Calibration of Stochastic Convenience Yield Models for Crude Oil Using the Kalman Filter
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Agenda

0.1. Introduction
0.2. Initial stochastic processes
0.3. Kalman filter
0.4. Calibration results
0.5. Pricing put options
0.6. Pricing results
0.7. Discussion and further research
1. Introduction

Every year over 600 million contracts are traded on different boards.

- Chicago Board Of Trade,

- New York Cotton Exchange,

- New York Merchantile Exchange.

A futures contract is a contract, traded on a futures exchange, to buy or sell a certain underlying instrument at a certain date in the future, at a specified price. This price is called the future price and is denoted by

\[ F(t, T), \] (1)

with \( t \) the current time and \( T \) the maturity time.
Futures prices are simply given by

\[ F(t, T) = S_t e^{\int_t^T (r - \delta_s) ds}, \]  

(2)

with \( S_t \) the spot price, \( r \) the risk-free interest rate and \( \delta_t \) the convenience yield.

Convenience yield is the premium associated with holding an underlying product or physical good, rather than the contract or derivative product.

Both spot price and convenience yield are believed to have stochastic processes.

**Problem:**

These processes are not observable in the market.

**Solution:**

Use of the Kalman filter.
2. Initial stochastic processes

2.1. Convenience yield follows an Ornstein-Uhlenbeck process (OU)

Empirical evidence shows that the convenience yield follows a mean-reverting pattern, i.e.

\[ d\delta_t = k(\alpha - \delta_t)dt + \sigma_\delta dW^\delta_t, \]

and for the spot price we have a geometrical Brownian motion,

\[ dS_t = (\mu - \delta_t)S_t dt + \sigma_S S_t dW^S_t, \]

- \( \alpha \) is the long rate mean to which \( \delta_t \) tends to revert
- \( k \) is the speed of adjustment
- \( \mu \) is the drift term
- \( \sigma_S \) and \( \sigma_\delta \) are the volatilities of \( dS_t \) and \( d\delta_t \) respectively
- \( dW^\delta_t dW^S_t = \rho dt, \rho \) is the correlation term.
2.2. Convenience yield follows a Cox-Ingersoll-Ross process (CIR)

\[
\begin{align*}
\frac{dS_t}{S_t} &= (\mu - \delta_t)dt + \sigma_S \sqrt{\delta_t} dW_t, \\
\ d\delta_t &= k(\alpha - \delta_t)dt + \sigma_\delta \sqrt{\delta_t} dZ_t.
\end{align*}
\] (3)

In this way, the convenience yield can not be negative. However, as we shall see later, the CIR process gives very unaccurate results. For both the OU and the CIR process for the convenience yield and the GBM for the spot price we can get a closed form solution for the future prices.
3. Closed form solution of the futures prices

Considering the OU process as the initial stochastic process for the convenience yield and a GMB for the spot price we can derive the closed form solution of the future prices explicitly by assuming an affine form, i.e.

\[ F_t(S, \delta, \tau) = S_t e^{A(\tau) - B(\tau) \delta_t}, \quad (4) \]

where

\[ A(\tau) = \left[ \left( r - \tilde{\alpha} + \frac{\sigma^2}{2k^2} - \frac{\sigma_S \sigma \delta \rho}{k} \right) \tau \right] + \left[ \frac{\sigma^2}{4} \frac{1 - e^{-2k\tau}}{k^3} \right] + \left[ \left( \tilde{\alpha}k + \sigma_S \sigma_S \rho - \frac{\sigma^2}{k} \right) \left( \frac{1 - e^{-k\tau}}{k^2} \right) \right] \]

and

\[ B(\tau) = \frac{1 - e^{-k\tau}}{k}, \quad \tilde{\alpha} = \alpha - \frac{\lambda}{k}, \]

By looking at equation (4) we see that the futures prices are linear in the log of the state variable \( S_t \). As we shall see, this comes in handy when applying the Kalman filter. A similar form exists for the future prices if the convenience yield follows a CIR process.
4. The Kalman filter

The main idea of the Kalman filter is to use observable variables to reconstitute the value of the non-observable variables (the spot price and convenience yield). Since the futures prices are widely observed and traded in the market, we consider these our observable variables.

