The use of CONTACT in dynamical simulations

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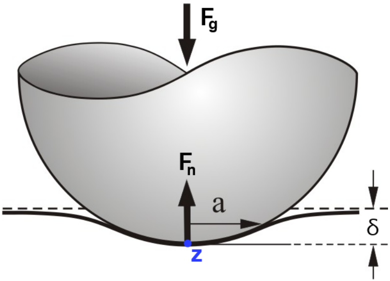
About VORtech

- Founded in 1996 in Delft.
- Specialized in mathematical consultancy and development of high performance scientific software.
- Broad range of customers.

CONTACT

- Software that solves contact problems between two objects (e.g. train wheel & rails).
- Main problem: simulation of a train over a bridge.
- Research question: how can CONTACT be used for dynamical contact problems?

Rigid body motion



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- Height ball $z(t)$ measured from a reference point.
- How to compute $z(t)$?

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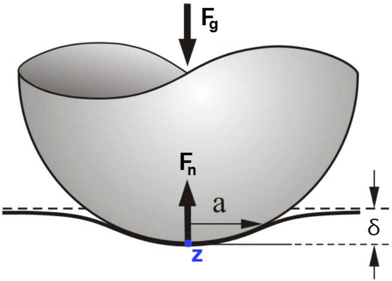

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- Gravitational force

$$F_g = mg$$

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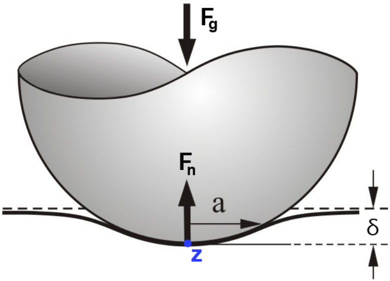

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- How to compute $z(t)$?

- Gravitational force

$$F_g = mg$$

- Normal force exerted by the surface

$$F_n = \frac{4}{3} E^* \sqrt{R} \delta(t)^{3/2}$$

Possibilities CONTACT

CONTACT is capable of

- Computing the normal force F_n.
- Computing the distribution of the pressure p_n in the contact area.
- Computing the (quasi-)stationary elastic deformation of the surface.

Can be done for arbitrarily shaped objects by supplying the penetration.

Rigid body motion

- The resulting force is $F(z) = F_n(z) - F_g$.
- By Newton's second law:

$$\begin{aligned} m\ddot{z} &= F(z) \\ &= F_n(z) - mg \end{aligned}$$

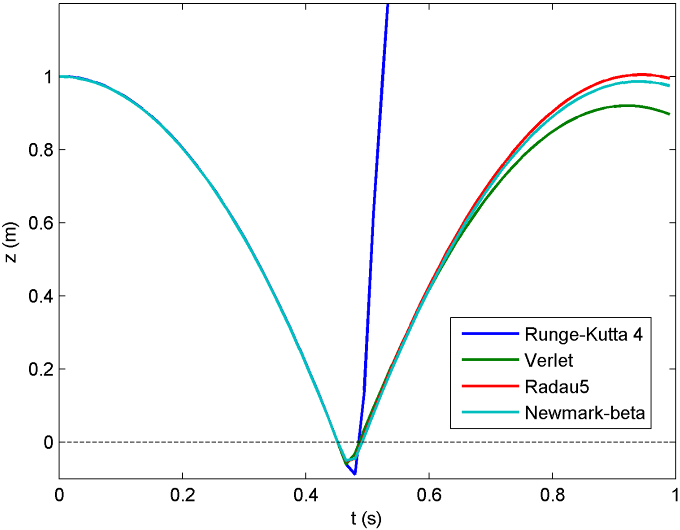
Can be solved using a time integration scheme.

Time integration schemes

Different kind of integration schemes:

- Runge-Kutta schemes, (Forward Euler, RK4, . . .)
- Radau schemes, (Backward Euler, Radau5, . . .)
- Verlet,
- Leapfrog,
- Adams methods,
- Backward differentiation formulas,
- Newmark-beta,
- HHT, and
- Generalized-α integration.

