Pricing multi-asset financial products with tail dependence using copulas

Jan de Kort

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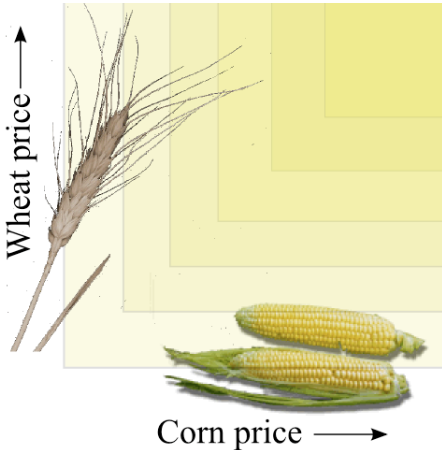
Worst-of corn and wheat

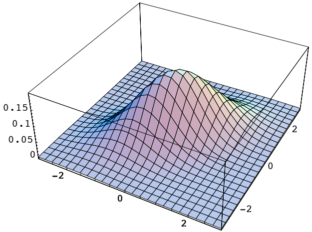
$$\text{Payoff} = \min(\text{ Corn price, Wheat price })$$



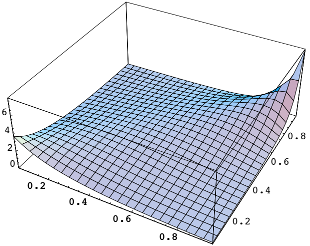
What is tail dependence mathematically?

$X \sim F$, $Y \sim G$ random variables

$$\textbf{Upper tail dependence} \quad := \quad \lim_{u \uparrow 1} \mathbb{P}[\, F(X) > u \,|\, G(Y) > u \,]$$

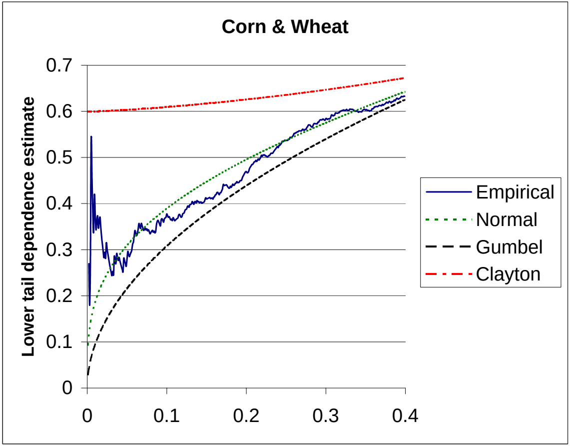


No tail dependence
(Gaussian density)



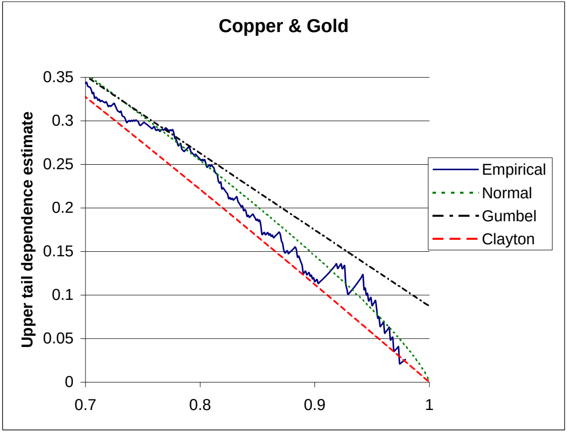
Upper tail dependence
(Gumbel density)

