Fast solver for the three-factor Heston-Hull/White problem

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Chapter 1

Introduction

1.1 Stochastic Models

It was in 1973 that Myron Scholes and Fischer Black came up with their Black-Scholes formula to price options. Until then a few over-the-counter options and some exchange-traded warrants were traded, but from that moment on options exchanges spring up in Chicago, New York and Philadelphia. Later on in London, Paris and Tokyo. Nowadays there are exchanges in many locations, such as the Netherlands and Germany. Options are traded in a world wide market.
The Black-Scholes formula

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0, \tag{1.1}$$

(σ: volatility, r: interest rate, q: dividend yield,S:underlying equity) is a very simple equation to price options. In this simplicity lays the elegance of the formula. How can an instrument, which follows a process that cannot be described, be priced by such a simple formula? It is here that Black and Scholes had to make some assumptions. If those assumptions are taken for granted then the Black-Scholes equation can be used without a doubt. However, as Fischer Black remarks in [1], these assumptions are simple and unrealistic and when Black and Scholes tried to make money with the formula, by simple buying options that were underpriced, all they gained was a loss. One has to be very careful when using the Black-Scholes formula.
This was also discovered by traders in 1987. Until then options were priced using a constant volatility (another assumption of the Black-Scholes formula) and this was one of the reasons for the big crash in October 1987. Options at the money have a different volatility than options in or out the money, the so called volatility smile or skew. This means that the assumption of constant volatility is not a good one and from that moment on it was tried to make a model that fitted the smile, while still having realistic underlying dynamics. All sorts of models for the volatility were tried, p.e. local volatility, implied volatility, historical volatility and furthermore their were numerous scientists who tried to model the volatility by assuming that volatility varies in a random way (stochastic volatility), such as Heston; Hull and White; Cox, Ingersoll and Ross. These models fit the smile and have realistic underlying dynamics (observing the market data). From these models (and other, potential more realistic ones) the Heston model (which is a version of the square root process described by Cox, Ingersoll and Ross) is the most popular, because of the existence of a fast and easily implemented quasi-closed form solution for European options [2].
It is therefore that volatility will be modeled using the Heston model.

1.2 The Heston Model

The partial differential equation for an equity underlying with constant volatility and interest is given by (1.1). Since volatility need not be constant this model can be extended with stochastic volatility. For the stochastic volatility the Heston model is used.
The Heston model is a mean reverting Ornstein-Uhlenbeck process (as observed in the market). The dynamics for the underlying equity and the Heston stochastic volatility are given by:

$$dS_t = (r-q)S_t dt + \sqrt{v_t} S_t dZ_1, \quad (1.2)$$

$$dv_t = -\lambda(v_t - \overline{v})dt + \eta\sqrt{v_t} dZ_2, \quad (1.3)$$

$$Cov(dZ_1, dZ_2) = \rho_{12} dt, \quad (1.4)$$

where r and q are resp. the interest rate and the dividend yield of the stock. λ is the speed of reversion of the instantaneous variance v_t to its long term mean $\overline{v}$ and η is the volatility of the volatility. Last but not least ρ_{12} is the correlation between random stock price returns and changes in v_t. These processes can be simulated using numerical techniques [2] and Figures (1.1) and (1.2) show a numerical simulation of these processes. The figures are obtained by a numerical simulation using an Euler discretization for the equity process S_t and a Millstein scheme for the variance process v_t (The Millstein scheme for the variance process is taken to avoid negative values). This discretization looks as follows

$$S_{i+1} = S_i + (r-q)S_i\Delta t + S_i\sqrt{v_i}\sqrt{\Delta t}N(0,1), \ i = 1,..,N$$

$$v_{i+1} = (\sqrt{v_i} + (\frac{\eta}{2})\sqrt{\Delta t}N(0,1))^2 - \lambda(v_i - \overline{v})\Delta t - \frac{\eta^2}{4}\Delta t, \ i = 1,..,N$$

(where Δt is the timestep size and $N(0,1)$ is the standard normal probability density function)
Figure (1.1) is a simulation of v_t only. The parameters for this simulation are $\lambda = 1, \ \eta = 0.5, \ v_0 = 0.35^2, \ \overline{v} = 0.35^2, \ T = 1$. In this figure it is seen that the variance process v_t is indeed mean reverting.
In Figure (1.2) a simulation of the equity underlying with stochastic volatility is shown. The paramters used for this simulation are $\lambda = 1, \ \eta = 0.5, \ v_1 = 0.35^2, \ \overline{v} = 0.35^2, \ r = 0.05, \ q = 0.03, \ S_1 = 1, \ T = 1$.

