{p}}{\partial t}. \tag{4.9}$$

We know that r is constant (see Figure 5), so rewriting leads to:

$$\mathbf{m} \cdot \mathbf{v} = \mathbf{m} \cdot \frac{\partial \mathbf{p}}{\partial t} = \mathbf{m} \cdot r \begin{pmatrix} -\sin(\phi_p) \\ \cos(\phi_p) \end{pmatrix} \frac{\partial \phi_p}{\partial t} = \mathbf{m} \cdot \mathbf{J}\mathbf{p} \frac{\partial \phi_p}{\partial t}. \tag{4.10}$$

After using the addition and subtraction formulas it leads to

$$\frac{\partial \phi_p}{\partial t} = \frac{\mathbf{m} \cdot \mathbf{v}}{\mathbf{m} \cdot \mathbf{J}\mathbf{p}} = \frac{\sin(\phi_v - \phi_s)}{\cos(\phi_p - \phi_s)} \frac{V}{r}. \tag{4.11}$$

4.3 General form of the conservation law in a moving spinning line

We can construct a general form of a conservation law. The conservation law consists of three parts, which add up to zero. These parts are the transport Q, the change of the total amounts M and the local production/destruction Ψ in a small piece of the spinning line.

$$\frac{\partial M}{\partial t} = Q_{R_1} - Q_{R_2} + \Psi. \tag{4.12}$$

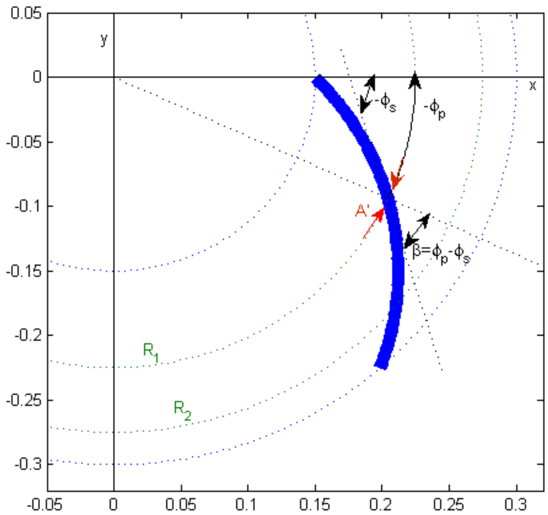

Figure 6: A small piece of the spinning line to construct the conservation law.

4.3.1 Transport

We take a cross-section A', this cross-section is not perpendicular to the spinning line, see Figure 6, so

$$A' = \frac{A}{\cos(\phi_p - \phi_s)} \tag{4.13}$$

The normal direction to the cross-section A' is given by the angle ϕ_p. We can determine the transport Q through cross-section A' by

$$Q = A'\mathbf{f} \cdot \begin{pmatrix} \cos\phi_p \\ \sin\phi_p \end{pmatrix}. \tag{4.14}$$

$\mathbf{f}$ denotes a transport flux, $\mathbf{f} = f\begin{pmatrix} \cos\phi_f \\ \sin\phi_f \end{pmatrix}$. So

$$Q = A'\mathbf{f}\begin{pmatrix} \cos\phi_p \\ \sin\phi_p \end{pmatrix} = A'f\cos(\phi_p - \phi_f) = \frac{\cos(\phi_p - \phi_f)}{\cos(\phi_p - \phi_s)}Af. \tag{4.15}$$

The netto transport into the spinning line section between $r = R_1$ and $r = R_2$ is now given by

$$Q_{R_1} - Q_{R_2} = -\left[\frac{\cos(\phi_p - \phi_f)}{\cos(\phi_p - \phi_s)}Af\right]_{R_1}^{R_2}. \tag{4.16}$$

4.3.2 Total amounts

The total amounts are calculated from a line integral along the spinning line. The local density of a given quantity is written as d, given per unit volume. The total amount M

of the given quantity in the spinning line segment between any two coordinates R_1 and R_2 is now found by the line integral (see Figure 6):

$$M = \int_{R_1}^{R_2} Ad\mathrm{d}s = \int_{R_1}^{R_2} Ad\|\frac{\partial \mathbf{p}}{\partial r}\|\mathrm{d}r = \int_{R_1}^{R_2} \frac{Ad}{\cos(\phi_p - \phi_s)}\mathrm{d}r. \tag{4.17}$$

We have used that

$$\mathrm{d}\mathbf{s} = \mathbf{p}(r + \mathrm{d}r) - \mathbf{p}(r) = \frac{\partial \mathbf{p}}{\partial r}\mathrm{d}r. \tag{4.18}$$

So

$$\mathrm{d}s = \|\frac{\partial \mathbf{p}}{\partial r}\|\mathrm{d}r. \tag{4.19}$$

Notice that the $\mathrm{d}\mathbf{s}$ from equation (4.18) and the $\mathrm{d}s$ for equation (4.19) are not the same. With equation(4.4) we get:

$$\begin{aligned}
\|\frac{\partial \mathbf{p}}{\partial r}\| &= \|\frac{\partial}{\partial r}\left(r\begin{pmatrix}\cos\phi_p \\ \sin\phi_p\end{pmatrix}\right)\| = \|\begin{pmatrix}\cos\phi_p \\ \sin\phi_p\end{pmatrix} + r\begin{pmatrix}-\sin(\phi_p)\frac{\partial\phi_p}{\partial r} \\ \cos(\phi_p)\frac{\partial\phi_p}{\partial r}\end{pmatrix}\| = \\
&= \|\begin{pmatrix}\cos\phi_p \\ \sin\phi_p\end{pmatrix} + r\begin{pmatrix}-\sin(\phi_p)\frac{1}{r}(-\tan(\phi_p - \phi_s)) \\ \cos(\phi_p)\frac{1}{r}(-\tan(\phi_p - \phi_s))\end{pmatrix}\| = \\
&= \sqrt{(\cos\phi_p + \sin\phi_p\tan(\phi_p - \phi_s))^2 + (\sin\phi_p - \cos\phi_p\tan(\phi_p - \phi_s))^2} = \\
&= \sqrt{\tan^2(\phi_p - \phi_s) + 1} = \frac{1}{\cos(\phi_p - \phi_s)}.
\end{aligned} \tag{4.20}$$

By introducing the coordinate r, we said the spinning line does not curve back towards the rotor, so ϕ_p is not perpendicular to ϕ_s. Then $\phi_p - \phi_s \neq \frac{\pi}{2}$ and it is allowed to divide by $\cos(\phi_p - \phi_s)$, as in equation (4.17).

4.3.3 Production and destruction

In the same way as the total amount M in equation (4.17) the production can be found:

$$\Psi = \int_{R_1}^{R_2} \frac{S}{\cos(\phi_p - \phi_s)}\mathrm{d}r, \tag{4.21}$$

where S denotes the production intensity.

4.3.4 The general conservation law

The general form of a conservation law can be found by combining the change of the total amount, the transport and the production terms.

$$\frac{\partial M}{\partial t} = Q_{R_1} - Q_{R_2} + \Psi. \tag{4.22}$$

So

$$\frac{\partial}{\partial t}\int_{R_1}^{R_2} \frac{Ad}{\cos(\phi_p - \phi_s)}\mathrm{d}r = -\left[\frac{\cos(\phi_p - \phi_f)}{\cos(\phi_p - \phi_s)}Af\right]_{R_1}^{R_2} + \int_{R_1}^{R_2} \frac{S}{\cos(\phi_p - \phi_s)}\mathrm{d}r. \tag{4.23}$$

Because this must hold for every R_1 and R_2, we can rewrite this conservation law into

$$\frac{\partial}{\partial t}\left(\frac{Ad}{\cos(\phi_p-\phi_s)}\right)+\frac{\partial}{\partial r}\left(\frac{\cos(\phi_p-\phi_f)}{\cos(\phi_p-\phi_s)}Af\right)=\frac{S}{\cos(\phi_p-\phi_s)}. \tag{4.24}$$

4.4 The conservation laws in a moving spinning line

In this section we will apply the general conservation law (4.24) to mass and to the x and y-momentum. The local densities, given as d in (4.17) and the flux $\mathbf{f}$ are as follows:

Quantity	Density (d)	unit
mass	ρ	kg/m^3
momentum	$\rho\mathbf{v}$	kg/sm^2
kinetic energy	$\frac{1}{2}\rho \mid \mathbf{v} \mid^2$	J/m^3

4.4.1 The continuity equation

The continuity equation is found by applying the conservation law to mass with density $d=\rho$, flux $\mathbf{f}=\rho\mathbf{v}$, so $f=\rho V$ and $\phi_f=\phi_v$ and production $S=0$ yields to:

$$\frac{\partial}{\partial t}\left(\frac{\rho A}{\cos(\phi_p-\phi_s)}\right)+\frac{\partial}{\partial r}\left(\frac{\cos(\phi_p-\phi_v)}{\cos(\phi_p-\phi_s)}\rho AV\right)=0. \tag{4.25}$$

The mass transport term is known as $\Phi=\frac{\cos(\phi_p-\phi_v)}{\cos(\phi_p-\phi_s)}\rho AV$. In this model the mass transport flux is not constant in contrast to the model in Section 3.

4.4.2 The momentum equation

To find the momentum equation in the x and y direction, we apply the conservation law to the x- and y-momentum.
We know, $u=\begin{pmatrix}1\\0\end{pmatrix}\mathbf{v}$ and $v=\begin{pmatrix}0\\1\end{pmatrix}\mathbf{v}$.
So in the x direction the density is $d=\rho\begin{pmatrix}1\\0\end{pmatrix}\mathbf{v}$,
the flux is $\mathbf{f}=\rho\begin{pmatrix}1\\0\end{pmatrix}\cdot\mathbf{v}\mathbf{v}^T-\frac{\eta}{2}\begin{pmatrix}1\\0\end{pmatrix}\left(\nabla\mathbf{v}+\nabla\mathbf{v}^T\right)$
and the production $S=0$.
The cosine of the angle difference is determined by

$$f\cos(\phi_p-\phi_f)=\mathbf{f}\begin{pmatrix}\cos\phi_p\\ \sin\phi_p\end{pmatrix}. \tag{4.26}$$

Substitution of these terms in the general conservation law (4.24) results in a momentum equation in the x and y direction.

$$\frac{\partial}{\partial t}\left(\frac{\rho Au}{\cos(\phi_p-\phi_s)}\right)+\frac{\partial}{\partial r}\left(\Phi u-\frac{\eta A}{2\cos(\phi_p-\phi_s)}\begin{pmatrix}1\\0\end{pmatrix}\left(\nabla\mathbf{v}+\nabla\mathbf{v}^T\right)\begin{pmatrix}\cos\phi_p\\ \sin\phi_p\end{pmatrix}\right)=0. \tag{4.27}$$

$$\frac{\partial}{\partial t}\left(\frac{\rho A v}{\cos(\phi_p-\phi_s)}\right)+\frac{\partial}{\partial r}\left(\Phi v-\frac{\eta A}{2\cos(\phi_p-\phi_s)}\begin{pmatrix}0\\1\end{pmatrix}\left(\nabla\mathbf{v}+\nabla\mathbf{v}^T\right)\begin{pmatrix}\cos\phi_p\\ \sin\phi_p\end{pmatrix}\right)=0 \tag{4.28}$$

Combination of these two equations gives the momentum equation in vector notation:

$$\frac{\partial}{\partial t}\left(\frac{\rho A\mathbf{v}}{\cos(\phi_p-\phi_s)}\right)+\frac{\partial}{\partial r}\left(\Phi\mathbf{v}-\frac{\eta A}{2\cos(\phi_p-\phi_s)}\left(\nabla\mathbf{v}+\nabla\mathbf{v}^T\right)\begin{pmatrix}\cos\phi_p\\ \sin\phi_p\end{pmatrix}\right)=0. \tag{4.29}$$

From the momentum balance for two halves of the spinning line (see [4], appendix A), the viscosity operator is given by

$$\left(\nabla\mathbf{v}+\nabla\mathbf{v}^T\right)\begin{pmatrix}\cos\phi_p\\ \sin\phi_p\end{pmatrix}=2\left(\mathbf{s}\cdot\frac{\partial\mathbf{v}}{\partial r}\right)\mathbf{s}\cos(\phi_p-\phi_s)^2. \tag{4.30}$$

The viscous force works only in the spinning line direction $\mathbf{s}$, because the matrix-vector product of the viscous tensor $\left(\nabla\mathbf{v}+\nabla\mathbf{v}^T\right)$ and an arbitrary vector is always parallel to the spinning line.