Example: Corn and wheat



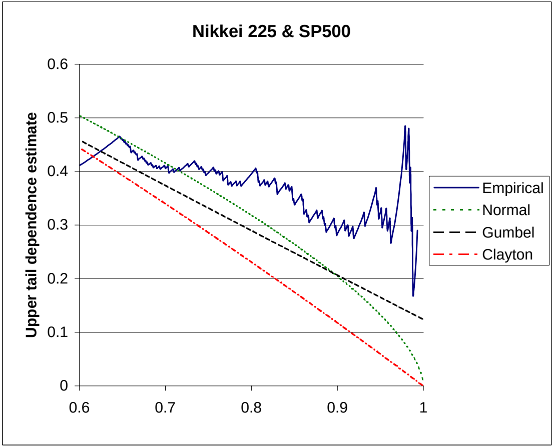
Lower tail dependence

Example: Copper and gold



No upper tail dependence

Example: Nikkei 225 and SP 500



Upper tail dependence

Outline

Outline

Copula definition

A function $C : [0,1]^2 \rightarrow \mathbb{R}$ is called a 2-**copula** if

- for all $(u, v) \in [0,1]^2$

$$C(u,0) = 0\,,$$
$$C(0,v) = 0\,,$$
$$C(u,1) = u\,,$$
$$C(1,v) = v\,,$$

- and for every $[x_1, x_2] \times [y_1, y_2] \in [0,1]^2$

$$C(x_2,y_2) - C(x_2,y_1) - C(x_1,y_2) + C(x_1,y_1) \geq 0\,.$$

Sklar's Theorem

Let H be a joint distribution function with continuous margins F and G such that

$$\text{Ran } F = \text{Ran } G = [0,1],$$

then

$$\exists!_{\textbf{Copula C}} \quad : \quad \mathbf{H}(\mathbf{x},\mathbf{y}) = \mathbf{C}(\,\mathbf{F}(\mathbf{x}),\, \mathbf{G}(\mathbf{y})\,)$$

for all $(x, y) \in \mathbb{R}$.

Copulas and dependence

Correlation	Association along **linear** function
Measure of concordance	Association along **monotone** function

Measures of concordance are a function of the copula only.

Example: Spearman's rank correlation

$$\text{Spearman's } \rho := 12 \iint_{I^2} C(u,v)\, du\, dv - 3$$

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Sample 1		Sample 2	
Observation	Rank	Observation	Rank
1	1	0.2	2
12	2	0.3	3
123	3	0.1	1
1234	4	0.4	4

Copulas and tail dependence

- Tail dependence is a property of the copula only.
- Construct right amount of tail dependence by using linear combination of copulas (Hu, 2002).

Outline

Calibration criterion?

- **Likelihood** of observing the sample given the model
- $\mathbf{L^2}$ **distance** to empirical copula
- **Measures of concordance** (e.g. Spearman's rho)

Likelihood (1)

Differentiating the joint distribution

$$H(x,y) \;=\; C(\,F(x),\,G(y)\,)$$

with respect to x and y gives the **joint density function**

$$h(x,y) \quad = \quad \left.\frac{\partial^2 C(u,v)}{\partial u \partial v}\right|_{(u,v)=(F(x),G(y))} \frac{\partial F}{\partial x}(x) \;\; \frac{\partial G}{\partial y}(y)$$

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$$:= \quad c(F(x),\;G(y)) \qquad f(x) \qquad g(y)$$

Likelihood (2)

Likelihood of observing a sample $\{(x_i, y_i)\}_{i=1}^n$ from (X, Y) where $X \sim F$ and $Y \sim G$ is defined as

$$\textbf{Likelihood} := \prod_{i=1}^{n} c(F(x_i), G(y_i))\, f(x_i)\, g(y_i)\,.$$

It is equivalent to maximize

$$\log(\,\textbf{Likelihood}\,) = \sum_{i=1}^{n} \log c(F(x_i),\, G(y_i)) + \sum_{i=1}^{n} \log f(x_i)\, g(y_i)\,.$$

Likelihood (3)

$$\log(\textbf{Likelihood}) = \underbrace{\sum_{i=1}^{n} \log c(F(x_i), G(y_i))}_{I} + \underbrace{\sum_{i=1}^{n} \log f(x_i)\, g(y_i)}_{II}$$

Approach 1 ("Inference For the Margins")

- Choose parametric form for F, G and C
- Maximize term II, this fixes F and G
- Maximize term I

Approach 2 ("Canonical Maximum Likelihood")

- Choose parametric form for C
- Replace F and G by their empirical counterparts
- Maximize term I

Likelihood (4)

If a mix of copulas is used, i.e.

$$c_{\text{mix}}(u,v) = \alpha_1\, c_1(u,v) \;+\; \alpha_2\, c_2(u,v) \;+\; \dots\;,$$

one has to maximize

$$\sum_{\substack{\text{observations}\\ k}} \log \sum_{\substack{\text{components}\\ i}} \alpha_i\;\; c_i\left(\,F^{\text{emp}}(x_k),\; G^{\text{emp}}(y_k)\,\right)\,.$$

Use Expectation Maximization (EM) algorithm because of good global convergence characteristics.