Numerical results



Computing the deformation

The deformation of the surface is described by:

$$\begin{cases} \rho \dfrac{\partial^2 u_i}{\partial t^2} = \displaystyle\sum_{j=1}^{3} \dfrac{\partial \sigma_{ij}}{\partial x_j} + F_i & i = 1,2,3 \\ e_{ij} = \dfrac{1}{2}\left(\dfrac{\partial u_i}{\partial x_j} + \dfrac{\partial u_j}{\partial x_i}\right) & i,j = 1,2,3 \\ \sigma_{ij} = 2Ge_{ij} + \lambda\delta_{ij}\displaystyle\sum_{k=1}^{3} e_{kk} & i,j = 1,2,3 \end{cases}$$

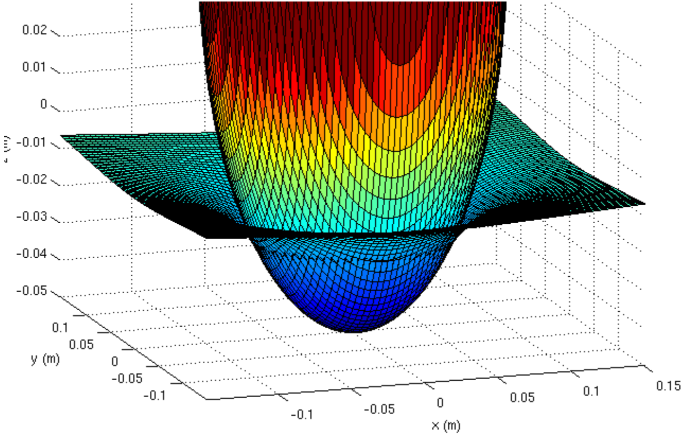
This can be solved using a Finite Element approach.

- Expensive, need to discretise w.r.t. z direction.
- Accurate, inertia is taken into account.

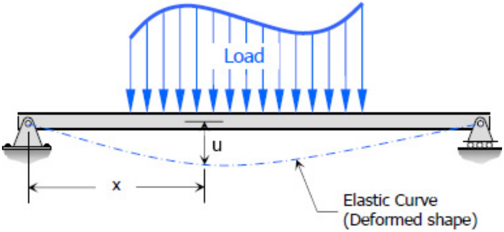
The quasi-static deformation

Using CONTACT

- Computational inexpensive.
- Quasi-static, inertia at the surface elements is ignored.



Global deformation of a bridge



The (1D) Euler-Bernoulli beam equation:

$$\frac{\partial^2}{\partial x^2}\left(E(x)I(x)\frac{\partial^2 u}{\partial x^2}\right) = -\rho(x)\frac{\partial^2 u}{\partial t^2} + p(x,t)$$

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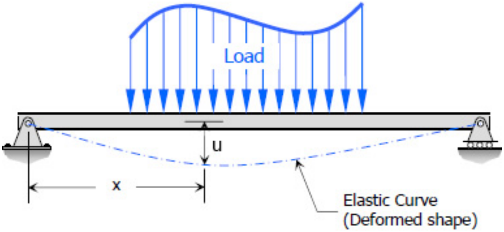

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$$EI\frac{\partial^4 u}{\partial x^4} = -\rho\frac{\partial^2 u}{\partial t^2} + p(x,t)$$

Modal analysis

Mode shapes are natural vibrations of the beam.

Substitute $u(x,t) = e^{i\lambda t} w(x)$ into

$$\begin{aligned} EI \frac{\partial^4 u}{\partial x^4} &= -\rho \frac{\partial^2 u}{\partial t^2} \\ \Longrightarrow \quad EI w^{(4)}(x) &= \rho \lambda^2 w(x) \end{aligned}$$