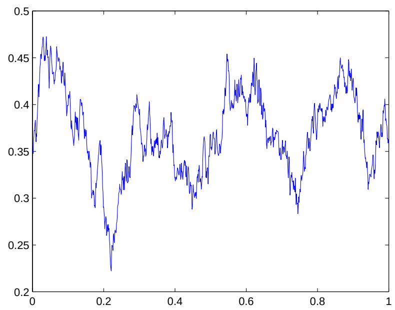

Figure 1.1: Simulation of the Heston process

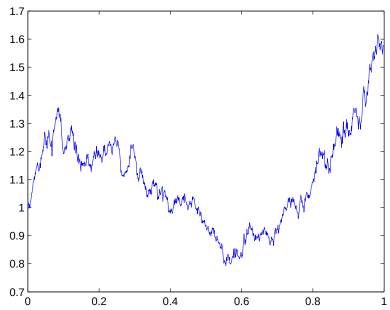

Figure 1.2: Simulation of the stock price with Heston stochastic volatility

1.2.1 The pricing equation for the two-factor model with stochastic volatility and an equity underlying

To price an instrument with the underlying processes given by (1.2), (1.3) and the correlation given by (1.4) we can derive a pricing equation. Solving this equation gives the price of the instrument.

There are two approaches to find the pricing equation for the two-factor model with stochastic volatility and an equity underlying. The first one is by setting up a portfolio Π containing the option being priced denoted by V(S,v,t), $-\Delta$ of the underlying equity and $-\Delta_1$ of another equity whose value depend on volatility. After this portfolio is constructed one hedges the portfolio to make it riskfree and since it is riskfree the return $d\Pi$ of the portfolio should equal $r\Pi dt$. Filling in all terms gives the pricing equation for the two-factor model. In [2] this approach is followed and the pricing equation for the two-factor model is obtained. It's worth reading this approach. However, in this thesis another approach is used, since this is, to my opinion, more easily extendable to the three-factor models.
The second approach is the **Feynman-Kac** approach. Feynman and Kac derived a relationship between stochastic differential equations and partial differential equation. The **Feynman-Kac** theorem is given by

Theorem 1.1 (Feynman-Kac) *Suppose the underlying processes $y_1(t), y_2(t), .., y_n(t)$ follow the stochastic differential equation:*

$$dy_i = \mu_i(y_1, y_2, .., y_n, t) + \sigma_i(y_1, y_2, .., y_n, t)dW_i, \tag{1.5}$$

then the function

$$f(y_1, y_2, .., y_n, t) = E_{y_1,y_2,..,y_n,t}[F(y_1(T), .., y_n(T))], \tag{1.6}$$

is given by the solution of the partial differential equation

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial y_i} + \frac{1}{2} \sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j \frac{\partial f}{\partial y_i \partial y_j} = 0, \tag{1.7}$$

subject to

$$f(y_1, y_2, .., y_n, T) = F(y_1, y_2, .., y_n), \tag{1.8}$$

where $\rho_{ij} = cov(dW_i, dW_j)/dt$

The price of a claim on S_t paying $F(S(T))$ at maturity is given by

$$V(S_t, v_t, t) = E_{S,v,t}\left(F(T)e^{-\int_t^T r(s)ds}\right) \tag{1.9}$$

Since (for now) r is constant the term $e^{-\int_t^T r(s)ds}$ can be taken out of the expectation. Here after the following function can be defined

$$V(S_t, v_t, t) = e^{-r(T-t)}U(S_t, v_t, t) \tag{1.10}$$

Where $U(S_t, v_t, t)$ is $E_{S,v,t}(F(T))$ and to the function $U(S_t, v_t, t)$ the **Feynman-Kac** theorem can be applied. Transforming the $U(S_t, v_t, t)$ back to $V(S_t, v_t, t)$ gives the pricing equation of the two-factor model. Setting $y_1 = S_t$ and $y_2 = v_t$ and applying the transformation as described above and Theorem (1.1) to (1.2) and (1.3) gives:

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} - \lambda(v_t - \underline{v})\frac{\partial V}{\partial v} - rV + \frac{1}{2}v_t S^2 \frac{\partial^2 V}{\partial S^2}$$
$$+\rho_{12} S v_t \eta \frac{\partial^2 V}{\partial S \partial v} + \frac{1}{2}\eta^2 v_t \frac{\partial^2 V}{\partial v^2} = 0,\ S_t \in [0,\infty],\ v_t \in [0,1]. \tag{1.11}$$

To solve this problem numerically the domain must be truncated to the domain $[S_{min}, S_{max}]x[v_{min}, v_{max}]$ and as a consequence boundary conditions are necessary. However, it is not always known what these boundary conditions are. How this problem is solved will be discussed in Chapter (2).
In this section it was suggested to use a stochastic model for the volatility, because volatility varies in a random way. The Heston model was proposed, because of a number of advantages. However, volatility is not the only variable which varies in a random way. The $interest$ is also a variable which can vary in a random way and why not model this variable as a stochastic quantity?!
In the next section the stochastic interest rate will be discussed.

1.3 Stochastic interest

Volatility is not the only quantity that varies in a random way. It can be seen in the market that interest too varies as a random quantity and therefore interest rate will also be modeled as a stochastic quantity. To model the interest rate, another mean reverting Ornstein-Uhlenbeck process (as observed in the market) is used, namely the Hull-White interest rate model. This model describes the short-term interest rate and the dynamics are given by:

$$dr_t = (\Theta(t) - ar_t)dt + \sigma_r dW \tag{1.12}$$

where r is the short term interest rate, Θ is a function of time determining the average direction in which r moves ($\Theta_{max} \approx 0.07$), chosen such that movements in r are consistent with todays zero coupon yield curve [1], a is the mean reversion rate (which is taken constant or sometimes it is computed using historic data, $a \approx 0.05$), governing the relationship between short an long rate volatilities and σ_r is the annual standard deviation of the short rate (it is determined via calibration to caplets, $\sigma_r \approx 0.01$).
In figure (1.3) an Euler discretization simulation for the Hull-White process is given with parameters: $a = 0.05,\ \Theta = 0.07,\ \sigma_r = 0.01,\ r_{begin} = 0.03$.

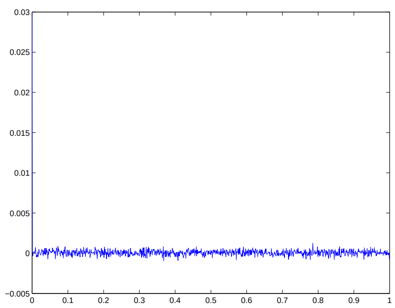

Figure 1.3: numerical simulation for Hull-White process

[1]www.powerfinance.com

1.4 The pricing equation for the three-factor Heston / Hull-White model

In the previous section it was suggested to model the interest rate and the volatility with stochastic models. In his section we will derive the pricing equation for the three-factor model with stochastic interest, stochastic volatility and an equity underlying.
The dynamics for the underlying equity S_t, the interest rate r_t and the variance v_t are given by:

$$\begin{aligned} dS_t &= (r_t - q)S_t dt + \sqrt{v_t} S_t dW_1, &(1.13)\\ dr_t &= (\Theta(t) - ar_t)dt + \sigma_r dW_2, &(1.14)\\ dv_t &= -\lambda(v_t - \underline{v})dt + \eta\sqrt{v_t} dW_3. &(1.15)\end{aligned}$$

The price of a claim on S_t paying $F(S(T))$ at maturity given by

$$V(S_t, v_t, t) = E_{S,v,t}\left(F(T)e^{-\int_t^T r(s)ds}\right). \quad (1.16)$$

However, this time the interest rate r is not constant and the term $e^{-\int_t^T r(s)ds}$ can not be pulled out of the expectation. This can be solved by defining an auxiliary process of the form $dz = -r(t)dt$. The function $V(S_t, v_t, r_t, t)$ is then equal to $e^{-z}E_{S_t,v_t,r_t,z,t}[F(T)e^{z(T)}]$. Defining $V(S_t, v_t, r_t, t) = e^{-z}U(S_t, v_t, r_t, t)$ and applying **Feynman-Kac** to $U(S_t, v_t, r_t, t)$ then gives the desired result. The whole derivation can be read in [4] or [3].
Applying Theorem (1.1) and the transformation as described above to the equations (1.13),(1.14) and (1.15) gives the following pricing equation for the three-factor Heston / Hull-White model:

$$\begin{aligned}&\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + (\Theta(t) - ar)\frac{\partial V}{\partial r} - \lambda(v_t - \underline{v})\frac{\partial V}{\partial v} - rV + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2}\\ &+\rho_{12}S\sqrt{v}\sigma_r\frac{\partial^2 V}{\partial S\partial r} + \rho_{13}Sv\eta\frac{\partial^2 V}{\partial S\partial v} + \frac{1}{2}\sigma_r^2\frac{\partial^2 V}{\partial r^2} + \rho_{23}\sigma_r\eta\sqrt{v}\frac{\partial^2 V}{\partial r\partial v}\\ &+\frac{1}{2}\eta^2 v\frac{\partial^2 V}{\partial v^2} = 0 \end{aligned} \quad (1.17)$$

where

$$\rho_{ij} = Cov(dW_i, dW_j)/dt \quad (1.18)$$

1.5 Goal of this thesis

The goal of this thesis is to implement a three dimensional solver, especially, but not only, to solve the partial differential equation (1.17). Three keywords for the solver are *fast*, *accurate* and *general* and no concession can be made on either three of them. However, solving three dimensional problems give rise to speed issues, which need to be resolved to obtain a useful finite difference solver. One very interesting topic of research is the boundary conditions. Since not all boundary conditions for the problem are known and since we like the set up to be general we like to go round the boundary condition. This can be done in two ways. One way is the use of pde boundary conditions. This gives the great advantage that an implicit method can be used, however the pde boundary conditions may give rise to other problems. Another way is by simply solving explicitly. This may, however, result in very slow methods and to a decrease of accuracy . Both methods will be examined for the 1 factor Black-Scholes, Hull/White and Heston model. Together with the boundary conditions we will investigate the other problems which may occur in these one-factor models, before we go to the three-factor model.

Chapter 2

Models for the asset price, interest rate and volatility

2.1 Black-Scholes Equation

The pricing equation for an option $V(S,t)$ with an underlying equity S is given by

$$(BS) = \begin{cases} V = V(S),\ t = T \\ \frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0,\ S \in [0,\infty),\ t \in [0,T) \end{cases} \tag{2.1}$$

Since (BS) is defined on the half space $\Re_+$, the behavior of $V(S)$ for $S \rightarrow \infty$ is entirely implied by the initial value, or payoff, at t = T. However, in numerical computations the domain must be truncated and a proper boundary condition has then to be chosen. Since the payoff implies the behavior for large S, the choice of theboundary condition has to be as ”weak” as possible to avoid its influence on the solution on the whole domain. There are two ways to achieve this:

1. The use of pde-boundary conditions. In this approach the whole PDE is discretized on the boundary grid node. The main benefit of this approach is that we can use implicit time-integration methods.
2. The use of explicit time integration methods on grids that are tree -shaped. In each time-integration step the boundary grid nodes are ”stripped” from the solution vector as they do not contribute to the solution in all further time-integration steps.

These approaches allow for general solvers which can be used for Heston / Hull-White models as well. In literature the ”linearity” condition is often used at boundaries where no conditions are known. It is, however, easily seen that these conditions not need be valid. Take for example a power option. The payoff of this option is $V(S) = S^2$ and this is obviously not linear at the boundary.
One remark that we make now, but will be repeated throughout this thesis is that the problem is solved backward in time. So it is solved from T to t and therefore the timestep is negative.

We will start with the discretization of the Black-Scholes equation. The first and second derivative for the interior points (resp. $\frac{\partial V_i}{\partial S}$ and $\frac{\partial^2 V_i}{\partial S^2}$ for $i = 2, .., N$) will be approximated by a second order accurate finite difference scheme using functions in the points S_{i-1}, S_i, S_{i+1} ($V_i = V(S_i)$). Since the grid is not necessarily uniform (later on we might use grid-stretching), the first and second derivative will be approximated resp. by an adjusted central and standard three point method, to guarantee second order

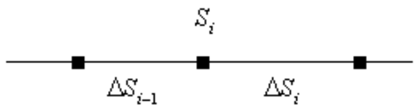

Figure 2.1: discretization of the interior points

accuracy.
The interior will be discretized using functions in the points $S_{i-1},\ S_i,\ S_{i+1}$ ($\Delta S_i = S_{i+1} - S_i$).