4.5 The instationary system in fixed r

The set of equations for the instationary case in a fixed coordinate system with coordinate r we have found is:

$$\phi_s=\arctan\left(\frac{\sin(\phi_p)+r\cos(\phi_p)\frac{\partial\phi_p}{\partial r}}{\cos(\phi_p)-r\sin(\phi_p)\frac{\partial\phi_p}{\partial r}}\right)\qquad \textit{or}\qquad \frac{\partial\phi_p}{\partial r}=\frac{-\tan(\phi_p-\phi_s)}{r}, \tag{4.31}$$

kinematic equation:

$$\frac{\partial\phi_p}{\partial t}=\frac{\sin(\phi_v-\phi_s)}{\cos(\phi_p-\phi_s)}\frac{V}{r}, \tag{4.32}$$

continuity equation:

$$\frac{\partial}{\partial t}\left(\frac{\rho A}{\cos(\phi_p-\phi_s)}\right)+\frac{\partial\Phi}{\partial r}=0, \tag{4.33}$$

momentum equations:

$$\frac{\partial}{\partial t}\left(\frac{\rho A\mathbf{v}}{\cos(\phi_p-\phi_s)}\right)+\frac{\partial}{\partial r}\left(\Phi\mathbf{v}-\eta A\cos(\phi_p-\phi_s)\left(\mathbf{s}\cdot\frac{\partial\mathbf{v}}{\partial r}\right)\mathbf{s}\right)=0. \tag{4.34}$$

With

$$\mathbf{p}=r\begin{pmatrix}\cos\phi_p\\ \sin\phi_p\end{pmatrix},\qquad \mathbf{v}=V\begin{pmatrix}\cos\phi_v\\ \sin\phi_v\end{pmatrix}, \tag{4.35}$$

$$\mathbf{s}=\begin{pmatrix}\cos\phi_s\\ \sin\phi_s\end{pmatrix},\qquad \mathbf{m}=\begin{pmatrix}-\sin\phi_s\\ \cos\phi_s\end{pmatrix}, \tag{4.36}$$

and mass transport Φ:

$$\Phi=\rho AV\frac{\cos(\phi_p-\phi_v)}{\cos(\phi_p-\phi_s)}. \tag{4.37}$$

This term is not constant because cross-section $\frac{A}{\cos(\phi_p - \phi_s)}$ is not perpendicular to the spinning line.

There are five differential equations (4.31-4.34) and we have to deal with five unknowns: ϕ_v, V, ϕ_s, ϕ_p and A.
The momentum equations are both second order, the other equations are all first order. Equation 4.31 is a first derivatie with respect to r, the kinematic equation is a first order derivative with respect to time and the continuity equation contains a first orer derivative with respect to time and one with respect to r. The momentem equation contains a first order time derivative and a second order derivative with respect to time. We have to deal with four time derivatives of first order, so we need four initial conditions. Six boundary conditions follow from the two first order derivatives with respect to r and the two second order derivatives with respect to r.

4.5.1 Boundary conditions

Assume again that the spinning line is perpendicular to the rotor when it leaves the orifice, and that the magnitude of the velocity V is 1 m/s at the orifice, so

$$V(R_{rot}, t) = 1\ m/s. \tag{4.38}$$

The angles ϕ_v, ϕ_s and ϕ_p are zero at the rotor:

$$\phi_v(R_{rot}, t) = 0, \qquad \phi_s(R_{rot}, t) = 0, \qquad \phi_p(R_{rot}, t) = 0, \tag{4.39}$$

and A is the area of the cross-section perpendicular to the spinning line. The radius of the orifice is 125 μm, so at the orifice

$$A(R_{rot}, t) = \pi(125 * 10^{-6})^2 = \frac{\pi}{64 * 10^{-6}}. \tag{4.40}$$

Those five boundary conditions are 'obvious' in this perpendicular case, but the other boundary condition is not that easy to find. You can try to find the magnitude of the final velocity at the coagulator v_e, then

$$V(R_{coag}, t) = v_e \tag{4.41}$$

Another option is to require that, at the coagulator, the spinning line has a slope equal to the slope of the coagulator at that point. Then one can derive (see Figure 7) that $\phi_p - \phi_s = \frac{\pi}{2}$, because $\phi_p \perp \phi_s$. Substituting this in equation (4.31) gives:

$$\frac{\partial \phi_p}{\partial r}(R_{coag}, t) = \frac{-\tan\left(\frac{\pi}{2}\right)}{R_{coag}} \tag{4.42}$$

Now we obtain a problem because $\tan\left(\frac{\pi}{2}\right) = \infty$. Another important thing to remember is that it is possible that the spinning line curves backwards, before it hits the coagulator.

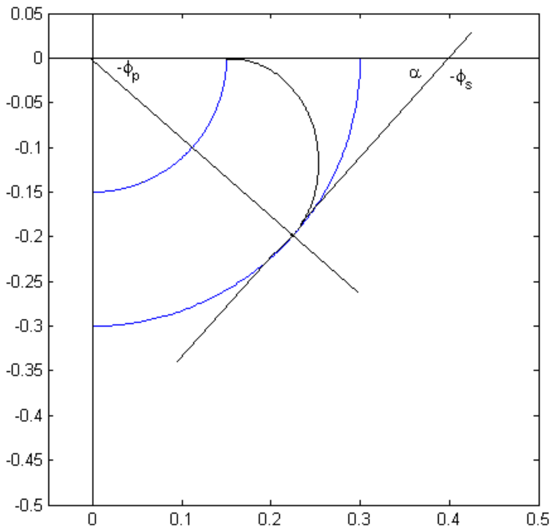

Figure 7: The angles when the slope of the spinning line at the coagulator is equal to the slope of the coagulator at that point.

4.5.2 Initial conditions

The four initial conditions we need are easier to find if one assume that the velocity V and the cross-section area A are constant in the whole spinning line. Then the initial conditions are:

$$\phi_v(r,0) = 0, \qquad V(r,0) = 1\ m/s, \qquad \phi_p(r,0) = 0, \tag{4.43}$$
$$A(r,0) = \pi(125 * 10^{-6})^2 = \frac{\pi}{64 * 10^{-6}}.$$

5 The instationary case with rotating r

In this section we transform the equations of the fixed coordinate system to a rotating coordinate system, in which the rotor is stationary and the coagulator rotates. First we transform the variables ϕ_p, $\mathbf{p}$, $\mathbf{s}$ and $\mathbf{v}$ such that we find new variables $\tilde{\phi_p}$, $\tilde{\mathbf{p}}$, $\tilde{\mathbf{s}}$ and $\tilde{\mathbf{v}}$. The new variables are chosen such that the original variables can be derived easily from them, and that a stationary solution may exist for the transformed variables. The original derivation can be found in the article of van 't Hof [4]. In Section 5.3 this derivation is written down in detail.
We can write the original variables in terms of the rotated variables by:

$$\mathbf{p} = \mathbf{C}\tilde{\mathbf{p}}, \qquad \mathbf{s} = \mathbf{C}\tilde{\mathbf{s}}, \qquad \mathbf{m} = \mathbf{C}\tilde{\mathbf{m}}, \tag{5.1}$$

where $\mathbf{C}$ a rotation matrix

$$\mathbf{C} = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix}. \tag{5.2}$$

Because $\mathbf{C}^T\mathbf{C} = \mathbf{I}$ it follows that

$$\tilde{\mathbf{p}} = \mathbf{C}^T\mathbf{p}, \qquad \tilde{\mathbf{s}} = \mathbf{C}^T\mathbf{s}, \qquad \tilde{\mathbf{m}} = \mathbf{C}^T\mathbf{m}. \tag{5.3}$$

We define the transformed angles by

$$\tilde{\phi_p} = \phi_p - \omega t, \tag{5.4}$$

$$\tilde{\phi_s} = \phi_s - \omega t. \tag{5.5}$$

Then

$$\tilde{\mathbf{p}} = \mathbf{C}^T\mathbf{p} = r\begin{pmatrix} \cos(\phi_p - \omega t) \\ \sin(\phi_p - \omega t) \end{pmatrix} = r\begin{pmatrix} \cos\tilde{\phi_p} \\ \sin\tilde{\phi_p} \end{pmatrix}. \tag{5.6}$$

$$\tilde{\mathbf{s}} = \mathbf{C}^T\mathbf{s} = \begin{pmatrix} \cos(\phi_s - \omega t) \\ \sin(\phi_s - \omega t) \end{pmatrix} = \begin{pmatrix} \cos\tilde{\phi_s} \\ \sin\tilde{\phi_s} \end{pmatrix}. \tag{5.7}$$

$$\tilde{\mathbf{m}} = \mathbf{C}^T\mathbf{m} = \begin{pmatrix} -\sin(\phi_s - \omega t) \\ \cos(\phi_s - \omega t) \end{pmatrix} = \begin{pmatrix} -\sin\tilde{\phi_s} \\ \cos\tilde{\phi_s} \end{pmatrix}. \tag{5.8}$$

The transformed velocity is given by the time derivative of the transformed parametrization of the spinning line $\tilde{\mathbf{p}}$, so

$$\tilde{\mathbf{v}} = \frac{\partial\tilde{\mathbf{p}}}{\partial t} = \frac{\mathrm{d}r}{\mathrm{d}t}\begin{pmatrix} \cos\tilde{\phi_p} \\ \sin\tilde{\phi_p} \end{pmatrix} + r\begin{pmatrix} -\sin\tilde{\phi_p} \\ \cos\tilde{\phi_p} \end{pmatrix}\left(\frac{\partial\phi_p}{\partial t} - \omega\right). \tag{5.9}$$

We know that

$$\mathbf{v} = \frac{\partial\mathbf{p}}{\partial t} = \frac{\mathrm{d}r}{\mathrm{d}t}\begin{pmatrix} \cos\phi_p \\ \sin\phi_p \end{pmatrix} + r\begin{pmatrix} -\sin\phi_p \\ \cos\phi_p \end{pmatrix}\frac{\partial\phi_p}{\partial t}, \tag{5.10}$$

then

$$\mathbf{C}^T\mathbf{v} = \frac{\mathrm{d}r}{\mathrm{d}t}\begin{pmatrix}\cos\tilde{\phi}_p\\ \sin\tilde{\phi}_p\end{pmatrix} + r\begin{pmatrix}-\sin\tilde{\phi}_p\\ \cos\tilde{\phi}_p\end{pmatrix}\frac{\partial\phi_p}{\partial t}. \tag{5.11}$$

Now, we can express the transformed velocity in terms of the original velocity by

$$\tilde{\mathbf{v}} = \mathbf{C}^T\mathbf{v} - \omega\mathbf{J}\tilde{\mathbf{p}}. \tag{5.12}$$

We can also express the velocity in the fixed coordinate system $\mathbf{v}$ in terms of the velocity in the rotating coordinate system $\tilde{\mathbf{v}}$:

$$\mathbf{v} = \mathbf{C}\tilde{\mathbf{v}} + \omega\mathbf{C}\mathbf{J}\tilde{\mathbf{p}}. \tag{5.13}$$

From the transformed velocity (5.12) it follows that

$$\tilde{\mathbf{v}} = \begin{pmatrix}\cos(\phi_v - \omega t)\\ \sin(\phi_v - \omega t)\end{pmatrix}V - \omega r\begin{pmatrix}-\sin\tilde{\phi}_p\\ \cos\tilde{\phi}_p\end{pmatrix}. \tag{5.14}$$

For later use, this is rewritten as

$$\tilde{\mathbf{v}} = \begin{pmatrix}\cos\tilde{\phi}_v\\ \sin\tilde{\phi}_v\end{pmatrix}\tilde{V}, \tag{5.15}$$

where $\tilde{V}$ the magnitude and $\tilde{\phi}_v$ the direction of the velocity.

5.1 The transformed mass flux and the transformed continuity equation

Now we know the separate terms, we can find the equations we need. The mass flux is given in transformed variables by

$$\Phi = \rho A\tilde{V}\frac{\cos(\tilde{\phi}_p - \tilde{\phi}_v)}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}, \tag{5.16}$$

and the transformed continuity equation is given by

$$\frac{\partial}{\partial t}\left(\frac{\rho A}{cos(\tilde{\phi}_p - \tilde{\phi}_s)}\right) + \frac{\partial\Phi}{\partial r} = 0. \tag{5.17}$$

5.2 The transformed kinematic equation

We know the time derivative of ϕ_p in terms of the transformed variables

$$\frac{\partial\phi_p}{\partial t} = \frac{\partial(\tilde{\phi}_p + \omega t)}{\partial t} = \frac{\partial\tilde{\phi}_p}{\partial t} + \omega. \tag{5.18}$$

We can rewrite the original equation (4.11):

$$\frac{\partial\phi_p}{\partial t} = \frac{\mathbf{m}\cdot\mathbf{v}}{\mathbf{m}\cdot\mathbf{J}\mathbf{p}} = \frac{(\mathbf{C}\tilde{\mathbf{v}})\cdot(\mathbf{C}\tilde{\mathbf{m}}) + \omega(\mathbf{C}\mathbf{J}\tilde{\mathbf{p}})\cdot(\mathbf{C}\tilde{\mathbf{m}})}{\mathbf{C}\tilde{\mathbf{m}}\cdot\mathbf{J}\mathbf{C}\tilde{\mathbf{p}}} = \frac{\sin(\tilde{\phi}_v - \tilde{\phi}_s)}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}\frac{\tilde{V}}{r} + \omega. \tag{5.19}$$

Combining those two (5.18-5.19) gives the transformed kinematic equation

$$\frac{\partial \tilde{\phi}_p}{\partial t} = \frac{\sin(\tilde{\phi}_v - \tilde{\phi}_s)}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)} \frac{\tilde{V}}{r}. \tag{5.20}$$

5.3 The transformed momentum equation

The last equation we have to transform is the momentum equation. After applying the product rule to the momentum equation (4.34) we find

$$\frac{\rho A}{\cos(\phi_s - \phi_p)} \frac{\partial \mathbf{v}}{\partial t} + \left(\frac{\partial}{\partial t} \frac{\rho A}{\cos(\phi_s - \phi_p)} + \frac{\partial \Phi}{\partial r} \right) \mathbf{v} + \Phi \frac{\partial \mathbf{v}}{\partial r} = \frac{\partial}{\partial r} \left(\eta A \cos(\phi_p - \phi_s) \left(\mathbf{s} \cdot \frac{\partial \mathbf{v}}{\partial r} \right) \mathbf{s} \right). \tag{5.21}$$