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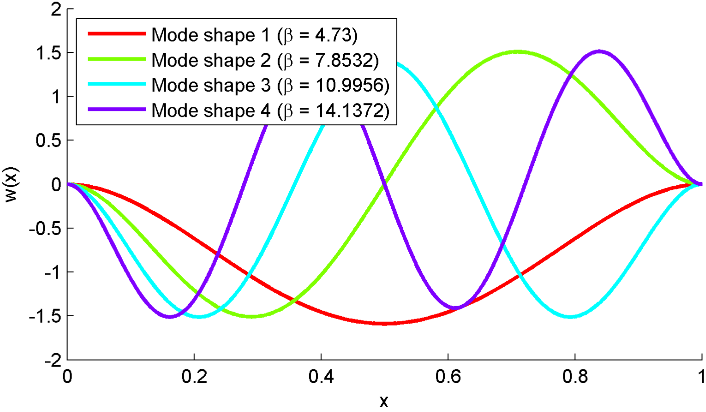
\Longrightarrow \quad & EI w^{(4)}(x) &=& \rho \lambda^2 w(x) \\
\Longrightarrow \quad & w(x) &=& c_1 \cos(\beta x) + c_2 \sin(\beta x) \\
& & & + c_3 \cosh(\beta x) + c_4 \sinh(\beta x)
\end{aligned}
$$

where $\beta = \left(\frac{\rho \lambda^2}{EI} \right)^{1/4}$

Modal analysis

The mode shapes for a clamped beam satisfy

$$\cos(\beta L)\cosh(\beta L) = 1$$



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The solution can be written as

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Idea: approximate u by

$$u(x,t) \approx u_m(x,t) = \sum_{i=1}^{m} c_i(t) w_i(x)$$

How do we compute $c_i(t)$?

Modal analysis

$$EI \sum_{i=1}^{\infty} c_i(t) w_i^{(4)}(x) = -\rho \sum_{i=1}^{\infty} c_i''(t) w_i(x) + p(x,t)$$

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We have $w_i^{(4)}(x) = \beta_i^4 w(x)$, so that

$$\sum_{i=1}^{\infty} w_i(x) \left[EI \beta_i^4 c_i(t) + \rho c_i''(t) \right] = p(x,t)$$

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Multiplying by $w_j(x)$ and integrating over $[0, L]$ yields

$$\sum_{i=1}^{\infty} \left(\left[EI \beta_i^4 c_i(t) + \rho c_i''(t) \right] \int_0^L w_i(x) w_j(x) \mathrm{d}x \right) = \int_0^L p(x,t) w_j(x) \mathrm{d}x$$

Modal analysis

We arrive at the differential equation:

$$\rho c_j''(t) = \int_0^L p(x,t) w_j(x) \mathrm{d}x - EI\beta_j^4 c_j(t)$$

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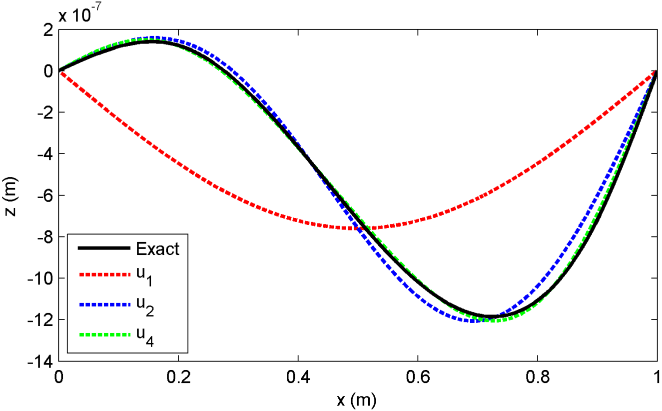
- A numerical integrator (such as a Newton-Cotes formula),
- A time integration scheme (such as Newmark-beta), and
- An iterative solver (such as Picard iteration).

Theorem

The error of the stationary modal solution satisfies

$$\|u - u_m\|_2 \leq \frac{L^4 \|p\|_2}{3\pi^4 EI m^3} = \mathcal{O}(m^{-3})$$



Combining local and global deformation

We described two different phenomena:

- Local deformation (occurring around the contact area), and
- Global deformation.