In state variable terms, we have in combination with $x_t = \ln S_t$,

$$\ln F_t(\tau) = x_t + A(\tau) - B(\tau)\delta_t + \epsilon_t.$$  \hspace{1cm} (5)

The error term $\epsilon_t$ has a diagonal variance matrix $H$. 
In combination with the relationship \( x_t = \ln S_t \) we have

\[
\begin{align*}
\left\{
\begin{array}{l}
dx_t &= (\mu - \delta_t - \frac{1}{2} \sigma_S^2)dt + \sigma_S dW_t^S, \\
d\delta_t &= k(\alpha - \delta_t)dt + \sigma_\delta dW_t^\delta.
\end{array}
\right.
\tag{6}
\end{align*}
\]

From (6) the transition equation follows immediately,

\[
\begin{bmatrix}
    x_{t_i} \\
    \delta_{t_i}
\end{bmatrix} = \begin{bmatrix}
    (\mu - \frac{1}{2} \sigma_S^2) \Delta t \\
    k \alpha \Delta t
\end{bmatrix} + \begin{bmatrix}
    1 & -\Delta t \\
    0 & 1 - k \Delta t
\end{bmatrix} \begin{bmatrix}
    x_{t_{i-1}} \\
    \delta_{t_{i-1}}
\end{bmatrix} + \begin{bmatrix}
    1 & 0 \\
    0 & 1
\end{bmatrix} \begin{bmatrix}
    \xi_{t_i}^1 \\
    \xi_{t_i}^2
\end{bmatrix},
\tag{7}
\]

where \( \xi_{t_i} \) takes into account the Brownian motions \( dW_t^S \) and \( dW_t^\delta \). \( \xi_{t_i} \) is assumed normal with mean zero and has a covariance-variance matrix given by

\[
V_{t_i} = \begin{bmatrix}
    \sigma_S^2 \Delta t & \rho \sigma_\delta \sigma_S \Delta t \\
    \rho \sigma_\delta \sigma_S \Delta t & \sigma_\delta^2 \Delta t
\end{bmatrix}
\tag{8}
\]

Note that the covariance-variance matrix does not depend on the state variables.
A similar procedure can be done in the case of the CIR process. However, the covariance-variance matrix does now depend on the state variable $\delta_t$ which will cause problems as we will see later.

$$V_{t_i} = \begin{bmatrix} \frac{\sigma^2_S \Delta t \delta_{t_{i-1}}}{\rho \sigma_S \sqrt{\Delta t} \sqrt{\delta_{t_{i-1}}} \sqrt{\text{Var}[\delta_{t_i} | \delta_{t_{i-1}}]}}, & \rho \sigma_S \sqrt{\Delta t} \sqrt{\delta_{t_{i-1}}} \sqrt{\text{Var}[\delta_{t_i} | \delta_{t_{i-1}}]} \sqrt{\text{Var}[\delta_{t_i} | \delta_{t_{i-1}}]} \end{bmatrix},$$

where

$$\text{Var}[\delta_{t_i} | \delta_{t_{i-1}}] = \alpha \left(\frac{\sigma^2_\delta}{2k}\right)(1 - e^{-k\Delta t})^2 + \delta_{t_{i-1}} \left(\frac{\sigma^2_\delta}{k}\right)(e^{-k\Delta t} - e^{-2k\Delta t}).$$ (9)
5. Iterative procedure

We now have a measurement equation and a transition equation that depend on the latent state variables $S_t$ and $\delta_t$ and on the observed variables, the futures prices $F_t$.

$$\varphi = \left\{ k, \mu, \alpha, \lambda, \sigma_S, \sigma_\delta, \rho, \{h_j\}_{j=1}^n \right\}.$$
Optimized parameter set = \( \varphi \)

This optimized parameter set is used for the last time to update the matrices in both the measurement and transition equation to generate the paths of the futures prices and the convenience yield.
6. Numerical results for the Ornstein-Uhlenbeck process for the commodity light crude oil

We use weekly observations of the light crude oil market from 01-02-2002 until 25-01-2008 (313 observations). At each observation we consider 7 monthly contracts.