Multiplying the whole equation with $\cos(\phi_s - \phi_p)$, dividing by ρA and using the continuity equation (4.33) leads to

$$\frac{\partial \mathbf{v}}{\partial t} + V \cos(\phi_p - \phi_v) \frac{\partial \mathbf{v}}{\partial r} = \frac{\cos(\phi_p - \phi_s)}{\rho A} \frac{\partial}{\partial r} \left(\eta A \cos(\phi_p - \phi_s) \left(\mathbf{s} \cdot \frac{\partial \mathbf{v}}{\partial r} \right) \mathbf{s} \right). \tag{5.22}$$

We can transform the terms separately, where is used that $\mathbf{JJ} = \mathbf{I}$

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} &= \frac{\partial}{\partial t} (\mathbf{C}\tilde{\mathbf{v}} + \omega \mathbf{CJ}\tilde{\mathbf{p}}) = \mathbf{C} \left(\frac{\partial \tilde{\mathbf{v}}}{\partial t} + \omega \mathbf{J} \frac{\partial \tilde{\mathbf{p}}}{\partial t} \right) + \frac{\partial \mathbf{C}}{\partial t} (\tilde{\mathbf{v}} + \omega \mathbf{J}\tilde{\mathbf{p}}) = \\ &= \mathbf{C} \frac{\partial \tilde{\mathbf{v}}}{\partial t} + \omega \mathbf{CJJ}\tilde{\mathbf{p}} \frac{\partial \tilde{\phi}_{\mathbf{p}}}{\partial t} + \omega \mathbf{CJ} (\tilde{\mathbf{v}} + \omega \mathbf{J}\tilde{\mathbf{p}}) = \\ &= \mathbf{C} \left(\frac{\partial \tilde{\mathbf{v}}}{\partial t} - \omega \tilde{\mathbf{p}} \frac{\sin(\tilde{\phi}_v - \tilde{\phi}_s)}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)} \frac{\tilde{V}}{r} + \omega \mathbf{J}\tilde{\mathbf{v}} - \omega^2 \tilde{\mathbf{p}} \right). \end{aligned} \tag{5.23}$$

We can find the radial derivative in the same way.

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial r} &= \frac{\partial}{\partial r} (\mathbf{C}\tilde{\mathbf{v}} + \omega \mathbf{CJ}\tilde{\mathbf{p}}) = \mathbf{C} \left(\frac{\partial \tilde{\mathbf{v}}}{\partial r} + \omega \mathbf{J} \frac{\partial \tilde{\mathbf{p}}}{\partial r} \right) = \\ &= \mathbf{C} \left(\frac{\partial \tilde{\mathbf{v}}}{\partial r} + \frac{\omega}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)} \mathbf{J}\tilde{\mathbf{s}} \right). \end{aligned} \tag{5.24}$$

The advective term $V \cos(\phi_p - \phi_v)$ in transformed variables reads

$$\begin{aligned} V \cos(\phi_p - \phi_v) &= \frac{\mathbf{v} \cdot \mathbf{p}}{r} = (\mathbf{C} (\tilde{\mathbf{v}} + \omega \mathbf{J}\tilde{\mathbf{p}})) \cdot \left(\frac{\mathbf{C}\tilde{\mathbf{p}}}{r} \right) = \\ &= (\tilde{\mathbf{v}} + \omega \mathbf{J}\tilde{\mathbf{p}}) \cdot \left(\frac{\tilde{\mathbf{p}}}{r} \right) = \frac{\tilde{\mathbf{v}} \cdot \tilde{\mathbf{p}}}{r} = \tilde{V} \cos(\tilde{\phi}_v - \tilde{\phi}_p). \end{aligned} \tag{5.25}$$

The left-hand side of the momentum equation (5.22) is therefore given by

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \tilde{V} \cos(\phi_p - \phi_v) \frac{\partial \mathbf{v}}{\partial r} &= \mathbf{C} \left(\frac{\partial \tilde{\mathbf{v}}}{\partial t} + \omega \mathbf{J}\tilde{\mathbf{v}} - \omega^2 \tilde{\mathbf{p}} + \tilde{V} \cos(\tilde{\phi}_p - \tilde{\phi}_v) \frac{\partial \tilde{\mathbf{v}}}{\partial r} \right) + \\ &+ \frac{\tilde{V}\omega}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)} \mathbf{C} \left(\cos(\tilde{\phi}_v - \tilde{\phi}_p) \tilde{\mathbf{m}} - \sin(\tilde{\phi}_v - \tilde{\phi}_s) \frac{\tilde{\mathbf{p}}}{r} \right), \end{aligned} \tag{5.26}$$

where use is made of $\tilde{\mathbf{m}} = \mathbf{J}\tilde{\mathbf{s}}$. We can expand the last term in brackets

$$\cos(\tilde{\phi}_v - \tilde{\phi}_p)\tilde{\mathbf{m}} - \sin(\tilde{\phi}_v - \tilde{\phi}_s)\frac{\tilde{\mathbf{p}}}{r} = \begin{pmatrix} -\sin\tilde{\phi}_v \\ \cos\tilde{\phi}_v \end{pmatrix} \left(\sin\tilde{\phi}_p \sin\tilde{\phi}_s + \cos\tilde{\phi}_s \cos\tilde{\phi}_p\right) = \frac{\mathbf{J}\tilde{\mathbf{v}}}{\tilde{V}} \cos(\tilde{\phi}_p - \tilde{\phi}_s). \tag{5.27}$$

So

$$\frac{\tilde{V}\omega}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}\mathbf{C}\left(\cos(\tilde{\phi}_v - \tilde{\phi}_p)\tilde{\mathbf{m}} - \sin(\tilde{\phi}_v - \tilde{\phi}_s)\frac{\tilde{\mathbf{p}}}{r}\right) = \mathbf{C}\omega\mathbf{J}\tilde{\mathbf{v}}. \tag{5.28}$$

Therefore, the left-hand side of the momentum equation is given by

$$\frac{\partial \mathbf{v}}{\partial t} + V\cos(\phi_p - \phi_v)\frac{\partial \mathbf{v}}{\partial r} = \mathbf{C}\left(\frac{\partial \tilde{\mathbf{v}}}{\partial t} + 2\omega\mathbf{J}\tilde{\mathbf{v}} - \omega^2\tilde{\mathbf{p}} + \tilde{V}\cos(\tilde{\phi}_p - \tilde{\phi}_v)\frac{\partial \tilde{\mathbf{v}}}{\partial r}\right). \tag{5.29}$$

The last thing we have to do is to transform the right-hand side of the momentum equation. Because of equation (5.24) and the fact that $\tilde{\mathbf{m}} = \mathbf{J}\tilde{\mathbf{s}}$, it follow that:

$$\frac{\partial \mathbf{v}}{\partial r}\cdot\mathbf{s} = \left(\mathbf{C}\frac{\partial \tilde{\mathbf{v}}}{\partial r}\right)\cdot(\mathbf{C}\tilde{\mathbf{s}}) + \left(\frac{\omega}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}\mathbf{C}\tilde{\mathbf{m}}\right)\cdot(\mathbf{C}\tilde{\mathbf{s}}) = \frac{\partial \tilde{\mathbf{v}}}{\partial r}\cdot\tilde{\mathbf{s}} + \frac{\omega}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}\tilde{\mathbf{m}}\cdot\tilde{\mathbf{s}} = \frac{\partial \tilde{\mathbf{v}}}{\partial r}\cdot\tilde{\mathbf{s}}. \tag{5.30}$$

Then

$$\frac{\cos(\phi_p - \phi_s)}{\rho A}\frac{\partial}{\partial r}\left(\eta A\cos(\phi_p - \phi_s)\left(\mathbf{s}\cdot\frac{\partial \mathbf{v}}{\partial r}\right)\mathbf{s}\right) = \frac{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}{\rho A}\frac{\partial}{\partial r}\left(\eta A\cos(\tilde{\phi}_p - \tilde{\phi}_s)\left(\tilde{\mathbf{s}}\cdot\frac{\partial \tilde{\mathbf{v}}}{\partial r}\right)\mathbf{C}\tilde{\mathbf{s}}\right). \tag{5.31}$$

Therefore, the transformed momentum equation is given by

$$\frac{\partial}{\partial t}\left(\frac{\rho A\tilde{\mathbf{v}}}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}\right) + \frac{\partial}{\partial r}\left(\Phi\tilde{\mathbf{v}} - \eta A\cos(\tilde{\phi}_p - \tilde{\phi}_s)\left(\tilde{\mathbf{s}}\cdot\frac{\partial \tilde{\mathbf{v}}}{\partial r}\right)\tilde{\mathbf{s}}\right) = \frac{\rho A\omega(\omega\tilde{\mathbf{p}} - 2\mathbf{J}\tilde{\mathbf{v}})}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}. \tag{5.32}$$

In the right hand-side we can recognize the Coriolis force and the centrifugal force. These forces appear because the coordinate system rotates.

5.4 The instationary system in rotating r

The set of equations for the instationary case in a rotating coordinate system with coordinate r we have found is:

$$\tilde{\phi}_s = \arctan\left(\frac{\sin(\tilde{\phi}_p) + r\cos(\tilde{\phi}_p)\frac{\partial\tilde{\phi}_p}{\partial r}}{\cos(\tilde{\phi}_p) - r\sin(\tilde{\phi}_p)\frac{\partial\tilde{\phi}_p}{\partial r}}\right) \quad \mathit{or} \quad \frac{\partial\tilde{\phi}_p}{\partial r} = \frac{-\tan(\tilde{\phi}_p - \tilde{\phi}_s)}{r}, \tag{5.33}$$

kinematic equation:

$$\frac{\partial \tilde{\phi}_p}{\partial t} = \frac{\sin(\tilde{\phi}_v - \tilde{\phi}_s)}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)} \frac{\tilde{V}}{r}, \tag{5.34}$$

continuity equation:

$$\frac{\partial}{\partial t}\left(\frac{\rho A}{cos(\tilde{\phi}_p - \tilde{\phi}_s)}\right) + \frac{\partial \Phi}{\partial r} = 0, \tag{5.35}$$

momentum equation:

$$\frac{\partial}{\partial t}\left(\frac{\rho A \tilde{\mathbf{v}}}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}\right) + \frac{\partial}{\partial r}\left(\Phi \tilde{\mathbf{v}} - \eta A \cos(\tilde{\phi}_p - \tilde{\phi}_s)\left(\tilde{\mathbf{s}} \cdot \frac{\partial \tilde{\mathbf{v}}}{\partial r}\right)\tilde{\mathbf{s}}\right) = \tag{5.36}$$

$$= \frac{\rho A \omega(\omega \tilde{\mathbf{p}} - 2\mathbf{J}\tilde{\mathbf{v}})}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}.$$

With

$$\tilde{\mathbf{p}} = r\begin{pmatrix}\cos\tilde{\phi}_p \\ \sin\tilde{\phi}_p\end{pmatrix}, \qquad \tilde{\mathbf{v}} = \tilde{V}\begin{pmatrix}\cos\tilde{\phi}_v \\ \sin\tilde{\phi}_v\end{pmatrix}, \tag{5.37}$$

$$\tilde{\mathbf{s}} = \begin{pmatrix}\cos\tilde{\phi}_s \\ \sin\tilde{\phi}_s\end{pmatrix}, \qquad \tilde{\mathbf{m}} = \begin{pmatrix}-\sin\tilde{\phi}_s \\ \cos\tilde{\phi}_s\end{pmatrix}, \tag{5.38}$$

and not constant mass transport term Φ:

$$\Phi = \rho A \tilde{V} \frac{\cos(\tilde{\phi}_p - \tilde{\phi}_v)}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}. \tag{5.39}$$

Just as in Section 4.5 the momentum equations are both second order and all the other equations are of first order. Equation (5.33) is first order with respect to r, the kinematic equation is a first order derivative with respect to time and the continuity equation contains a first orer derivative with respect to time and one with respect to r. The momentem equation contains a first order time derivative and a second order derivative with respect to r. So we need six boundary conditions and four initial conditions to solve the system of four differential equations (5.33-5.36) with unknowns $\tilde{\phi}_v$, $\tilde{V}$, $\tilde{\phi}_s$, $\tilde{\phi}_p$ and A.

5.4.1 Boundary conditions

Again we assume that the spinning line is perpendicular to the rotor, when it leaves the orifice. Just as in the previous sections the magnitude of the velocity $\tilde{V}$ is 1 m/s, so

$$\tilde{V}(R_{rot}, t) = 1\ m/s. \tag{5.40}$$

Because the spinning line leaves the rotor horizontal, the angles $\tilde{\phi}_v$, $\tilde{\phi}_s$ and $\tilde{\phi}_p$ are all zero:

$$\tilde{\phi}_v(R_{rot}, t) = 0, \qquad \tilde{\phi}_s(R_{rot}, t) = 0, \qquad \tilde{\phi}_p(R_{rot}, t) = 0. \tag{5.41}$$

The Area of the spinning line A is perpendicular to the spinning line, the radius of the orifice is 125 μm so

$$A(R_{rot}, t) = \pi(125 * 10^{-6})^2 = \frac{\pi}{64 * 10^{-6}} \tag{5.42}$$

In the same way as in Section 4, the final boundary condition is not that obvious to find. You can try to find the magnitude of the final velocity at the coagulator v_e, then

$$V(R_{coag}, t) = v_e \tag{5.43}$$

Just as in Section 4, at the coagulator we can find $\tilde{\phi}_p - \tilde{\phi}_s = \frac{\pi}{2}$, so another boundary condition follows from equation (5.33):

$$\frac{\partial \tilde{\phi}_p}{\partial r}(R_{coag}, t) = \frac{-\tan\left(\frac{\pi}{2}\right)}{R_{coag}} \tag{5.44}$$

Again we obtain a problem because $\tan\left(\frac{\pi}{2}\right) = \infty$.