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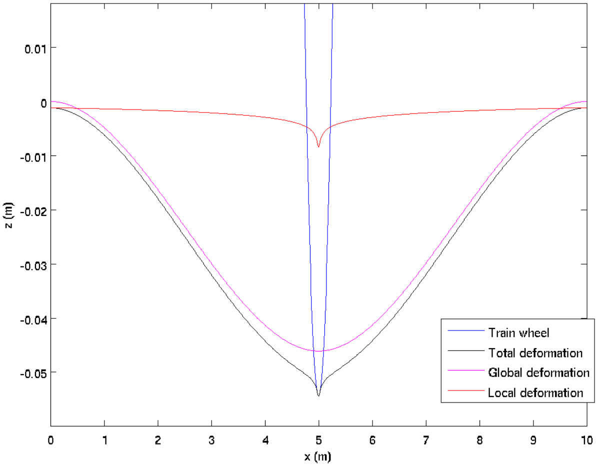
In reality, a bridge can deform both globally as locally:

$$u_{\text{tot}}(x,t) = u(x,t) + l(x,t)$$

Combining local and global deformation



Combining local and global deformation

- The global deformation can be derived by superposing the modal coefficients that satisfy

$$\rho c_j''(t) = \int_0^L p(x,t) w_j(x) \mathrm{d}x - EI\beta_j^4 c_j(t)$$

- The rigid height $z(t)$ of the wheel can be derived by solving

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$$m_w \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = F(t)$$

- Both $p(x,t)$ and $F(t)$ are derived using CONTACT by supplying the penetration

$$\delta(x,t) = \sum_{j=1}^{m} c_j(t) w_j(x) - [z(t) + g(x - st)]$$

Combining local and global deformation

Applied to Backward Euler:

$$\begin{cases} c_1^{k+1} = c_1^k + \Delta t \dot{c}_1^{k+1} \\ \dot{c}_1^{k+1} = \dot{c}_1^k + \dfrac{\Delta t}{\rho}\left[\displaystyle\int_0^L p(x, \mathbf{c}^{k+1}, z^{k+1}, t^{k+1}) w_1(x) \mathrm{d}x - EI\beta_1^4 c_1^{k+1}\right] \\ \quad\vdots \qquad\qquad \vdots \\ c_m^{k+1} = c_m^k + \Delta t \dot{c}_m^{k+1} \\ \dot{c}_m^{k+1} = \dot{c}_m^k + \dfrac{\Delta t}{\rho}\left[\displaystyle\int_0^L p(x, \mathbf{c}^{k+1}, z^{k+1}, t^{k+1}) w_m(x) \mathrm{d}x - EI\beta_m^4 c_m^{k+1}\right] \\ z^{k+1} = z_k + \Delta t \dot{z}^{k+1} \\ \dot{z}^{k+1} = \dot{z}^k + \Delta t \left[\dfrac{F(\mathbf{c}^{k+1}, z^{k+1}, t^{k+1})}{m_c} - g\right] \end{cases}$$

Combining local and global deformation

This can be written as

$$\begin{pmatrix} 1 & -\Delta t & & & & & \\ \Delta t\frac{EI}{\rho}\beta_1^4 & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & -\Delta t & & \\ & & & \Delta t\frac{EI}{\rho}\beta_m^4 & 1 & & \\ & & & & & 1 & -\Delta t \\ & & & & & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ \dot{c}_1 \\ \vdots \\ c_m \\ \dot{c}_m \\ z \\ \dot{z} \end{pmatrix}^{k+1} = \begin{pmatrix} c_1 \\ \dot{c}_1 \\ \vdots \\ c_m \\ \dot{c}_m \\ z \\ \dot{z} \end{pmatrix}^{k} + \Delta t \begin{pmatrix} 0 \\ \frac{1}{\rho}\int_0^L p(x, \mathbf{c}^{k+1}, z^{k+1}, t^{k+1}) w_1(x)\mathrm{d}x \\ \vdots \\ 0 \\ \frac{1}{\rho}\int_0^L p(x, \mathbf{c}^{k+1}, z^{k+1}, t^{k+1}) w_m(x)\mathrm{d}x \\ 0 \\ \frac{F(\mathbf{c}^{k+1}, z^{k+1}, t^{k+1})}{m_c} - g \end{pmatrix}$$

or

$$A\mathbf{y}^{k+1} = \mathbf{y}^k + \Delta t \mathbf{f}^{k+1}$$

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Picard approach:

$$\mathbf{y}_{j+1}^{k+1} = A^{-1}(\mathbf{y}^k + \Delta t \mathbf{f}_j^k)$$

For stiff materials, this doesn't converge!

The Quasi-Newton approach

We linearise $F(\mathbf{c}_j^{k+1}, z_j^{k+1}, t^{k+1})$ at each iteration j.