6.1. Details for starting the iterative procedure

- The nearest future price is retained as the spot price $S_0$.

- The $\delta_0$ is subtracted out of two futures contracts at $t = 0$.

\[
\begin{bmatrix}
\mu & \sigma_S & \alpha & k & \sigma_\delta & \rho & \lambda & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\
\end{bmatrix};
\]

lowerbound=\[
\begin{bmatrix}
-10 & 0.001 & -10 & 0.001 & 0.001 & -1 & -10 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
\end{bmatrix};
\]

upperbound=\[
\begin{bmatrix}
10 & 10 & 10 & 10 & 10 & 1 & 10 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{bmatrix};
\]
Numerical results for the Ornstein-Uhlenbeck process for the commodity light crude oil

<table>
<thead>
<tr>
<th>Prs</th>
<th>Ini parset</th>
<th>Opti parset</th>
<th>Ini parset</th>
<th>Opti parset</th>
<th>Ini parset</th>
<th>Opti parset</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
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<td>1.4221 (0.0372)</td>
<td>0.3</td>
<td>1.4221 (0.0380)</td>
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<td>1.4221 (0.0382)</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.2</td>
<td>0.3733 (0.1471)</td>
<td>0.2</td>
<td>0.3733 (0.1376)</td>
<td>0.2</td>
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</tr>
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<td>0.0699 (0.1025)</td>
<td>0.6</td>
<td>0.0699 (0.1082)</td>
</tr>
<tr>
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<td>-0.0183 (0.1602)</td>
<td>0.1</td>
<td>-0.0183 (0.1459)</td>
<td>0.01</td>
<td>-0.0183 (0.1543)</td>
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<tr>
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<td>0.3630 (0.0137)</td>
<td>0.4</td>
<td>0.3630 (0.0139)</td>
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<td>0.3630 (0.0153)</td>
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<tr>
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<td>0.4028 (0.0172)</td>
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<td>0.4028 (0.0181)</td>
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<td>0.8378 (0.0164)</td>
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<td>$</td>
<td>0.0246</td>
<td>0.0188 (0.0008)</td>
<td>0.0246</td>
<td>0.0188 (0.0007)</td>
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<td>0.0072 (0.0003)</td>
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<tr>
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<td>0.0022 (0.0001)</td>
<td>0.0291</td>
<td>0.0022 (0.0001)</td>
</tr>
<tr>
<td>$</td>
<td>h_4</td>
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<td>0.0000 (0.0001)</td>
<td>0.0313</td>
<td>0.0000 (0.0001)</td>
</tr>
<tr>
<td>$</td>
<td>h_5</td>
<td>$</td>
<td>0.0336</td>
<td>0.0006 (0.0000)</td>
<td>0.0336</td>
<td>0.0006 (0.0000)</td>
</tr>
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<td>$</td>
<td>h_6</td>
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<td>0.0357</td>
<td>0.0000 (0.0001)</td>
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<td>$</td>
<td>h_7</td>
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<td>0.0377</td>
<td>0.0014 (0.0001)</td>
</tr>
<tr>
<td>Log-Likelihood</td>
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<td>8744.6479</td>
<td>8744.6479</td>
<td>8744.6479</td>
<td>8744.6479</td>
<td>8744.6479</td>
</tr>
</tbody>
</table>

Table 1: Optimized parameter set (Opti parset) for different initial parameter sets (Ini parset). Standard errors in parentheses.
Figure 1: Comparison between the calibrated future prices and market future prices for contract $\tau_1$.