5.4.2 Initial conditions

The four initial conditions we need are easier to find. Assume that the velocity V and the cross-section area A are constant in the whole spinning line. Then the initial conditions are:

$$\begin{gathered} \tilde{\phi}_v(r,0) = 0, \qquad \tilde{V}(r,0) = 1\ m/s, \qquad \tilde{\phi}_p(r,0) = 0, \\ A(r,0) = \pi(125 * 10^{-6})^2 = \frac{\pi}{64 * 10^{-6}}. \end{gathered} \tag{5.45}$$

6 The stationary case with rotating r

In this section we will consider the instationary case in a rotating coordinate system with coordinate r, ignoring the time derivative. Then we get the stationary case. Section 6.1, is a detailed version of [4].

6.1 Deriving the stationary system from the instationary case

First we consider the kinematic equation (5.34). Ignoring the time derivative gives

$$\tilde{\phi}_v = \tilde{\phi}_s. \tag{6.1}$$

This means that the velocity vector is parallel to the spinning line. So

$$\tilde{\mathbf{v}} = \tilde{V}\tilde{\mathbf{s}}, \tag{6.2}$$

where $\tilde{V}$ the magnitude of the velocity. Substituting equation (6.1) in the mass transport term Φ (5.39) gives

$$\Phi = \rho A \tilde{V}. \tag{6.3}$$

Ignoring the time derivative in the continuity equation (5.35) says $\frac{\partial \Phi}{\partial r} = 0$, so the mass transport term is constant. Omitting the time derivative in the momentum equation (5.36) gives

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Phi\tilde{\mathbf{v}} - \eta A\cos(\tilde{\phi}_p - \tilde{\phi}_s)\left(\tilde{\mathbf{s}}\cdot\frac{\mathrm{d}\tilde{\mathbf{v}}}{\mathrm{d}r}\right)\tilde{\mathbf{s}}\right) = \frac{\rho A\omega(\omega\tilde{\mathbf{p}} - 2\mathbf{J}\tilde{\mathbf{v}})}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}. \tag{6.4}$$

Substituting $\tilde{\mathbf{v}} = \tilde{V}\tilde{\mathbf{s}}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Phi\tilde{V}\tilde{\mathbf{s}} - \eta A\cos(\tilde{\phi}_p - \tilde{\phi}_s)\left(\tilde{\mathbf{s}}\cdot\frac{\mathrm{d}\tilde{V}\tilde{\mathbf{s}}}{\mathrm{d}r}\right)\tilde{\mathbf{s}}\right) = \frac{\rho A\omega(\omega\tilde{\mathbf{p}} - 2\mathbf{J}\tilde{\mathbf{v}})}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}. \tag{6.5}$$

Rewriting this leads to

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Phi\tilde{V}\tilde{\mathbf{s}} - \eta A\cos(\tilde{\phi}_p - \tilde{\phi}_s)\frac{\mathrm{d}\tilde{V}}{\mathrm{d}r}\tilde{\mathbf{s}}\right) = \frac{\rho A\omega(\omega\tilde{\mathbf{p}} - 2\mathbf{J}\tilde{\mathbf{v}})}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}, \tag{6.6}$$

where we have used that $\tilde{\mathbf{s}}\cdot\tilde{\mathbf{s}} = 1$ and $\tilde{\mathbf{s}}\cdot\frac{\mathrm{d}\tilde{\mathbf{s}}}{\mathrm{d}r} = 0$. This last assumption can be made clear when we take

$$\tilde{\mathbf{s}} = \begin{pmatrix}\cos\tilde{\phi}_s\\ \sin\tilde{\phi}_s\end{pmatrix} = \begin{pmatrix}\tilde{\mathbf{s}}_1(r)\\ \tilde{\mathbf{s}}_2(r)\end{pmatrix}.$$

Then it follows that $\tilde{\mathbf{s}_1}^2 + \tilde{\mathbf{s}_2}^2 = 1$. Taking the derivative leads to

$$2\tilde{\mathbf{s}}_1\frac{\mathrm{d}\tilde{\mathbf{s}}_1}{\mathrm{d}r} + 2\tilde{\mathbf{s}}_2\frac{\mathrm{d}\tilde{\mathbf{s}}_2}{\mathrm{d}r} = 0.$$

Because

$$\tilde{\mathbf{s}}\cdot\frac{\mathrm{d}\tilde{\mathbf{s}}}{\mathrm{d}r} = \tilde{\mathbf{s}}_1\frac{\mathrm{d}\tilde{\mathbf{s}}_1}{\mathrm{d}r} + \tilde{\mathbf{s}}_2\frac{\mathrm{d}\tilde{\mathbf{s}}_2}{\mathrm{d}r}$$

it follows that $\tilde{\mathbf{s}} \cdot \frac{\mathrm{d}\tilde{\mathbf{s}}}{\mathrm{d}r} = 0$. The momentum equation, after using $A = \frac{\Phi}{\rho \tilde{V}}$, is now given by

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Phi\left(\tilde{V} - \frac{\eta\cos(\tilde{\phi}_p - \tilde{\phi}_s)}{\rho\tilde{V}}\frac{\mathrm{d}\tilde{V}}{\mathrm{d}r}\right)\tilde{\mathbf{s}}\right) = \frac{\omega\Phi}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}\left(\frac{\omega}{\tilde{V}}\tilde{\mathbf{p}} - 2\tilde{\mathbf{m}}\right). \tag{6.7}$$

6.2 The stationary system in rotating r

We assume that $\tilde{\phi}_v$ and A can be calculated when the system is solved, because of equation (6.1) and (6.3). Then the set of equations for the stationary case in a rotating coordinate system with coordinate r we have found is:

$$\tilde{\phi}_s = \arctan\left(\frac{\sin(\tilde{\phi}_p) + r\cos(\tilde{\phi}_p)\frac{\mathrm{d}\tilde{\phi}_p}{\mathrm{d}r}}{\cos(\tilde{\phi}_p) - r\sin(\tilde{\phi}_p)\frac{\mathrm{d}\tilde{\phi}_p}{\mathrm{d}r}}\right) \qquad \textit{or} \qquad \frac{\mathrm{d}\tilde{\phi}_p}{\mathrm{d}r} = \frac{-\tan(\tilde{\phi}_p - \tilde{\phi}_s)}{r}, \tag{6.8}$$

momentum equations:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Phi\left(\tilde{V} - \frac{\eta\cos(\tilde{\phi}_p - \tilde{\phi}_s)}{\rho\tilde{V}}\frac{\mathrm{d}\tilde{V}}{\mathrm{d}r}\right)\tilde{\mathbf{s}}\right) = \frac{\omega\Phi}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}\left(\frac{\omega}{\tilde{V}}\tilde{\mathbf{p}} - 2\tilde{\mathbf{m}}\right). \tag{6.9}$$

With

$$\tilde{\mathbf{p}} = r\begin{pmatrix}\cos\tilde{\phi}_p \\ \sin\tilde{\phi}_p\end{pmatrix}, \tag{6.10}$$

$$\tilde{\mathbf{s}} = \begin{pmatrix}\cos\tilde{\phi}_s \\ \sin\tilde{\phi}_s\end{pmatrix}, \qquad \tilde{\mathbf{m}} = \begin{pmatrix}-\sin\tilde{\phi}_s \\ \cos\tilde{\phi}_s\end{pmatrix}, \tag{6.11}$$

$$\Phi = \rho A\tilde{V}. \tag{6.12}$$

Here, the mass transport term Φ is constant.
We have three equations and three unknowns $\tilde{V}$, $\tilde{\phi}_s$ and $\tilde{\phi}_p$. The momentum equations are both second order, the other equation is of first order. Therefore we need five boundary conditions to solve the system.

6.2.1 Boundary conditions

Assume that the spinning line is perpendicular to the rotor, when it leaves the orifice. Again the magnitude of the velocity $\tilde{V}$ is 1 m/s, so

$$\tilde{V}(R_{rot}) = 1\ m/s. \tag{6.13}$$

From equation (6.1) we know that $\tilde{\phi}_s = \tilde{\phi}_v$ and the spinning line leaves the rotor horizontal:

$$\tilde{\phi}_p(R_{rot}) = 0, \qquad \tilde{\phi}_s(R_{rot}) = 0. \tag{6.14}$$

As seen before (Section 4), at the coagulator we can find $\tilde{\phi}_p - \tilde{\phi}_s = \frac{\pi}{2}$, so a boundary condition follows from equation (6.8):

$$\frac{\partial \tilde{\phi}_p}{\partial r}(R_{coag}) = \frac{-\tan\left(\frac{\pi}{2}\right)}{R_{coag}} \tag{6.15}$$

Again we obtain a problem because $\tan\left(\frac{\pi}{2}\right) = \infty$. You can try to find the magnitude of the final velocity at the coagulator v_e, then

$$\tilde{V}(R_{coag}) = v_e \tag{6.16}$$

6.3 Comparison of the two stationary cases in a rotating coordinate system

From Section 3, we know the system in the stationary case with rotating coordinate system in s (3.21-3.24). We can introduce $\tilde{\phi}_p$ and $\tilde{\phi}_s$ as shown in Figure 8.

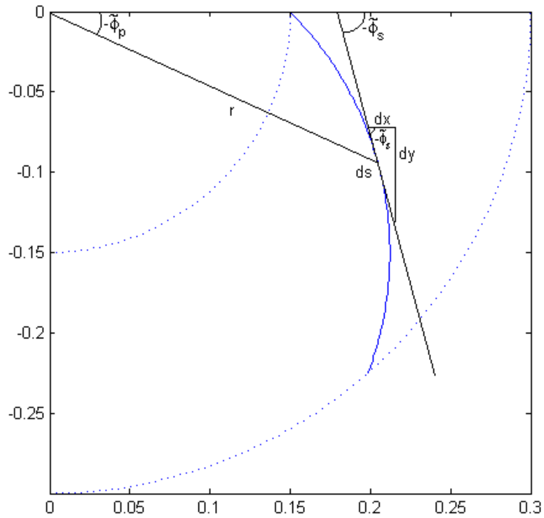

Figure 8: The angles $\tilde{\phi}_p$ and $\tilde{\phi}_s$.

Introducing polar coordinates gives:

$$x = r\cos(-\tilde{\phi}_p) = r\cos\tilde{\phi}_p, \tag{6.17}$$

$$y = -r\sin(-\tilde{\phi}_p) = r\sin\tilde{\phi}_p. \tag{6.18}$$

So with polar coordinates the derivatives with respect to s become:

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\mathrm{d}r}{\mathrm{d}s}\cos\tilde{\phi}_p - r\frac{\mathrm{d}\tilde{\phi}_p}{\mathrm{d}s}\sin\tilde{\phi}_p, \tag{6.19}$$

$$\frac{\mathrm{d}y}{\mathrm{d}s} = \frac{\mathrm{d}r}{\mathrm{d}s}\sin\tilde{\phi}_p + r\frac{\mathrm{d}\tilde{\phi}_p}{\mathrm{d}s}\cos\tilde{\phi}_p. \tag{6.20}$$

From Figure 8 you can find:

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos(-\tilde{\phi}_s) = \cos\tilde{\phi}_s, \tag{6.21}$$

$$\frac{\mathrm{d}y}{\mathrm{d}s} = -\sin(-\tilde{\phi}_s) = \sin\tilde{\phi}_s. \tag{6.22}$$

Combining these two sets of equations leads to

$$\frac{\mathrm{d}r}{\mathrm{d}s}\cos\tilde{\phi}_p - r\frac{\mathrm{d}\tilde{\phi}_p}{\mathrm{d}s}\sin\tilde{\phi}_p = \cos\tilde{\phi}_s, \tag{6.23}$$

$$\frac{\mathrm{d}r}{\mathrm{d}s}\sin\tilde{\phi}_p + r\frac{\mathrm{d}\tilde{\phi}_p}{\mathrm{d}s}\cos\tilde{\phi}_p = \sin\tilde{\phi}_s. \tag{6.24}$$

Multiplying equation (6.23) by $\sin\tilde{\phi}_s$ and add this to equation (6.24) multiplied by $-\cos\tilde{\phi}_s$ leads to

$$-r\frac{\mathrm{d}\tilde{\phi}_p}{\mathrm{d}s} = \frac{\mathrm{d}r}{\mathrm{d}s}\tan(\tilde{\phi}_p - \tilde{\phi}_s). \tag{6.25}$$

Multiplying by $\frac{\mathrm{d}s}{\mathrm{d}r}$ and using $\frac{\mathrm{d}\tilde{\phi}_p}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}r} = \frac{\mathrm{d}\tilde{\phi}_p}{\mathrm{d}r}$ gives

$$\frac{\mathrm{d}\tilde{\phi}_p}{\mathrm{d}r} = -\frac{\tan(\tilde{\phi}_p - \tilde{\phi}_s)}{r} \tag{6.26}$$

This equation is exactly the same as equation (6.8). We found this equation from the relationship between the derivatives (6.21-6.22) and polar coordinates. Therefore, equations (3.21-3.24) together have to result in the momentum equation (6.9) from Section 6.2.