L^2 distance to empirical copula

$$||C - C^{\text{emp}}||_{L^2}^2 = \iint_{I^2} |C(u,v) - C^{\text{emp}}(u,v)|^2 \, du \, dv$$

Application: NIKKEI 225 and SP 500

Copula		Likelihood	L^2-dist.	Spearman's ρ
100.00%	Normal (ρ=0.239)	14.18	0.0387	0.158
100.00%	Gumbel (θ=1.201)	21.26	0.0368	0.142
100.00%	Gumbel survival (θ=1.144)	10.09	0.0417	
100.00%	Clayton (θ=0.201)	6.39	0.0447	
100.00%	Clayton survival (θ=0.394)	20.72	0.0363	0.110
100.00%	Frank (θ=1.403)	13.11	0.0382	
23.62% 76.38%	Normal (ρ=−0.230) Clayton survival (θ=0.641)	22.19	0.0373	0.156
23.74% 76.26%	Gumbel (θ=1.812) Clayton survival (θ=0.211)	21.71	0.0361	0.140

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Outline

Pricing model

- Building block: multivariate Gaussian model
- **Hack 1**: Keep Gaussian copula, replace margins
- **Hack 2**: Replace copula, replace margins

Model calibration: Instantaneous vs. terminal dependence

Terminal dependence

- **Dependence between price levels**
- This is what matters for pricing!
- Autocorrelation between consecutive levels usually high

Calibration method assumes observations to be time-independent

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Instantaneous dependence

- **Dependence between (daily, weekly, monthly) returns**
- Autocorrelation in returns usually low

Marginal distributions

- Multivariate Gaussian model prices back one vanilla option (i.e. one strike)
- We need to price back a continuum of options (all posible strikes)
- Therefore, use volatility parametrization instead of constant volatility

Differences Hack 1 – Hack 2

	Hack 1	**Hack 2**
Copula	Gaussian	Mix / Archimedean
Calibration	Spearman's rho	Maximum likelihood
Pricing method	Sample from distribution of levels. Terminal covariances taken from Black-Scholes.	Sample from distribution of returns. Add up daily increments.

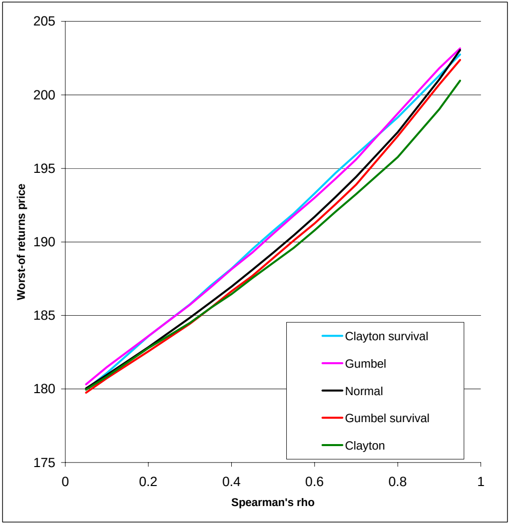
Pricing algorithm (Hack 2)

1. **Calibrate** a copula to historical Δt periodical forward returns
2. For $k = \Delta t, 2\Delta t, \ldots$ to maturity
   - (i) **Simulate** an observation **from the copula** obtained in step 1.
   - (ii) Transform these numbers into daily increments and **update forwards**
3. **Calculate the option price** (at maturity, forward = spot)
4. **Repeat** steps 2 and 3 **and average** the option price.

Application: Worst-of returns NIKKEI 225 and SP500

Copula	Daily returns		Monthly returns		Levels	
	Price	Rel. diff.	Price	Rel. diff.	Price	Rel. diff.
Normal	5.05		5.00		6.97	
Gumbel	4.98	-1.39%	5.07	1.40%	7.05	1.17%
Clayton surv.	4.88	-3.37%	5.12	2.40%	6.98	0.18%
Normal Clayton surv.	5.04	-0.20%	5.03	0.60%	7.06	1.37%
Gumbel Clayton surv.	5.00	-0.99%	5.13	2.60%		