Figure 2: Comparison between filtered $\delta_{t,i}$ and market convenience yield, $\delta_{\text{market}}(\tau_1, \tau_2)$. 
7. Errors

Figure 3: Innovation corresponding to $F_t(\tau_1)$. Mean = $-8.1734\times10^{-4}$, Variance = 0.0022.

Figure 4: Innovation corresponding to $F_t(\tau_2)$. Mean = $8.6078\times10^{-4}$, Variance = 0.0019.
8. **Numerical results for the Cox-Ingersoll-Ross process for the commodity light crude oil**

As noted earlier, the matrix $V_t$ given in (9) depends on the state variables. This results in:

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Ini parset</th>
<th>Opti parset</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
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<td>10.0000</td>
</tr>
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<td>$\mu$</td>
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<td>0.8616</td>
</tr>
<tr>
<td>$\alpha$</td>
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</tr>
<tr>
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</tr>
<tr>
<td>$\sigma_S$</td>
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</tr>
<tr>
<td>$\sigma_\delta$</td>
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<td>10.0000</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.8378</td>
<td>0.2196</td>
</tr>
<tr>
<td>$</td>
<td>h_1</td>
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<td>h_2</td>
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</tr>
<tr>
<td>$</td>
<td>h_3</td>
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<tr>
<td>$</td>
<td>h_4</td>
<td>$</td>
</tr>
<tr>
<td>$</td>
<td>h_5</td>
<td>$</td>
</tr>
<tr>
<td>$</td>
<td>h_6</td>
<td>$</td>
</tr>
<tr>
<td>$</td>
<td>h_7</td>
<td>$</td>
</tr>
<tr>
<td>Log-Likelihood</td>
<td></td>
<td>5104.3040</td>
</tr>
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</table>

Table 2: Optimized parameter set (Opti parset) for different initial parameter sets (Ini parset). Standard errors in parentheses.
Figure 5: Comparison between the calibrated future prices and market future prices for contract $\tau_1$.

Figure 6: Comparison between the calibrated $\delta_t$ and the market implied convenience yield.


9. **Improvements**

If we shift the data such that $\delta_0^{market} > 0$ we have

<table>
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<tr>
<th>Parameters</th>
<th>Ini parset</th>
<th>Opti parset</th>
<th>Ini parset</th>
<th>Opti parset</th>
<th>Ini parset</th>
<th>Opti parset</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1.6118 (0.0485)</td>
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<td>1.6118 (0.0285)</td>
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<tr>
<td>$\mu$</td>
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<td>0.2932 (0.1221)</td>
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<td>$\alpha$</td>
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<td>0.0590 (0.0843)</td>
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<td>0.0590 (0.0880)</td>
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<tr>
<td>$\sigma_s$</td>
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<tr>
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<td>0.0189 (0.0008)</td>
<td>0.0246</td>
<td>0.0189 (0.0008)</td>
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<tr>
<td>$</td>
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<td>0.0073 (0.0003)</td>
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<td>$</td>
<td>h_3</td>
<td>$</td>
<td>0.0291</td>
<td>0.0022 (0.0001)</td>
<td>0.0291</td>
<td>0.0022 (0.0001)</td>
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<tr>
<td>$</td>
<td>h_4</td>
<td>$</td>
<td>0.0313</td>
<td>0.0000 (0.0001)</td>
<td>0.0313</td>
<td>0.0000 (0.0001)</td>
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<tr>
<td>$</td>
<td>h_5</td>
<td>$</td>
<td>0.0336</td>
<td>0.0006 (0.0000)</td>
<td>0.0336</td>
<td>0.0006 (0.0000)</td>
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<td>0.0000 (0.0001)</td>
<td>0.0357</td>
<td>0.0000 (0.0001)</td>
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<tr>
<td>$</td>
<td>h_7</td>
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<td>0.0015 (0.0001)</td>
<td>0.0377</td>
<td>0.0015 (0.0001)</td>
</tr>
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</table>

Log-Likelihood | $8341.0028$ | $8341.0028$ | $8341.0028$

Table 3: Optimized parameter set (Opti parset) for different initial parameter sets (Ini parset).
Standard errors in parentheses.
Figure 7: Comparison between the calibrated future prices and market future prices for contract $\tau_1$.