Equation (3.24) is derived from Pythagoras (3.1), multiplying this equation by $\left(\frac{\mathrm{d}s}{\mathrm{d}r}\right)^2$ leads to

$$\left(\frac{\mathrm{d}s}{\mathrm{d}r}\right)^2 = \left(\frac{\mathrm{d}x}{\mathrm{d}r}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}r}\right)^2. \tag{6.27}$$

With polar coordinates we can find the derivate of x and y with respect to r:

$$\frac{\mathrm{d}x}{\mathrm{d}r} = \cos\tilde{\phi}_p - r\frac{\mathrm{d}\tilde{\phi}_p}{\mathrm{d}r}\sin\tilde{\phi}_p, \tag{6.28}$$

$$\frac{\mathrm{d}y}{\mathrm{d}r} = \sin\tilde{\phi}_p + r\frac{\mathrm{d}\tilde{\phi}_p}{\mathrm{d}r}\cos\tilde{\phi}_p. \tag{6.29}$$

Substituting this in equation (6.27) leads to

$$\frac{\mathrm{d}s}{\mathrm{d}r} = \frac{1}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)} \tag{6.30}$$

The momentum balances in the x and y direction (3.21-3.22) can be rewritten, we found equation (3.15) and (3.16):

$$-\Phi\frac{d}{ds}\left(v\frac{dx}{ds}\right) + \rho A\omega^2 x + 2\rho A\omega v\frac{dy}{ds} + \frac{d}{ds}\left(F\frac{dx}{ds}\right) = 0,$$

$$-\Phi\frac{\mathrm{d}}{\mathrm{d}s}\left(v\frac{\mathrm{d}y}{\mathrm{d}s}\right)+\rho A\omega^2 y-2\rho A\omega v\frac{\mathrm{d}x}{\mathrm{d}s}+\frac{\mathrm{d}}{\mathrm{d}s}\left(F\frac{\mathrm{d}y}{\mathrm{d}s}\right)=0.$$

Rewriting this, using $\Phi = \rho A v$ and assume the polymer to be Newtonian $F = \frac{\eta\Phi}{\rho v}\frac{\mathrm{d}v}{\mathrm{d}s}$ (3.23), leads to

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\Phi\left(\frac{\eta}{\rho v}\frac{\mathrm{d}v}{\mathrm{d}s}-v\right)\frac{\mathrm{d}x}{\mathrm{d}s}\right)=\omega\Phi\left(-\frac{\omega}{v}x-2\frac{\mathrm{d}y}{\mathrm{d}s}\right) \tag{6.31}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\Phi\left(\frac{\eta}{\rho v}\frac{\mathrm{d}v}{\mathrm{d}s}-v\right)\frac{\mathrm{d}y}{\mathrm{d}s}\right)=\omega\Phi\left(-\frac{\omega}{v}y+2\frac{\mathrm{d}x}{\mathrm{d}s}\right) \tag{6.32}$$

Those momentum balances in the x and y direction can be written in vector form in the same way as the momentum balance (6.9):

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\Phi\left(\frac{\eta}{\rho v}\frac{\mathrm{d}v}{\mathrm{d}s}-v\right)\begin{pmatrix}\frac{\mathrm{d}x}{\mathrm{d}s}\\ \frac{\mathrm{d}y}{\mathrm{d}s}\end{pmatrix}\right)=\omega\Phi\left(-\frac{\omega}{v}\begin{pmatrix}x\\ y\end{pmatrix}+2\begin{pmatrix}-\frac{\mathrm{d}y}{\mathrm{d}s}\\ \frac{\mathrm{d}x}{\mathrm{d}s}\end{pmatrix}\right). \tag{6.33}$$

Multiplying by -1 gives

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\Phi\left(v-\frac{\eta}{\rho v}\frac{\mathrm{d}v}{\mathrm{d}s}\right)\begin{pmatrix}\frac{\mathrm{d}x}{\mathrm{d}s}\\ \frac{\mathrm{d}y}{\mathrm{d}s}\end{pmatrix}\right)=\omega\Phi\left(\frac{\omega}{v}\begin{pmatrix}x\\ y\end{pmatrix}-2\begin{pmatrix}-\frac{\mathrm{d}y}{\mathrm{d}s}\\ \frac{\mathrm{d}x}{\mathrm{d}s}\end{pmatrix}\right). \tag{6.34}$$

We can rewrite equation (6.34) in terms of $\frac{\mathrm{d}}{\mathrm{d}r}$, $\tilde{\phi}_p$ and $\tilde{\phi}_s$ by using the fact $\frac{\mathrm{d}}{\mathrm{d}s}=\frac{\mathrm{d}}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}s}$ and substituting the polar coordinates (6.17-6.18), the derivatives of x and y with respect to s (6.21-6.22) and equation (6.30) leads to

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Phi\left(v-\frac{\eta}{\rho v}\cos(\tilde{\phi}_p-\tilde{\phi}_s)\frac{\mathrm{d}v}{\mathrm{d}r}\right)\begin{pmatrix}\cos\tilde{\phi}_s\\ \sin\tilde{\phi}_s\end{pmatrix}\right)=\frac{\omega\Phi}{\cos(\tilde{\phi}_p-\tilde{\phi}_s)}\left(\frac{\omega}{v}\begin{pmatrix}r\cos\tilde{\phi}_p\\ r\sin\tilde{\phi}_p\end{pmatrix}-2\begin{pmatrix}-\sin\tilde{\phi}_s\\ \cos\tilde{\phi}_s\end{pmatrix}\right). \tag{6.35}$$

Repeating equation (6.9) gives

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Phi\left(\tilde{V}-\frac{\eta}{\rho\tilde{V}}\cos(\tilde{\phi}_p-\tilde{\phi}_s)\frac{\mathrm{d}\tilde{V}}{\mathrm{d}r}\right)\tilde{\mathbf{s}}\right)=\frac{\omega\Phi}{\cos(\tilde{\phi}_p-\tilde{\phi}_s)}\left(\frac{\omega}{\tilde{V}}\tilde{\mathbf{p}}-2\tilde{\mathbf{m}}\right), \tag{6.36}$$

with

$$\tilde{\mathbf{p}}=r\begin{pmatrix}\cos\tilde{\phi}_p\\ \sin\tilde{\phi}_p\end{pmatrix},\qquad \tilde{\mathbf{s}}=\begin{pmatrix}\cos\tilde{\phi}_s\\ \sin\tilde{\phi}_s\end{pmatrix},\qquad \tilde{\mathbf{m}}=\begin{pmatrix}-\sin\tilde{\phi}_s\\ \cos\tilde{\phi}_s\end{pmatrix}.$$

and $\tilde{V}$ the magnitude of the velocity. Now you can see that both systems of equations are the same.

6.3.1 Viscous force

In equation (6.34) the dependent variables are x, y and v, in equation (6.36) $\tilde{\phi}_p$, $\tilde{\phi}_s$ and $\tilde{V}$ are dependent variables.

In both equations you can recognize the viscous force, the Coriolis force and the centrifugal force.
In equation (6.34), in the left-hand side you can recognize the term $\Phi v - F$. In the other equation the term on that place you can find is $\Phi\left(\tilde{V} - \frac{\eta\cos(\tilde{\phi}_p - \tilde{\phi}_s)}{\rho\tilde{V}}\frac{\mathrm{d}\tilde{V}}{\mathrm{d}r}\right)$. Using $\frac{\mathrm{d}\tilde{V}}{\mathrm{d}r} = \frac{\mathrm{d}\tilde{V}}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}r}$ and equation (6.30) it is easy to see that $\Phi v - F$ and $\Phi\left(\tilde{V} - \frac{\eta\cos(\tilde{\phi}_p - \tilde{\phi}_s)}{\rho\tilde{V}}\frac{\mathrm{d}\tilde{V}}{\mathrm{d}r}\right)$ describe the same thing.
Later on, in Section 7 the term $\Phi\left(\tilde{V} - \frac{\eta\cos(\tilde{\phi}_p - \tilde{\phi}_s)}{\rho\tilde{V}}\frac{\mathrm{d}\tilde{V}}{\mathrm{d}r}\right)\tilde{\mathbf{s}}$ from equation (6.36) is called $\mathbf{k}$. The magnitude of this vector is called k, so $\mathbf{k} = k\tilde{\mathbf{s}}$. It follows that $F - \Phi v = -k$.

7 Reformulation of the stationary case with rotating r

It is obvious to start with a stationary model for continuation research. In this section we choose the stationary case in a rotating coordinate system, with coordinate r from Section 6. Here we rewrite the system of equations in a more convenient form, and make it dimensionless, just as done in [4]. There are introduced symbols $\mathbf{k}$ and k to denote certain terms. The scalar k will perform an important role when the system is solved numerically.

The set of equations we have found in Section 6 is

$$\tilde{\phi}_s = \arctan\left(\frac{\sin(\tilde{\phi}_p) + r\cos(\tilde{\phi}_p)\frac{\partial\tilde{\phi}_p}{\partial r}}{\cos(\tilde{\phi}_p) - r\sin(\tilde{\phi}_p)\frac{\partial\tilde{\phi}_p}{\partial r}}\right) \qquad \textit{or} \qquad \frac{\partial\tilde{\phi}_p}{\partial r} = \frac{-\tan(\tilde{\phi}_p - \tilde{\phi}_s)}{r}, \tag{7.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\Phi\left(\tilde{V} - \frac{\eta\cos(\tilde{\phi}_p - \tilde{\phi}_s)}{\rho\tilde{V}}\frac{\mathrm{d}\tilde{V}}{\mathrm{d}r}\right)\tilde{\mathbf{s}}\right) = \frac{\omega\Phi}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}\left(\frac{\omega}{\tilde{V}}\tilde{\mathbf{p}} - 2\tilde{\mathbf{m}}\right), \tag{7.2}$$

with constant mass transport

$$\Phi = \rho A\tilde{V}. \tag{7.3}$$

The momentum equation can be written as a system of first order equations, as follows:

$$\mathbf{k} \equiv \Phi\tilde{V}\tilde{\mathbf{s}} - \frac{\eta\Phi\cos(\tilde{\phi}_p - \tilde{\phi}_s)}{\rho\tilde{V}}\frac{\mathrm{d}\tilde{V}}{\mathrm{d}r}\tilde{\mathbf{s}}, \tag{7.4}$$

$$\frac{\mathrm{d}\mathbf{k}}{\mathrm{d}r} = \frac{\omega^2\Phi}{\tilde{V}\cos(\tilde{\phi}_p - \tilde{\phi}_s)}\tilde{\mathbf{p}} - \frac{2\omega\Phi}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}\tilde{\mathbf{m}}. \tag{7.5}$$

Here $\mathbf{k}$ is a vector equal to the integral of the right hand side of equation (7.2), it is also equal to the momentum transport in the transformed equations. It follows from equation (7.4) that $\mathbf{k}$ and $\tilde{\mathbf{s}}$ are parallel. Therefore $\mathbf{k} = k\tilde{\mathbf{s}}$.

The derivative of the scalar k is found using the product rule and combining the result with equation (7.5), then

$$\frac{\mathrm{d}\mathbf{k}}{\mathrm{d}r} = \tilde{\mathbf{s}}\frac{\mathrm{d}k}{\mathrm{d}r} + k\frac{\mathrm{d}\tilde{\mathbf{s}}}{\mathrm{d}r} = \frac{\omega^2\Phi}{\tilde{V}\cos(\tilde{\phi}_p - \tilde{\phi}_s)}\tilde{\mathbf{p}} - \frac{2\omega\Phi}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}\tilde{\mathbf{m}}. \tag{7.6}$$

Because $\tilde{\mathbf{s}}\cdot\tilde{\mathbf{s}} = 1$, $\tilde{\mathbf{m}}\cdot\tilde{\mathbf{s}} = 0$, $\tilde{\mathbf{p}}\cdot\tilde{\mathbf{s}} = r\cos(\tilde{\phi}_p - \tilde{\phi}_s)$ and $\tilde{\mathbf{s}}\cdot\frac{\mathrm{d}\tilde{\mathbf{s}}}{\mathrm{d}r} = 0$ we can derive an expression for $\frac{\mathrm{d}k}{\mathrm{d}r}$:

$$\frac{\mathrm{d}k}{\mathrm{d}r} = \frac{\omega^2\Phi}{\tilde{V}\cos(\tilde{\phi}_p - \tilde{\phi}_s)}\tilde{\mathbf{p}}\cdot\tilde{\mathbf{s}} = \frac{\omega^2\Phi r}{\tilde{V}}. \tag{7.7}$$

In the same way we can derive an expression for $\frac{\mathrm{d}\tilde{\mathbf{s}}}{\mathrm{d}r}$, by taking the scalar product of equation (7.6) and the vector $\tilde{\mathbf{m}}$. Because $\tilde{\mathbf{m}}\cdot\tilde{\mathbf{m}} = 1$ and $\tilde{\mathbf{m}}\cdot\tilde{\mathbf{s}} = 0$ it follows that

$$k\frac{\mathrm{d}\tilde{\mathbf{s}}}{\mathrm{d}r}\cdot\tilde{\mathbf{m}} = \frac{\omega^2\Phi}{\tilde{V}\cos(\tilde{\phi}_p - \tilde{\phi}_s)}\tilde{\mathbf{p}}\cdot\tilde{\mathbf{m}} - \frac{2\omega\Phi}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}. \tag{7.8}$$