Application: Worst-of returns, fix Spearman's rho



Contracts studied

$$\textbf{Best-of returns} = \max\left(0, \max\left(\frac{S_1(T)}{S_1(0)}, \frac{S_2(T)}{S_2(0)}\right) - 1\right)$$

$$\textbf{Worst-of returns} = \max\left(0, \min\left(\frac{S_1(T)}{S_1(0)}, \frac{S_2(T)}{S_2(0)}\right) - 1\right)$$

$$\textbf{At-the-money spread} = \max\Big(0, S_1(T) - S_2(T) - S_1(0) + S_2(0)\Big)$$

$$\textbf{Bivariate digital} = \mathbb{1}(\ S_1(T) > K_1\ )\ \mathbb{1}(\ S_2(T) > K_2\ )$$

Outline

Why hedging?

- Bank sells options to others who are interested in gambling
- Bank itself does not want to take on any risk
- Replicate option by buying ‘right amount’ of underlyings

What is hedging?

Option value $V(A, B)$ depends on value underlyings A and B.

Our aim is to duplicate the option using the underlyings, i.e.

$$\frac{\partial}{\partial A}\left[V(A,B) + \Delta_A \cdot A + \Delta_B \cdot B\right] = 0\,,$$
$$\frac{\partial}{\partial B}\left[\text{―――――― '' ――――――}\right] = 0\,.$$

Therefore, set

$$\Delta_A = -\frac{\partial V(A,B)}{\partial A}, \quad \Delta_B = -\frac{\partial V(A,B)}{\partial B}\,.$$

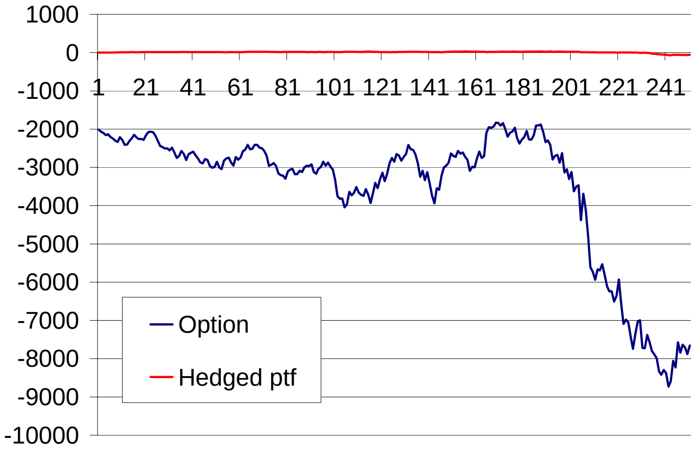
Delta hedging with futures

- The option is sold and the premium is put in a money account earning the overnight rate.
- The portfolio is delta hedged using futures on the underlying assets and zero coupon bonds.
- The portfolio is revalued and rebalanced in the same way on each day of the simulation period. Every day the hedging instruments are liquidated and replaced to re-establish delta-neutrality.

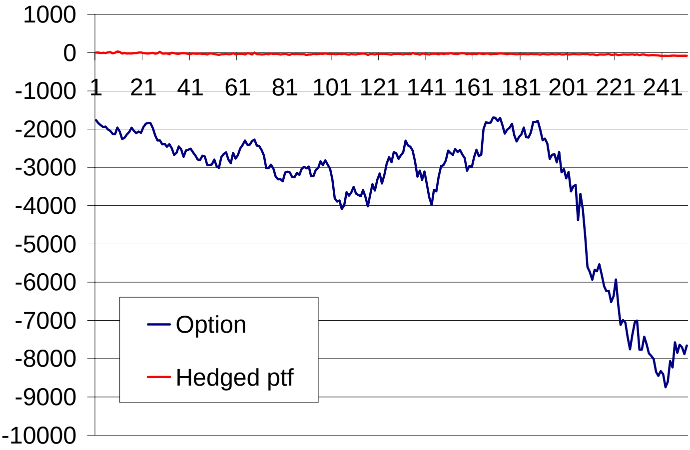
Measuring hedging performance

- Average hedged position should be close to zero
- Variance of hedged portfolio considerably smaller than variance of naked option position