Figure 8: Comparison between the calibrated $\delta_{t_i}$ and the market convenience yield, $\delta_{\text{market}}(\tau_1, \tau_2)$. 
10. Errors

Figure 9: Innovation corresponding to $F_t(\tau_1)$. Mean = -8.2412e-004, Variance = 0.0003.

Figure 10: Innovation corresponding to $F_t(\tau_2)$. Mean = 7.6238e-004, Variance = 0.0009.
11. Pricing put options on futures contracts

The value of a contingent claim $V$, where its payoff is given by a call i.e. $V(F, T) = \max(F - K)^+$, yields

$$V(F, t) = e^{-r(T-t)}F \mathcal{N}(d_1) - e^{-r(T-t)}K \mathcal{N}(d_2),$$

with

$$d_1 = \frac{\ln(F/K) + \frac{1}{2} \hat{\sigma}_F^2(T-t)}{\hat{\sigma}_F \sqrt{T-t}}, \quad d_2 = \frac{\ln(F/K) - \frac{1}{2} \hat{\sigma}_F^2(T-t)}{\hat{\sigma}_F \sqrt{T-t}}$$

and

$$\hat{\sigma}_F^2(t, T) = \frac{1}{T-t} \left\{ \frac{2 \rho \sigma_S \delta}{k^2} (1 - \theta) - \frac{2 \rho \sigma_S \delta}{k} (T-t) + \sigma_S^2(T-t) + \frac{\sigma_\delta^2}{2k^3} (1 - \theta^2) - \frac{2 \sigma_\delta^2}{k^3} (1 - \theta) + \frac{\sigma_\delta^2}{k^2} (T-t) \right\}.$$ (11)
12. **Contract details**

<table>
<thead>
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<th>Contract</th>
<th>Date at t=0</th>
<th>Date at t=T</th>
<th>Last obs. day</th>
<th>Observed trading days</th>
<th>Total trading days</th>
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</thead>
<tbody>
<tr>
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<td>20/11/2007</td>
<td>20/05/2008</td>
<td>17/03/2008</td>
<td>79</td>
<td>125</td>
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<tr>
<td>Z8</td>
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<td>17/03/2008</td>
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<td>258</td>
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<tr>
<td>M9</td>
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<td>19/05/2009</td>
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<td>386</td>
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Table 4: Contract details.
13. Results following a naïve approach

Figure 11: Comparison between the calibrated future prices and market future prices for contract M8.

Figure 12: Comparison between the calibrated future prices and market future prices for contract Z8.
<table>
<thead>
<tr>
<th>Parameters</th>
<th>Ini parset</th>
<th>Opti parset</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
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<td>0.8046 (0.1012)</td>
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<tr>
<td>$\mu$</td>
<td>0.3733</td>
<td>0.5374 (0.4453)</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.0699</td>
<td>0.0851 (0.0169)</td>
</tr>
<tr>
<td>$\lambda$</td>
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<td>-0.0754 (0.1405)</td>
</tr>
<tr>
<td>$\sigma_{S}$</td>
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<td>0.3493 (0.0305)</td>
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<td>$\sigma_{\delta}$</td>
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<tr>
<td>$</td>
<td>h_1</td>
<td>$</td>
</tr>
<tr>
<td>$</td>
<td>h_2</td>
<td>$</td>
</tr>
<tr>
<td>$</td>
<td>h_3</td>
<td>$</td>
</tr>
<tr>
<td>Log-Likelihood</td>
<td></td>
<td>902.7301</td>
</tr>
</tbody>
</table>

Table 5: Optimized parameter set (Opti parset). Standard errors in parentheses.
Figure 13: Comparison between the calibrated put prices and market put prices for contract M8 with K=85.