Next we want to know the derivatives of the angles $\tilde{\phi}_p$ and $\tilde{\phi}_s$. Writing out the scalar product in equation (7.8) and dividing by k leads to the derivative of $\tilde{\phi}_s$:

$$\begin{aligned}\frac{\mathrm{d}\tilde{\phi}_s}{\mathrm{d}r} = \frac{\mathrm{d}\tilde{\mathbf{s}}}{\mathrm{d}r} \cdot \tilde{\mathbf{m}} &= \frac{\omega^2 \Phi}{k\tilde{V}\cos(\tilde{\phi}_p - \tilde{\phi}_s)} \sin(\tilde{\phi}_p - \tilde{\phi}_s) - \frac{2\omega\Phi}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)} \\ &= \frac{\omega^2\Phi}{k\tilde{V}} \tan(\tilde{\phi}_p - \tilde{\phi}_s) - \frac{2\omega\Phi}{\cos(\tilde{\phi}_p - \tilde{\phi}_s)}.\end{aligned} \tag{7.9}$$

The derivative of $\tilde{\phi}_p$ is known from equation (7.1) which simply states that the curve $\tilde{\mathbf{p}}$ points in the direction $\tilde{\mathbf{s}}$,

$$\frac{\mathrm{d}\tilde{\phi}_p}{\mathrm{d}r} = \frac{1}{r}\begin{pmatrix} -\sin\phi_p \\ \cos\phi_p \end{pmatrix} \cdot \frac{\mathrm{d}\tilde{\mathbf{p}}}{\mathrm{d}r} = -\frac{1}{r}\tan(\tilde{\phi}_p - \tilde{\phi}_s). \tag{7.10}$$

Define $\beta = \tilde{\phi}_p - \tilde{\phi}_s$, then the system of equations written in terms of the angle difference β, velocity $\tilde{V}$ and momentum transport k becomes:

$$\frac{\mathrm{d}\beta}{\mathrm{d}r} = \frac{2\omega\Phi}{k\cos\beta} - \left(\frac{1}{r} + \frac{\omega^2 r\Phi}{k\tilde{V}}\right)\tan\beta, \tag{7.11}$$

$$\frac{\mathrm{d}k}{\mathrm{d}r} = \frac{\omega^2 r\Phi}{\tilde{V}}, \tag{7.12}$$

$$\frac{\mathrm{d}\tilde{V}}{\mathrm{d}r} = \frac{\rho\tilde{V}}{\eta\cos\beta}\left(\tilde{V} - \frac{k}{\Phi}\right). \tag{7.13}$$

After the system has been solved, the spinning line position can be calculated from equation (7.10).

7.1 Reducing the system

The system may be further reduced by considering $kr\sin(\beta)$. Taking the derivative of this function with respect to r and substitution of equation (7.11) and (7.12) leads to a simple equation for β:

$$\frac{\mathrm{d}}{\mathrm{d}r}(kr\sin\beta) = k\sin\beta + r\frac{\mathrm{d}k}{\mathrm{d}r}\sin\beta + kr\frac{\mathrm{d}\beta}{\mathrm{d}r}\cos\beta = 2\omega\Phi r. \tag{7.14}$$

Integrating this equation leads to

$$kr\sin\beta = \omega\Phi(r^2 - C_\beta), \tag{7.15}$$

where C_β the integration constant.

This equation can also be found from the balance perpendicular to the tangent (3.31) from Section 3.4:

$$\left((F - \Phi v)\left(y\frac{\mathrm{d}x}{\mathrm{d}s} - x\frac{\mathrm{d}y}{\mathrm{d}s}\right)\right) = -\Phi\omega\left(y^2 + x^2\right) + C_1,$$

and polar coordinates introduced in Section 6.3:

$$y\frac{\mathrm{d}x}{\mathrm{d}s} - x\frac{\mathrm{d}y}{\mathrm{d}s} = r\sin(\tilde{\phi_p} - \tilde{\phi_s}) = r\sin\beta.$$

With $F - \Phi v = -k$ follows

$$kr\sin\beta = \omega\Phi(r^2 - C_1) = \omega\Phi(r^2 - C_\beta). \tag{7.16}$$

When the momentum transport k is negative near the rotor and positive near the coagulator there is a radius r at which $k = 0$, this point is called $r_{k=0}$. This point is introduced because you know at this point an extra condition: equation (7.19). However, then the position $r_{k=0}$ is unknown. Then, using the shooting method, the model can be solved more easily.
At $r_{k=0}$ the left-hand-side of equation (7.15) is zero

$$0 = \omega\Phi(r_{k=0}^2 - C_\beta), \tag{7.17}$$

so $C_\beta = r_{k=0}^2$. Then the angle difference β is given by

$$\sin\beta = \frac{\omega\Phi}{kr}\left(r^2 - r_{k=0}^2\right) \tag{7.18}$$

At the point $r_{k=0}$ the numerator and the denominator are both zero in this equation. Taking the limit gives with L'Hospital's rule:

$$\sin\beta = \frac{\omega\Phi 2r_{k=0}}{r_{k=0}\frac{\mathrm{d}k}{\mathrm{d}r}|_{k=0}} = \frac{2\tilde{V}_{k=0}}{r_{k=0}\omega}. \tag{7.19}$$

At the point $r_{k=0}$ we know two things, $k = 0$ and $\sin\beta = \frac{2\tilde{V}_{k=0}}{r_{k=0}\omega}$. However, we do not know the value of r corresponding with the point where $k = 0$.

Equation (7.19) can also be derived from the balance perpendicular to the tangent, assuming $F - \Phi v = 0$ (3.29):

$$x\frac{\mathrm{d}y}{\mathrm{d}s} - y\frac{\mathrm{d}x}{\mathrm{d}s} = -\frac{2v}{\omega}.$$

and polar coordinates:

$$x\frac{\mathrm{d}y}{\mathrm{d}s} - y\frac{\mathrm{d}x}{\mathrm{d}s} = -r\sin(\tilde{\phi_p} - \tilde{\phi_s}) = -r\sin\beta.$$

Then with $v = \tilde{V}$:

$$\sin\beta = \frac{2\tilde{V}_{k=0}}{\omega r_{k=0}}. \tag{7.20}$$

Now, the equations for $\tilde{V}$ and k are given by

$$\frac{\mathrm{d}k}{\mathrm{d}r} = \frac{\omega^2 r\Phi}{\tilde{V}}, \tag{7.21}$$

$$\frac{\mathrm{d}\tilde{V}}{\mathrm{d}r} = \frac{rk\rho\tilde{V}}{\eta\sqrt{r^2k^2 - \omega^2\Phi^2\left(r^2 - r_{k=0}^2\right)^2}}\left(\tilde{V} - \frac{k}{\Phi}\right). \tag{7.22}$$

After solving this system, the spinning line shape can be calculated from the angle $\tilde{\phi}_p$. This angle is found by solving

$$\sin\beta = \frac{\omega\Phi\left(r^2 - r_{k=0}^2\right)}{rk}, \tag{7.23}$$

$$\frac{\mathrm{d}\tilde{\phi}_p}{\mathrm{d}r} = -\frac{1}{r}\tan\beta, \tag{7.24}$$

where $\beta = \tilde{\phi}_p - \tilde{\phi}_s$.

The spinning line position angle $\tilde{\phi}_{rot}$ at the rotor is needed to calculate the spinning line shape, using equation (7.23-7.24). Substituting $r = L\hat{r}$, $k = K\hat{k}$ and $\tilde{V} = U\hat{\tilde{V}}$ gives us the scaled equations:

$$\frac{\hat{\tilde{V}}}{\hat{r}}\frac{\mathrm{d}\hat{k}}{\mathrm{d}\hat{r}} = \frac{\omega^2L^2\Phi}{UK}, \tag{7.25}$$

$$\frac{1}{\hat{r}\hat{k}\hat{\tilde{V}}}\frac{\mathrm{d}\hat{\tilde{V}}}{\mathrm{d}\hat{r}} = \frac{L^2K\rho}{\eta\sqrt{L^2\hat{r}^2K^2\hat{k}^2 - \omega^2\Phi^2\left(L^2\hat{r}^2 - r_{k=0}^2\right)^2}}\left(U\hat{\tilde{V}} - \frac{K\hat{k}}{\Phi}\right). \tag{7.26}$$

Using the scaling

$$\hat{K} = \Phi R_{rot}\omega, \qquad \hat{U} = R_{rot}\omega, \qquad \hat{L} = R_{rot}, \qquad \hat{r}_{k=0} = \frac{r_{k=0}}{R_{rot}}, \qquad \eta = \mu\rho R_{rot}^2\omega.$$

gives us the dimensionless set of equations.

7.2 Dimensionless stationary case with rotating r

The dimensionless system becomes:

$$\frac{\mathrm{d}\hat{k}}{\mathrm{d}\hat{r}} = \frac{\hat{r}}{\hat{\tilde{V}}}, \tag{7.27}$$

$$\frac{\mathrm{d}\hat{\tilde{V}}}{\mathrm{d}\hat{r}} = \frac{1}{\mu}\frac{\hat{r}\hat{k}\hat{\tilde{V}}\left(\hat{\tilde{V}} - \hat{k}\right)}{\sqrt{\hat{r}^2\hat{k}^2 - \left(\hat{r}^2 - \hat{r}_{k=0}^2\right)^2}}. \tag{7.28}$$

The shape of the spinning line may be calculated from

$$\sin\beta = \frac{\hat{r}^2 - \hat{r}_{k=0}^2}{\hat{r}\hat{k}}, \tag{7.29}$$

$$\frac{\mathrm{d}\tilde{\phi}_p}{\mathrm{d}\hat{r}} = -\frac{1}{\hat{r}}\tan\beta. \tag{7.30}$$

Notice that there is a singularity when $\hat{r} = \hat{r}_{k=0}$ in equation (7.28). So this equation satisfies only for $1 < \hat{r} < \hat{r}_{k=0}$ and $\hat{r}_{k=0} < \hat{r} < \hat{R}_{coag}$. Because of continuity this is not a problem. But in this point, $r_{k=0}$, there are, next to continuity, two conditions to satisfy: $k = 0$ and equation (7.19).
We have four equations (7.27-7.30) and four unknown variables $\hat{\tilde{V}}$, $\hat{k}$, $\tilde{\phi}_p$ and β. From these equations, only equation (7.29) is not a differential equation. $\hat{k}$, $\hat{\tilde{V}}$ and $\tilde{\phi}_p$ occur all in first order term. So we need three boundary conditions. And we need also the value $r_{k=0}$ to solve the system.

If there is no point $\hat{r}_{k=0}$ such that $\hat{R}_{rot} \leq \hat{r}_{k=0} \leq \hat{R}_{coag}$ you need four boundary conditions.

7.2.1 Boundary conditions

Boundary conditions are needed for $\hat{k}$, $\hat{\tilde{V}}$ and $\tilde{\phi}_p$ and to determine $\hat{r}_{k=0}$. When we assume that the spinning line leaves the orifice perpendicular to the rotor, we know from Section 6 that

$$\tilde{\phi}_p(R_{rot}) = 0, \qquad \tilde{\phi}_s(R_{rot}) = 0, \qquad \tilde{V}(R_{rot}) = 1\ m/s. \tag{7.31}$$

From this section we know that $\hat{R}_{rot} = 1$, then

$$\tilde{\phi}_p(1) = 0, \qquad \tilde{\phi}_s(1) = 0. \tag{7.32}$$

$\tilde{V}$ is scaled by $\hat{U}\hat{\tilde{V}}$ with $\hat{U} = R_{rot}\omega$, so

$$\hat{\tilde{V}}(1) = \frac{\tilde{V}(1)}{R_{rot}\omega} = \frac{1}{0.3 * 262} = \frac{1}{78.6} \tag{7.33}$$

The boundary conditions we have found are:

$$\tilde{\phi}_p(1) = 0, \qquad \beta(1) = \tilde{\phi}_p(1) - \tilde{\phi}_s(1) = 0, \qquad \hat{\tilde{V}}(1) = \frac{1}{78.6}. \tag{7.34}$$

Now we know a boundary condition for $\tilde{\phi}_p$, β and $\hat{\tilde{V}}$, but still we do not know a condition for $\hat{k}$ The dimensionless viscosity coefficient μ has value $\mu = 0.1197$.

8 Numerical methods for differential equations

8.1 Numerical methods for initial value problems

The problem in the s coordinate from Section 3 is only dependent of s. In that case single-step methods can be used to solve the problem numerically, if the problem is an initial value problem. The problem from Section 3 is a problem with both initial and boundary conditions. The most practical single-step methods are higher order Runge-Kutta methods. A well known method is the Euler method, although this method is seldom used in practice. The Euler forward method is a first order Runge-Kutta process. The fourth order Runge-Kutta process is often used, however lower or higher order processes are also possible.
Almost every numerical method is designed for first order differential equations. So a higher order differential equation has to be rewritten to a system of first order differential equations.
In this section we will use equidistant stepsize h for simplicity. All the calculations can also be made with non-equidistant stepsize h_n.