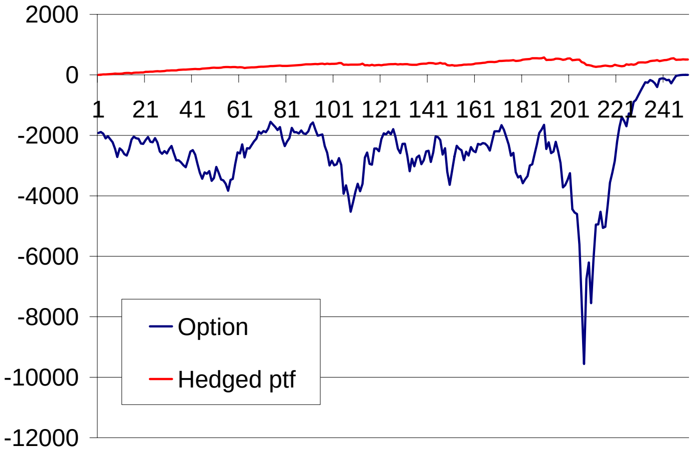
Worst-of corn and wheat — Gaussian copula



Worst-of corn and wheat — Copula w/ tail dependence



ATM Spread corn and wheat — Gaussian copula



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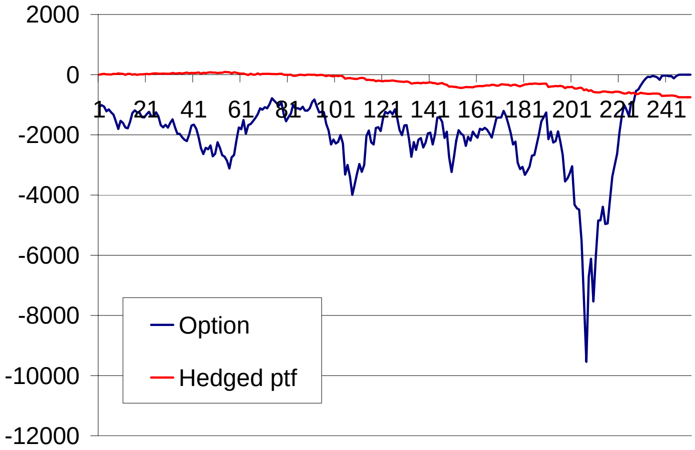


Outline

Conclusions

- Experiments suggests that price shift due to changing copulas is small for best-of, worst-of and spread contracts. Heuristic explanation:
  - Terminal distribution converges to Gaussian.
  - Low strike does not emphasize bivariate tail.
- Hedging performance for products with tail dependent underlyings is acceptable if a Gaussian copula is used.

Recommendations

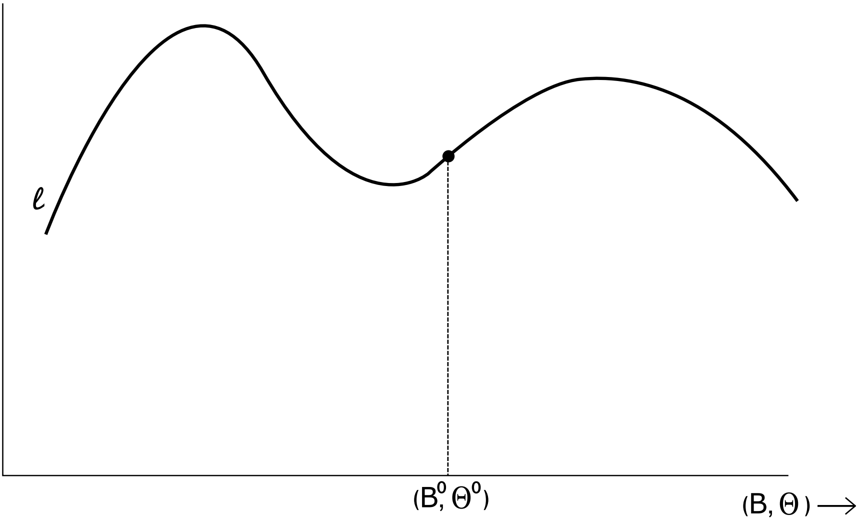
Study impact tail dependence on path-dependent products.

This is more difficult, since:

- Consistency with marginal price processes
- Higher dimensional Archimedean copulas have identical bivariate margins

Questions?