Figure 14: Comparison between the calibrated put prices and market put prices for contract M8 with K=80.
## 14. Results for a matured M8 contract for different strikes

<table>
<thead>
<tr>
<th>Contract</th>
<th>$K$</th>
<th>Date at t=0</th>
<th>Date at t=T</th>
<th>Observed trading days</th>
<th>Total trading days</th>
</tr>
</thead>
<tbody>
<tr>
<td>M8</td>
<td>85</td>
<td>20/11/2007</td>
<td>20/05/2008</td>
<td>125</td>
<td>125</td>
</tr>
<tr>
<td>M8</td>
<td>110</td>
<td>05/12/2007</td>
<td>20/05/2008</td>
<td>115</td>
<td>115</td>
</tr>
<tr>
<td>M8</td>
<td>115</td>
<td>10/03/2008</td>
<td>20/05/2008</td>
<td>51</td>
<td>51</td>
</tr>
<tr>
<td>M8</td>
<td>125</td>
<td>13/03/2007</td>
<td>20/05/2008</td>
<td>48</td>
<td>48</td>
</tr>
</tbody>
</table>

Table 6: Contract details.
<table>
<thead>
<tr>
<th>Parameters</th>
<th>K=85</th>
<th>K=110</th>
<th>K=115</th>
<th>K=125</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>0.2089</td>
<td>0.2144</td>
<td>0.0014</td>
<td>0.0630</td>
</tr>
<tr>
<td>$\mu$</td>
<td>1.4354</td>
<td>1.3117</td>
<td>6.556</td>
<td>3.9980</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>11.2299</td>
<td>8.0303</td>
<td>0.2753</td>
<td>109.2796</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>3.1413</td>
<td>1.9467</td>
<td>7.5792</td>
<td>18.3986</td>
</tr>
<tr>
<td>$\sigma_S$</td>
<td>0.2613</td>
<td>0.2587</td>
<td>0.3013</td>
<td>0.1649</td>
</tr>
<tr>
<td>$\sigma_\delta$</td>
<td>0.0303</td>
<td>0.0455</td>
<td>3.0048</td>
<td>1.6310</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.9999</td>
<td>0.9999</td>
<td>0.6635</td>
<td>-0.9999</td>
</tr>
<tr>
<td>$</td>
<td>h_1</td>
<td>$</td>
<td>0.0341</td>
<td>0.0231</td>
</tr>
</tbody>
</table>

Table 7: Optimized parameter set for different strike prices for the contract M8.
<table>
<thead>
<tr>
<th>Parameters</th>
<th>K=85</th>
<th>K=110</th>
<th>K=115</th>
<th>K=125</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_S$</td>
<td>0.3737</td>
<td>0.3756</td>
<td>0.3090</td>
<td>0.2415</td>
</tr>
<tr>
<td>$\sigma_\delta$</td>
<td>0.6701</td>
<td>0.4923</td>
<td>3.8011</td>
<td>1.7195</td>
</tr>
<tr>
<td>$</td>
<td>h_1</td>
<td>$</td>
<td>0.0145</td>
<td>0.0159</td>
</tr>
</tbody>
</table>

Table 8: Optimized parameter set for different strike prices for the contract M8.
Figure 15: Comparison between the calibrated put prices and market put prices for contract M8 K=85.

Figure 16: Comparison between the calibrated put prices and market put prices for contract M8 K=110.
Figure 17: Comparison between the calibrated put prices and market future put for contract M8 K=115.

Figure 18: Comparison between the calibrated put prices and market future put for contract M8 K=125.
Figure 19: Comparison between $\tilde{\sigma}_F(t, T)$ and $\sigma_{RD}(t, T)$ for contract M8 with K=85.