8.1.1 The Runge-Kutta formula

Generally, the explicit Runge-Kutta formula can be written as

$$u_{j+1} = u_j + h \sum_{i=1}^{s} b_i k_i, \tag{8.1}$$

where

$$\begin{aligned} k_1 &= f(x_j, u_j), \\ k_i &= f\left(x_j + c_i h, u_j + h \sum_{n=1}^{i-1} a_{in} k_n\right), \qquad i = 2, 3, ..., s. \end{aligned} \tag{8.2}$$

Here b_i, c_i and a_{in} are the Runge-Kutta parameters, and s is the number of stages. This Runge-Kutta formula is called explicit because the k_i depend only on previous k_n, $n = 1, 2, ...i - 1$.

8.1.2 Euler's method

Consider an initial value problem:

$$y' = f(t, y), \qquad y(0) = y_0. \tag{8.3}$$

The time is divided in intervals h, $t_j = jh$, $j = 0, 1, ..., n$. In this section the exact solution will be denoted by $y_j = y(t_j)$ and the numerical approximation of this exact solution will be denoted by u_j. For this problem, Euler's method, a first order Runge-Kutta method, is defined by

$$\frac{u_{j+1} - u_j}{h} = f(t_j, u_j), \qquad u_0 = y_0. \tag{8.4}$$

Rewriting this leads to

$$u_{j+1} = u_j + hf(t_j, u_j). \tag{8.5}$$

Every single step method can be written in the form

$$u_{j+1} = u_j + \Phi(t_j, t_{j+1}, u_j, uj + 1, h) \tag{8.6}$$

If Φ does not depend on u_{j+1}, the method is called explicit, otherwise you have to deal with an implicit method. Euler's method is an explicit method. In this case, Euler Forward is used. Another possibility is Euler Backward. In that case equation (8.4) can be replaced by

$$\frac{u_j - u_{j-1}}{h} = f(t_j, u_j). \tag{8.7}$$

8.1.3 Runge-Kutta order four

A common used numerical method is the Runge-Kutta order four. Consider again the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0. \tag{8.8}$$

This problem can be solved by using the fourth order Runge-Kutta method. Define step size h, then $t_j = jh$ and take u_j the numerical approximation of the exact solution $y(t_j)$, $j = 0, 1, ..., n$. Then a Runge-Kutta order four method is given by

$$\begin{aligned}
k_1 &= hf(t_j, u_j) \qquad (8.9)\\
k_2 &= hf(t_{j+\frac{1}{2}}, u_j + \frac{1}{2}k_1)\\
k_3 &= hf(t_{j+\frac{1}{2}}, u_j + \frac{1}{2}k_2)\\
k_4 &= hf(t_{j+1}, u_j + k_3)\\
u_{j+1} &= u_j + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)
\end{aligned}$$

8.2 Error definitions

8.2.1 Local truncation error

Definition 8.1 *The local truncation error of a single-step method is given by*

$$e_{j+1} = \frac{y_{j+1} - y_j}{h} - \Phi(y_{j+1}, y_j, t_{j+1}, t_j, h). \tag{8.10}$$

Assuming that the method was exact at the previous step size, the local truncation error e_j represents the accuracy of the method at step j. The local truncation error depends on the step size, the particular step in the approximation and the differential equation.

Definition 8.2 *A single-step method with local truncation error e_j at step j is called consistent with the differential equation it approximates if*

$$\lim_{h\to 0}\left(\max_{0\le j\le n} |e_j|\right) = 0 \tag{8.11}$$

8.2.2 Stability

An initial value problem is said to be stable if a small change in the initial conditions does not result in big changes in the final solution. Assume an initial condition containing an error ϵ_0. The disturbed solution $\tilde{y}$ satisfies

$$\tilde{y}' = f(t, y'), \qquad \tilde{y}(0) = y_0 + \epsilon_0. \tag{8.12}$$

Then define $\epsilon(t) = \tilde{y}(t) - y(t)$.

Definition 8.3 *The initial value problem is called stable if*

$$\lim_{t\to\infty} |\epsilon(t)| < \infty. \tag{8.13}$$

Definition 8.4 *The initial value problem is absolutely stable if*

$$\lim_{t\to\infty} |\epsilon(t)| = 0. \tag{8.14}$$

8.2.3 Global error

Definition 8.5 *The global error of the numerical solution is defined by*

$$E_j = y_j - u_j. \tag{8.15}$$

So the global error represents the difference between the overall true solution and the overall numerical approximated solution.

Definition 8.6 *A numerical process is called convergent if*

$$\lim_{h\to 0}\left(\max_{0\le j\le n} |E_j|\right) = 0 \tag{8.16}$$

From a practical point of view, convergence implies that if the step size reduces, the global error will be reduced too. If a numerical scheme is stable and consistent, then the numerical solution converges to the exact solution. In that case, the global error is of the same order as the local truncation error. (See [7] for a proof)

8.3 The local truncation error for Euler's method and Rung-Kutta order four

Assume again the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0. \tag{8.17}$$

Then the local truncation error for Euler's method is given by

$$e_{j+1} = \frac{y_{j+1} - y_j}{h} - f(y_j, t_j). \tag{8.18}$$

Expanding y_{j+1} into Taylor series and substituting this in (8.18) gives $e_{j+1} = O(h)$. So the local truncation error for Euler's method for this initial value problem is of order one.
The Runge-Kutta order four method from Section 8.1.3, has local truncation error $O(h^4)$ and the arithmetic cost is four evaluations per step. In the second order Runge-Kutta methods the local truncation errors are $O(h^2)$, and the cost is two functional evaluations per step. When you take a higher stage Runge-Kutta method, the order is no longer of the same size as the stage. For example, a ninth stage Runge-Kutta method is of order seven. A fourth order Runge-Kutta method with step size h gives more accuracy than a second order Runge-Kutta method with step size $\frac{1}{2}h$, because the fourth order method requires twice as many evaluations per step.

8.4 Numerical methods for boundary value problems

8.4.1 Finite difference method

Consider the boundary-value problem

$$-y'' + p(x)y' + q(x)y = f(x), \quad 0 < x < 1, \quad y(0) = 0, \quad y(1) = 0. \tag{8.19}$$

You can use difference formula to find a system of equations, which you can solve. Therefore you have to divide the interval $[0, 1]$ into n pieces of size h. The derivatives y' and y'' can be approximated by the centered-difference formula, so the differential equation can be approximated by:

$$-\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + p_j \frac{u_{j+1} - u_{j-1}}{2h} + q_j u_j = f_j, \qquad 1 \leq j \leq n-1, \quad u_0 = 0, \quad u_n = 0 \tag{8.20}$$

To find the approximated solution of the boundary-value problem (8.19) you have to solve the system

$$A\mathbf{u} = \mathbf{f}. \tag{8.21}$$

Here A is a matrix of the form:

$$A = \begin{pmatrix} \frac{2}{h^2} + q_1 & -\frac{1}{h^2} + \frac{p_1}{2h} & 0 & \cdots & \cdots & \cdots \\ -\frac{1}{h^2} - \frac{p_2}{2h} & \frac{2}{h^2} + q_2 & -\frac{1}{h^2} + \frac{p_2}{2h} & 0 & \cdots & \cdots \\ 0 & \ddots & \ddots & \ddots & & \end{pmatrix}, \tag{8.22}$$

and $\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_{n-1} \end{pmatrix}$ and $\mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_{n-1} \end{pmatrix}$.

We have seen a problem with Dirichlet boundary conditions. When you have to deal with Neumann boundary conditions, you need to introduce a virtual point, because you need u_{-1} or u_{n+1} Not always centered-difference formulas give acceptable answers. This varies for different problems. Another possibility is to use upwind differences.

8.4.2 Definitions

In this subsection some terms are introduced which we need to analyse a finite difference scheme.

Definition 8.7 *The condition number K of matrix A is defined by*

$$K(A) = \|A\|\|A^{-1}\| \tag{8.23}$$

A matrix is well-conditioned if $K(A)$ is close to 1. If $K(A) >> 1$, the matrix is ill-conditioned.

Definition 8.8 *The truncation error of the finite difference scheme $A\mathbf{u} = \mathbf{f}$ is given by*

$$e_j = (A\mathbf{y} - \mathbf{f})_j, \qquad j = 1, ..., n-1, \tag{8.24}$$

with y_j the exact solution.

Definition 8.9 *The scheme is called consistent if*

$$\lim_{h\to 0} \|e_j\| = 0. \tag{8.25}$$

with in limit j such that jh constant. Another notation sais a numerical method is consistent if

$$\Phi(x, y, 0) = f(x, y). \tag{8.26}$$

When a finite difference scheme is stable, the system has an unambiguous solution.

Definition 8.10 *A finite difference scheme is said to be stable if there exists a constant M, independent of step size h, such that*

$$\|A^{-1}\| \leq M, \qquad h \to 0. \tag{8.27}$$

Definition 8.11 *A finite difference scheme is convergent if the global error $\mathbf{y}_j - \mathbf{u}_j$ satisfies*

$$\lim_{h\to 0} \|\mathbf{y}_j - \mathbf{u}_j\| = 0. \tag{8.28}$$

8.5 Non-linear systems

For a non-linear initial value problem, you can use Euler's method. Because the system of equations is not linear, an iterative process, like Newton's method or quasi-Newton, is required to solve it.
A non-linear boundary value problem can be discretized with finite difference methods, after that you need an iterative process to solve the discretized system. For instance you can use Picard iteration or Newton's method.

Example 8.1 *Assume the non-linear boundary value problem*

$$y'' = f(x, y, y'), \qquad y(a) = \alpha, \qquad y(b) = \beta, \qquad a \leq x \leq b. \tag{8.29}$$

Discretizing with finite difference and rewriting gives:

$$\begin{gathered} -u_{i+1} + 2u_i - w_{i-1} + h^2 f\left(x_i, u_i, \frac{u_{i+1} - u_{i-1}}{2h}\right) = 0, \\ u_0 = \alpha, \quad u_n + 1 = \beta, \quad i = 0, ..., n+1, \end{gathered} \tag{8.30}$$

where $h = \frac{b-a}{n+1}$.
Then you can solve this system, which has the form $\mathbf{F}(\mathbf{x}) = \mathbf{0}$*, for example by using Newton's method.*

Definition 8.12 *Newton's method is defined by:*

$$\mathbf{x}^{(p)} = \mathbf{x}^{(p-1)} - J\left(\mathbf{x}^{(p-1)}\right)^{-1} \mathbf{F}(\mathbf{x}^{(p-1)}), \tag{8.31}$$

where $J(\mathbf{x})$ *the Jacobian of* $\mathbf{F}$*, so*

$$J(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}\mathbf{x} & \dots & \frac{\partial f_1}{\partial x_n}\mathbf{x} \\ & \vdots & \\ \frac{\partial f_n}{\partial x_n}\mathbf{x} & \dots & \frac{\partial f_1}{\partial x_n}\mathbf{x} \end{pmatrix}.$$

9 Perturbation Theory

In the eighteenth century perturbation theory came up. It has it roots in physics. In that time, the most important application of perturbation theory was in celestial mechanics. The equations describing the motions of celestial bodies consist of a part containing the mutual attraction of the earth and the sun and a part with small perturbation terms. Those new equations were difficult to study, but started off the development of perturbation theory. Poincaré was the first to discuss this subject systematically.

9.1 Regular perturbation method

In the mathematical model, the small perturbation term is due to a small parameter ϵ. First we will consider a simple example, a harmonic oscillation. In the first case, the effect of friction has been neglected. After that, we will introduce a friction term which will be small. The equation without friction reads:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + x = 0. \tag{9.1}$$

Take into account the effect of friction gives

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\epsilon\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0. \tag{9.2}$$

$\epsilon\frac{\mathrm{d}x}{\mathrm{d}t}$ is called the 'friction term' or 'damping term' and this particular simple form of the friction term has been based on certain assumptions concerning the mechanics of friction. [6].
Equation (9.1) is called the 'unperturbed problem' and equation (9.2) the 'perturbed problem', where $0 \leq \epsilon \ll 1$. If you put $\epsilon = 0$ in the perturbed problem you get the unperturbed problem.

To solve a perturbed problem, physicists developed an approach for calculating the quantities in the form of an expansion into powers of ϵ. Consider the vector function $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. The function $f(t, x; \epsilon)$ is continuous in the variables $t \in \mathbb{R}$ and $x \in D \subset \mathbb{R}^n$ and ϵ is a small parameter. The function f has to be expanded with respect to a small parameter ϵ. The most naive procedure to construct an approximation for f is to assume that it is possible to expand f in an asymptotic expression:

$$f(t,x;\epsilon) = f_0(t,x)+\epsilon f_1(t,x)+\epsilon^2 f_2(t,x)+...+\epsilon^n f_n(t,x)+... = \sum_{n=0}^{N}\epsilon^n f_n(x,t)+R_N(x,t;\epsilon), \tag{9.3}$$

with coefficients $f_1,\ f_2, ...$ which depend on t and x, ϵ^n are called order functions and $R_N(x,t;\epsilon)$ is the error. If it is possible to estimate the error, then the method for obtaining the asymptotic expression for f is called the 'regular perturbation method'.