- Cannot be done analytically, but instead we can set

$$\frac{\partial F_j}{\partial z_j^{k+1}} \approx \frac{F(\mathbf{c}_j^{k+1}, z_j^{k+1}, t^{k+1}) - F(\mathbf{c}_j^{k+1}, z_j^{k+1} - \alpha, t^{k+1})}{\alpha}$$

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- To achieve stability, we should also linearise w.r.t to $\mathbf{c}_j^{k+1}$.

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- To achieve stability, we should also linearise w.r.t to $\mathbf{c}_j^{k+1}$.
- This is computationally very expensive.

Idea: we only linearise with respect to the approach δ_j^{k+1}.

$$F_{j+1}^{k+1} = F_j^{k+1} + \left(\frac{\partial F}{\partial \delta}\right)_j^{k+1} \cdot (\delta_{j+1}^{k+1} - \delta_j^{k+1})$$

The Quasi-Newton approach

$$\delta(t) = \max_{0\leq x\leq L} [u(x,t) - w(x,t)]$$

$$= \max_{0\leq x\leq L} \left[\sum_{i=1}^{m} c_i(t) w_i(x) - g(x - st)\right] - z(t)$$

The Quasi-Newton approach

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$$= \max_{0\le x\le L} \left[\sum_{i=1}^{m} c_i(t) w_i(x) - g(x - st)\right] - z(t)$$

After iteration j of time step $k+1$, we have

$$\delta_{j+1}^{k+1} = \max_{0\le x\le L} \left[\sum_{i=1}^{m} c_{i,j+1}^{k+1} w_i(x) - g(x - st^{k+1})\right] - z_{j+1}^{k+1}$$

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$$\delta(t) = \max_{0\le x\le L} [u(x,t) - w(x,t)]$$
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$$\approx \sum_{i=1}^{m} c_{i,j+1}^{k+1} w_i(x_j^{k+1}) - g(x_j^{k+1} - st^{k+1}) - z_{j+1}^{k+1}$$

The Quasi-Newton approach

After linearising, we arrive at

$$\dot{c}_{i,j+1}^{k+1} + \frac{\Delta t}{\rho}\left[EI\beta_i^4 c_{i,j+1}^{k+1} + \left(\frac{\partial I}{\partial \delta}\right)_{i,j}^{k+1} \cdot \left(z_{j+1}^{k+1} - \sum_{l=1}^{m} c_{l,j+1}^{k+1} w_l(x_j^{k+1})\right)\right]$$

$$= \dot{c}_i^k + \frac{\Delta t}{\rho}\left[I_{i,j}^{k+1} + \left(\frac{\partial I}{\partial \delta}\right)_{i,j}^{k+1} \cdot \left(z_j^{k+1} - \sum_{l=1}^{m} c_{l,j}^{k+1} w_l(x_j^{k+1})\right)\right]$$

$$\dot{z}_{j+1}^{k+1} + \frac{\Delta t}{m_c} \cdot \frac{\partial}{\partial \delta} F_j^{k+1} \cdot \left(z_{j+1}^{k+1} - \sum_{i=1}^{m} c_{i,j+1}^{k+1} w_i(x_j^{k+1})\right)$$

$$= \dot{z}^k - \Delta t g + \frac{\Delta t}{m_c}\left[F_j^{k+1} + \frac{\partial}{\partial \delta} F_j^{k+1} \cdot \left(z_j^{k+1} - \sum_{i=1}^{m} c_{i,j}^{k+1} w_i(x_j^{k+1})\right)\right]$$