Expectation Maximization algorithm



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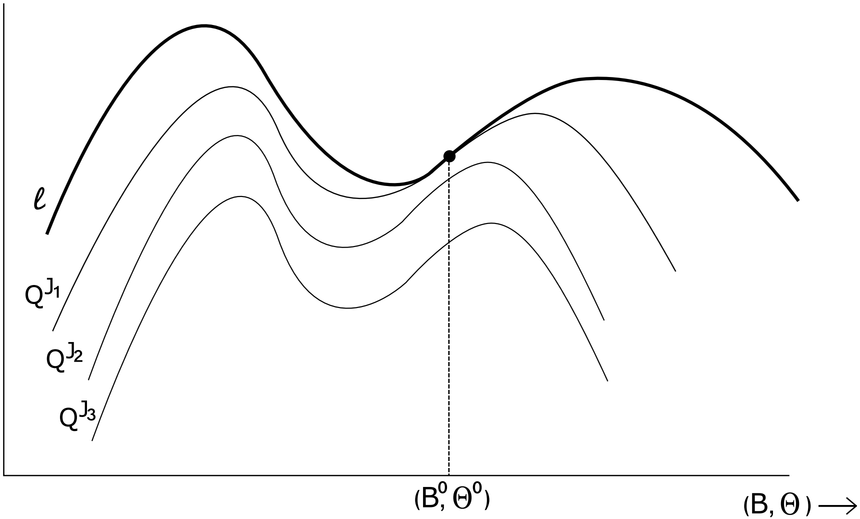

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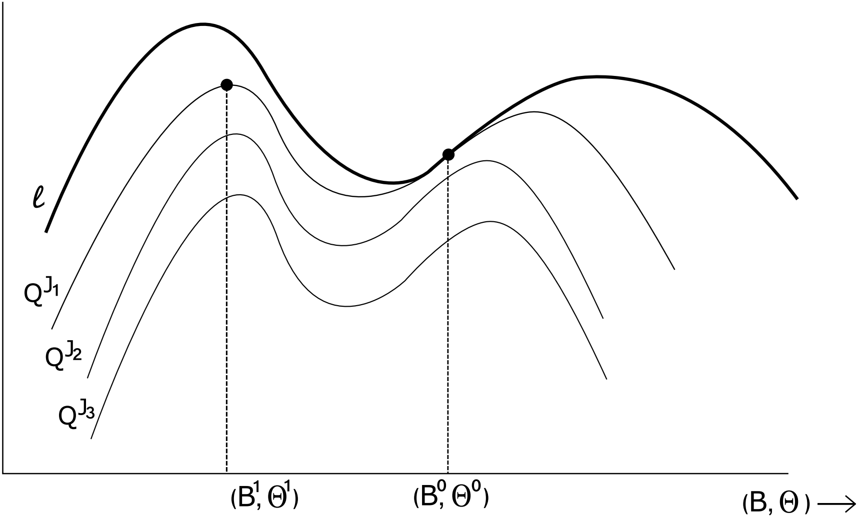

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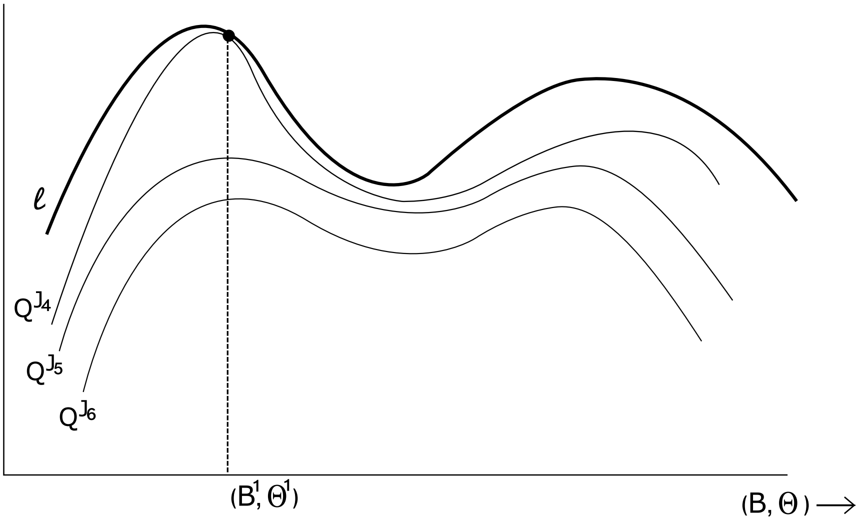

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