Example 9.1 *Consider again an oscillator with friction:*

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2\epsilon \frac{\mathrm{d}x}{\mathrm{d}t} + x = 0, \tag{9.4}$$

with initial values $x(0) = a$ *and* $\frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0$*. Then we can expand* x *in:*

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \epsilon^3 ... \tag{9.5}$$

Substituting this in the original equation (9.4) gives a big expression in terms of ϵ*, collecting the terms with equal powers of* ϵ *leads to*

$$\begin{aligned} &\frac{\mathrm{d}^2 x_0}{\mathrm{d}t^2} + x_0 = 0 \\ &\frac{\mathrm{d}^2 x_n}{\mathrm{d}t^2} + x_n = -2\frac{\mathrm{d}x_{n-1}}{\mathrm{d}t}, \qquad n = 1, 2, ... \end{aligned} \tag{9.6}$$

From the initial values, which do not depend on ϵ*, follows*

$$\begin{aligned} x_0(0) = a, &\qquad \frac{\mathrm{d}x_0}{\mathrm{d}t}(0) = 0 \\ x_n(0) = 0, \qquad \frac{\mathrm{d}x_n}{\mathrm{d}t}(0) = 0, &\qquad n = 1, 2, ... \end{aligned} \tag{9.7}$$

Then you can find the solutions of the equations:

$$\begin{aligned} x_0(t) &= a\cos t \\ x_1(t) &= a\sin t - at\cos t, \qquad \textit{etc.} \end{aligned} \tag{9.8}$$

Substituting this in the expansion of x *(equation (9.5)) gives us the formal expansion*

$$x(t) = a\cos t + \epsilon(a\sin t - at\cos t) + \epsilon^2 ... \tag{9.9}$$

This expansion corresponds with an oscillation with increasing amplitude. When you substitute $t = 1/\epsilon$ into the formal expansion, you can see that the second term is no longer $O(\epsilon)$, when $t = O(1/\epsilon^2)$ the second term blows up. Terms of those type are called 'secular terms'. Because it is not possible to give a regular asymptotic expansion uniformly valid in the whole time interval, the initial value problem (9.4) is called a 'singular perturbation problem'.
The naive regular perturbation method is only applicable for finite time intervals, so we need another method to construct asymptotic approximations uniformly valid in 'large' time intervals, e.g. $O(1/\epsilon)$ or even for all values of $t \geq 0$.

9.2 Strained coordinate method

The method used in the previous section is only valid for finite time intervals. To get asymptotic approximations uniformly valid in 'large' time intervals, we need another method to solve the problem. It is possible to modify the regular perturbation method by using a stretched time coordinate.

$$t = (1 + \epsilon\omega_1 + \epsilon^2\omega_2 + ...)\tau \tag{9.10}$$

Then you can determine the constants ω_i such that the solution of the perturbed problem does no longer contain secular terms. So you can obtain an asymptotic expansion valid for time intervals of $O(1/\epsilon)$. This method is called the Poincaré-Lindstedt method.

Another method is introduced by Lighthill, this method is useful for perturbation problems for which the reduced differential equation with $\epsilon = 0$ contains a singularity, e.g. $(t + \epsilon u)\frac{\mathrm{d}u}{\mathrm{d}t} + q(t)u = r(t), \quad t \geq 0$. In that procedure the stretched coordinate is given by

$$t = \tau + \epsilon f_1(\tau) + \epsilon^2 f_2(\tau) + ... \tag{9.11}$$

Here, the stretching functions f_i are chosen in such a way that an asymptotic expansion of the solution of the perturbation problem becomes possible.
Here we will explain the method of the strained coordinate with an example.

Example 9.2 *Consider the initial value problem describing a nonlinear spring:*

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x + \epsilon x^3 = 0, \qquad t \geq 0 \tag{9.12}$$

$$x(0) = \alpha, \qquad \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0$$

Then we can find, by using the regular perturbation theory,

$$x(t) = \cos t + \epsilon\left(-\frac{3}{8}t\sin t + \frac{1}{32}(\cos 3t - \cos t)\right) + O(\epsilon^2) \tag{9.13}$$

The second term is a secular term, so this solution has only a meaning whenever t is bounded. Now we substitute the stretched time coordinate $t = (1 + \epsilon\omega_1 + \epsilon^2\omega_2 + ...)\tau$ and find the initial value problem for x as a function of τ:

$$\frac{1}{(1 + \epsilon\omega_1 + \epsilon^2\omega_2 + ...)^2}\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} + x + \epsilon x^3 = 0, \qquad \tau \geq 0 \tag{9.14}$$

$$x(0) = \alpha, \qquad \frac{\mathrm{d}x}{\mathrm{d}\tau}(0) = 0$$

Whenever $1 + \epsilon\omega_1 + \epsilon^2\omega_2 + ...$ is asymptotically convergent, we can apply the regular perturbation method. So expand $x(\tau)$ by

$$x(\tau) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + ... \tag{9.15}$$

Substituting this expansion into equation (9.14) and taking equal powers of ϵ leads to

$$\begin{aligned}
&\frac{\mathrm{d}^2x_0}{\mathrm{d}\tau^2} + x_0 = 0, && x_0(0) = \alpha, && \frac{\mathrm{d}x_0}{\mathrm{d}\tau}(0) = 0 \\
&\frac{\mathrm{d}^2x_1}{\mathrm{d}\tau^2} + x_1 = -x_0^3 - 2\omega_1 x_0, && x_1(0) = \alpha, && \frac{\mathrm{d}x_1}{\mathrm{d}\tau}(0) = 0 \\
&\dots && \dots && \dots
\end{aligned} \tag{9.16}$$

The solution of the first equation of this set of linear initial value problems is

$$x_0(\tau) = \alpha \cos \tau \tag{9.17}$$

Then the initial value problem for $x_1(\tau)$ becomes

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}\tau^2} + x_1 = -x_0^3 - 2\omega_1 x_0 = -\alpha^3 \cos^3 \tau - 2\omega_1 \alpha \cos \tau = -\alpha \left(\frac{3}{4}\alpha^2 + 2\omega_1 \right) \cos \tau - \frac{\alpha^3}{4} \cos 3\tau \tag{9.18}$$

To eliminate the secular term in $x_1(\tau)$, we have to take $\omega_1 = -\frac{3}{8}\alpha^2$. Then the solution of the second differential equation of the set becomes

$$x_1(\tau) = \frac{\alpha^3}{32}(\cos 3\tau - \cos \tau) \tag{9.19}$$

In this way you can solve the set of differential equations as far as you like. Up to the second order we have

$$x(\tau) = \alpha \cos \tau + \epsilon \frac{\alpha^2}{32}(\cos 3\tau - \cos \tau) + O(\epsilon^2) \tag{9.20}$$

uniformly valid in any finite τ interval $0 \leq \tau \leq \tau_0$ with τ_0 independent of ϵ, with (when you calculate ω_2 in the same way as ω_1) $\tau = \left(1 - \frac{3}{8}\alpha^2\epsilon + \frac{57}{256}\alpha^4\epsilon^2 + ...\right)^{-1} t$

Except a method for finite time intervals, as seen in Section 9.1, now we know also a method valid in 'large' time intervals.

10 Further research

10.1 The model

Initially, we have to compare the several models with the s and the r coordinate. The easiest way to start is to compare the stationary case with rotating coordinate system in s (Section 3) and r (Section 6) as done in Section 6.3. What boundary conditions are needed, and which description is more accurate?

In the problem description with the r coordinate (see Section 4), we assumed that the spinning line could not curve backwards to the rotor. Is that a correct assumption?

In Section 6.3 we compared $F - \Phi v$ and k: $-k = F - \Phi v$. In Section 3.5 we saw that if $F - \Phi v \equiv 0$, the spinning line will stick to the rotor. What happens if at some point $F - \Phi v = 0$ or $k = 0$?

In the dimensionless stationary case in coordinate r, $r_{k=0}$ is introduced (Section 7), what does this point mean? Can you measure this point in practice?
Can you explain this point in a mathematical way, does it agree with a boundary layer? How does this point looks like in the case with coordinate s?

10.2 Boundary conditions

An important part of the problem is to find correct boundary conditions. It seems to be logical that the spinning line leaves the rotor perpendicular, but it appears to be not that obvious. Research about how the spinning line leaves the rotor is needed to find the correct boundary condition. This research can be done by looking to the spinning process with a camera with very much pictures per second, for example $50,000$.

What is the value of F_0, the initial value for the viscous force, in the stationary case with rotating coordinate s, treated in Section 3. Or what is the initial value of k.

When you describe the problem as an initial boundary value problem, instead of an initial value problem, what are the boundary conditions on the coagulator? A problem is that we do not know the length of the spinning line, when it hits the coagulator (Section 3). What are the physical conditions on the coagulator.

10.3 Solving the systems

When boundary conditions are found, solve the several problems numerically. How does the solution look like? Do the various problems give comparable results?
Probably, another way to solve the problems is by using perturbation theory. Are the answers the same as by solving the systems numerically?

10.4 Model extension

A problem description is always a simplification of the real world. If the spinning line cools down very fast, there is a rapid change in viscosity. Then the model can be upgraded by introducing heat equations. To find out if the temperature of the spinning line changes fast one can use a heat camera.

Another point of attention is air friction. In the literature they never speak about such thin cylinders as the spinning line, with that high velocity. Perhaps, this can influence the path of the spinning line very much.

The water on the coagulator has a vertical velocity, so in the z-direction, is this a problem when you look only in the x,y-plane? Maybe it is not realistic to neglect the z-direction, and you need to introduce gravity.

We assumed the polymer to be Newtonian, in general polymers are not Newtonian. How does this assumption affect the model?

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A List of symbols

R_{rot}	m	radius of the rotor.
R_{coag}	m	radius of the coagulator.
Φ	kg/s	mass flux through the spinning line.
A	m^2	cross-section area of the spinning line, perpendicular to the spinning line.
ρ	$\mathrm{kg/m}^3$	mass density of the polymer.
v	m/s	velocity of the fluid in the spinning line.
s	m	coordinate s.
$\mathbf{F}_{centr}$	$\mathrm{kgm/s}^2$(=N)	centrifugal force.
$\mathbf{F}_{cor}$	$\mathrm{kgm/s}^2$	Coriolis force.
$\mathbf{F}_{visc}$	$\mathrm{kgm/s}^2$	viscous force.
m	kg	mass.
ω	rad/s	angular velocity of the rotor.
F	$\mathrm{kgm/s}^2$	norm of the viscous force vector at s.
ϵ		stress (elongation per unit length).
E	$\mathrm{N/m}^2$	Young's modulus.
$\mathbf{e}_\theta$		tangent unit vector.
$\mathbf{e}_r$		radius unit vector.
$\mathbf{e}_n$		normal unit vector.
$\mathbf{e}_x$		unit vector in the x direction.
$\mathbf{e}_y$		unit vector in the y direction.
θ		slope angle of the tangent tot the spinning line.
ϕ		polar angle of a point of the spinning line.
$\mathbf{I}_{in}$	$\mathrm{kgm/s}^2$	entering momentum flux.
$\mathbf{I}_{out}$	$\mathrm{kgm/s}^2$	leaving momentum flux.
$\mathbf{p}$	m	position of the spinning line.
ϕ_p		angular coordinate of the spinning line position.
t	s	time.
r	m	radial coordinate of the spinning line.
$\mathbf{v}$	m/s	velocity of the fluid in the spinning line.
ϕ_v		angular coordinate of the flow velocity.
V	m/s	fluid flow speed in the spinning line.
$\mathbf{s}$		unit tangent vector to the spinning line.
ϕ_s		angular coordinate of the direction of the spinning line.
$\mathbf{m}$		unit normal vector.
$\mathbf{J}$		rotation operator.
A'	m^2	cross-section through the spinning line, not perpendicular to the spinning line.
Q	differs by situation	transport
$\mathbf{f}$	differs by situation	transport flux.
S	differs by situation	production intensity.
Ψ	differs by situation	production term in the conservation law.
η	Pa s	viscosity coefficient.

$\tilde{\phi}_p$		transformed angular coordinate of the spinning line position.
$\tilde{\phi}_s$		transformed angular coordinate of the direction of the spinning line.
$\tilde{\phi}_v$		transformed angular coordinate of the flow velocity.
$\tilde{\mathbf{p}}$	m	transformed position of the spinning line.
$\tilde{\mathbf{v}}$	m/s	transformed velocity of the fluid in the spinning line.
$\tilde{\mathbf{s}}$		transformed unit tangent vector to the spinning line.
$\mathbf{C}$		rotation matrix.
$\tilde{V}$	m/s	fluid flow speed in the spinning line.
$\tilde{\mathbf{m}}$		transformed unit normal vector.
$\mathbf{k}$	kgm/s^2	momentum transport in the transformed equations.
k	kgm/s^2	scalar such that $\mathbf{k} = k\tilde{\mathbf{s}}$.
β		angle difference $\tilde{\phi}_p - \tilde{\phi}_s$.
C_β		integration constant.
L	1/m	scaling factor for length.
$\hat{r}$		dimensionless radius coordinate of the spinning line.
K	s^2/kgm	scaling factor for momentum transport.
$\hat{k}$		dimensionless momentum transport in the transformed equations.
U	s/m	scaling factor for velocity.
$\hat{\tilde{V}}$		dimensionless transformed velocity.
$r_{k=0}$	m	radius coordinate of the spinning line where the momentum transport is zero.
μ		dimensionless viscosity parameter.
$\hat{r}_{k=0}$		dimensionless radius coordinate where the momentum transport is zero.