The Quasi-Newton approach

This can be written as

$$\dot{\mathbf{x}}_{j+1}^{k+1} + \Delta t A_j^{k+1} \mathbf{x}_j^{k+1} = \dot{\mathbf{x}}^k + \Delta t \mathbf{g}_j^{k+1}, \text{ where}$$

$$A_j^{k+1} = \begin{pmatrix} \frac{EI}{\rho}\beta_1^4 - \frac{w_1(x_j^{k+1})}{\rho} \cdot \frac{\partial}{\partial \delta} I_{1,j}^{k+1} & \dots & -\frac{w_m(x_j^{k+1})}{\rho} \cdot \frac{\partial}{\partial \delta} I_{1,j}^{k+1} & \frac{1}{\rho} \cdot \frac{\partial}{\partial \delta} I_{1,j}^{k+1} \\ \vdots & \ddots & \vdots & \vdots \\ -\frac{w_1(x_j^{k+1})}{\rho} \cdot \frac{\partial}{\partial \delta} I_{m,j}^{k+1} & \dots & \frac{EI}{\rho}\beta_m^4 - \frac{w_m(x_j^{k+1})}{\rho} \cdot \frac{\partial}{\partial \delta} I_{m,j}^{k+1} & \frac{1}{\rho} \cdot \frac{\partial}{\partial \delta} I_{m,j}^{k+1} \\ -\frac{w_1(x_j^{k+1})}{m_c} \cdot \frac{\partial}{\partial \delta} F_j^{k+1} & \dots & -\frac{w_m(x_j^{k+1})}{m_c} \cdot \frac{\partial}{\partial \delta} F_j^{k+1} & \frac{1}{m_c} \cdot \frac{\partial}{\partial \delta} F_j^{k+1} \end{pmatrix}$$

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The matrix A_j^{k+1} is dense; the mode shapes c_i and the rigid height z are dependent on each other!

The Quasi-Newton approach

We arrive at the system

$$\begin{pmatrix} I & -\Delta t I \\ A_j^{k+1} & I \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix}_{j+1}^{k+1} = \begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix}^{k} + \Delta t \begin{pmatrix} \mathbf{0} \\ \mathbf{g}_j^{k+1} \end{pmatrix}$$

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Theorem

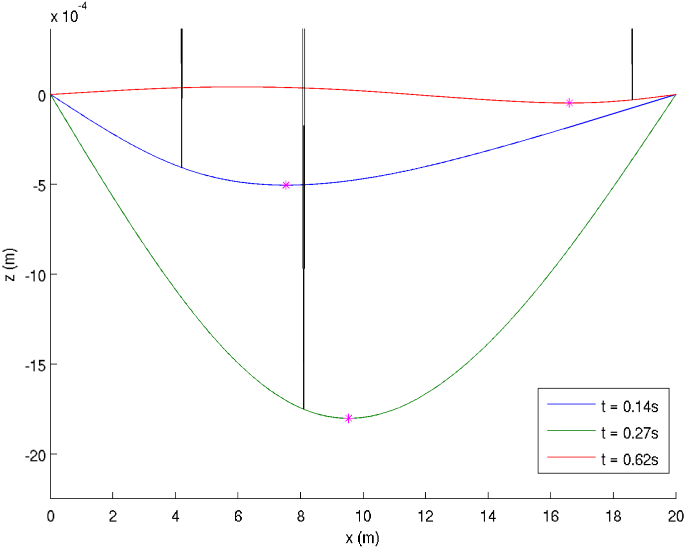
The solution of the system is given by

$$\begin{pmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{pmatrix}_{j+1}^{k+1} = \begin{pmatrix} Y_j^{k+1}(\mathbf{x}_j^{k+1} + \Delta t \dot{\mathbf{x}}_j^{k+1} + (\Delta t)^2 \mathbf{g}_j^{k+1}) \\ Y_j^{k+1}(\dot{\mathbf{x}}_j^{k+1} + \Delta t \mathbf{g}_j^{k+1} + \frac{1}{\Delta t}\mathbf{x}_j^{k+1}) - \mathbf{x}_j^{k+1} \end{pmatrix}$$

where

$$Y_j^{k+1} = D - \frac{\Delta t D \mathbf{e}_j^{k+1} (\mathbf{f}_j^{k+1})^T D}{1 + \Delta t (\mathbf{f}_j^{k+1})^T D \mathbf{e}_j^{k+1}}$$

Numerical results



Conclusion & Further Research

- Quasi-static deformation is computed using CONTACT.
- Combined with a time integration scheme such as Newmark-beta or Radau5.
- Global deformation is solved using modal analysis.
- The total deformation is solved using a Quasi-Newton approach.

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- Further research: taking friction into account as the result of rolling and sliding of wheels.

Any questions?