Figure 20: Comparison between $\tilde{\sigma}_F(t, T)$ and $\sigma_{RD}(t, T)$ for contract M8 with K=110.
Figure 21: Comparison between $\tilde{\sigma}_F(t, T)$ and $\sigma_{RD}(t, T)$ for contract M8 with K=115.

Figure 22: Comparison between $\tilde{\sigma}_F(t, T)$ and $\sigma_{RD}(t, T)$ for contract M8 with K=125.
Figure 23: Calibrated volatility contract M8 with different strike prices
15. Results for unmatured contracts

Ideas:

- $\sigma_S = \bar{\sigma}_S \tau^\beta$ and $\sigma_\delta = \bar{\sigma}_\delta \tau^\gamma$ to include more (time-dependent) structure into (11)

- $\tilde{\sigma}_F = a^2 \tau^b + c^2 \tau^\gamma + d^2$, $\gamma \leq 0$,

Results

![Graph](image1)

Figure 24: Comparison between $\tilde{\sigma}_F(t, T)$ and $\sigma_{\text{RD}}(t, T)$ for contract M8 with K=85.

![Graph](image2)

Figure 25: Comparison between $\tilde{\sigma}_F(t, T)$ and $\sigma_{\text{RD}}(t, T)$ for contract M8 with K=80.
16. Discussion

- Although the EKF is not optimal for the Ornstein-Uhlenbeck process, it is still acceptable and therefore we use this technique to price put options on futures contracts.

- Inadequacy of the CIR process.
  - CIR process was inspired by Robeiro and Hodges.
  - Period from 17th of March 1999 to the 24th of December 2003.
  - 20 observations.

- Initial values of the parameters unknown.
  - Tested [10,-10].
  - Converged to the same optimized parameter set.
When pricing a put option on a futures contract, there are some important details to be considered.

– Stochastic interest rates.

– The errors in our calibrated put prices can be explained by the use of the extended Kalman filter. When linearizing the system matrices, we introduce an approximation in the estimation.

High values of some parameters. What could be concluded?
17. Further research

Topics of further research are

- Starting from the future process, one does not know how to choose this volatility. In fact, the future price process may be given by

\[
\frac{dF(t, T)}{F(t, T)} = \sum_{i=1}^{n} \sigma_i dW^i_t,
\]

with \(i\)-independent Brownian motions.

Choose

\[
\frac{dF(t, T)}{F(t, T)} = \sigma_1(t, T)dW^1_t + \sigma_2(t, T)dW^2_t,
\]

with

\[
\sigma_1(t) = \sqrt{1 - \rho^2}\sigma_S \quad \text{and} \quad \sigma_2(t) = \rho\sigma_S
\]

and

\[
\begin{cases}
W^1_t = \frac{W^S_t}{\sqrt{1 - \rho^2}} - \frac{\rho W^\delta_t}{\sqrt{1 - \rho^2}}, \\
W^2_t = W^\delta_t.
\end{cases}
\]
Then

\[ \frac{dS_t}{S_t} = (r - \delta_t)dt + \sigma_S dW_t^S \]  \hspace{1cm} (15)

and

\[ d\delta_t = k(\alpha(t) - \delta_t)dt + \sigma_\delta dW_t^\delta, \]  \hspace{1cm} (16)

with

\[ \alpha(t) = \frac{1}{k} \frac{\partial^2}{\partial t^2} \ln F(0, t) + \frac{\partial}{\partial t} \ln F(0, t) + \frac{\sigma_\delta^2}{2k^2} (e^{-2kt} - 1) + \frac{\rho \sigma_S \sigma_\delta}{k}. \]  \hspace{1cm} (17)
• Another reason for initially assuming stochastic processes for the state variables is because of the Greeks.

• High values of parameters. Solutions:
  – accept
  – stochastic volatility
  – jump processes
  – time-dependent mean-reverting term
• Economic, weather, war.
18. Conclusion

It is a real challenge to end up with a realistic model for oil prices which, at the same time, can be dealt with in a fast and efficient way. The models presented in this work are a first step in that direction.