To attain second order accuracy the first derivative will be approximated by:

$$\frac{\partial V_i}{\partial S} = \alpha_{-1}^i V_{i-1} + \alpha_0^i V_i + \alpha_1^i V_{i+1}, \tag{2.2}$$
$$= \alpha_{-1}^i V(S_i - \Delta S_{i-1}) + \alpha_0^i V(S_i) + \alpha_1^i V(S_i + \Delta S_i). \tag{2.3}$$

The terms $V_{i-1} = V(S_i - \Delta S_{i-1})$ and $V_{i+1} = V(S_i + \Delta S_i)$ can be expanded in a Taylor serie:

$$V(S_i - \Delta S_{i-1}) = V_i - \Delta S_{i-1} V_i' + (\Delta S_{i-1})^2 \frac{1}{2} V_i'' - (\Delta S_{i-1})^3 \frac{1}{6} V_i''', \tag{2.4}$$
$$V(S_i + \Delta S_i) = V_i + \Delta S_i V_i' + (\Delta S_i)^2 \frac{1}{2} V_i'' + (\Delta S_i)^3 \frac{1}{6} V_i'''. \tag{2.5}$$

First the substitution of (2.4) and () in (2.3) is made. Thereafter the $\alpha_{-1}^i, \alpha_0^i, \alpha_1^i$ have to be chosen such that the first derivative is approximated with second order accuracy. Working out all terms it can be concluded that second order accuracy is reached if $\alpha_{-1}^i, \alpha_0^i, \alpha_1^i$ satisfy the following linear system:

$$\begin{bmatrix} 1 & 1 & 1 \\ -\Delta S_{i-1} & 0 & \Delta S_i \\ \frac{1}{2}\Delta S_{i-1}^2 & 0 & \frac{1}{2}\Delta S_i^2 \end{bmatrix} \begin{bmatrix} \alpha_{-1}^i \\ \alpha_0^i \\ \alpha_1^i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{2.6}$$

This procedure will be applied to determine the stencil for all following derivatives.
It is, however, not possible to use the central scheme for an explicit time-discretization scheme and in that case we will use upstream discretization. Since the problem we are investigating is solved backward in time upstream discretization is given by

$$\beta_i \frac{\partial V}{\partial S} = \begin{cases} \beta_i \frac{V_{i+1} - V_i}{\Delta S_i} & \beta_i > 0 \\ \beta_i \frac{V_i - V_{i-1}}{\Delta S_{i-1}} & \beta_i < 0 \end{cases} \tag{2.7}$$

This can be written in the following form

$$\frac{\partial V_i}{\partial S} = \alpha_{-1}^i V_{i-1} + \alpha_0^i V_i + \alpha_1^i V_{i+1}, \tag{2.8}$$
$$= \alpha_{-1}^i V(S_i - \Delta S_{i-1}) + \alpha_0^i V(S_i) + \alpha_1^i V(S_i + \Delta S_i), \tag{2.9}$$

with

$$\alpha_{-1}^{i} = -\frac{1}{2}\left(\frac{\beta_i}{\Delta S_i} - |\frac{\beta_n}{\Delta S_{i-1}}|\right), \tag{2.10}$$

$$\alpha_{1}^{i} = \frac{1}{2}\left(\frac{\beta_i}{\Delta S_i} + |\frac{\beta_n}{\Delta S_{i-1}}|\right), \tag{2.11}$$

$$\alpha_{0}^{i} = -\alpha_{-1}^{i} - \alpha_{1}^{i}. \tag{2.12}$$

The second derivative will be approximated by:

$$\frac{\partial^2 V_i}{\partial S^2} = \beta_{-1}^{i} V_{i-1} + \beta_0^i V_i + \beta_{-1}^{i} V_{i+1}, \tag{2.13}$$

second order accuracy is reached if the coefficients $\alpha_{-1}, \alpha_0, \alpha_1$ satisfy:

$$\begin{bmatrix} 1 & 1 & 1 \\ -\Delta S_{i-1} & 0 & \Delta S_i \\ \frac{1}{2}\Delta S_{i-1}^2 & 0 & \frac{1}{2}\Delta S_i^2 \end{bmatrix} \begin{bmatrix} \beta_{-1}^i \\ \beta_0^i \\ \beta_1^i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2.14}$$

Combined with the boundary conditions the following semi-discrete system is obtained.

$$\frac{du}{dt} + Au = 0, \tag{2.15}$$

where A is the discretization matrix of the spatial operator $L(V)$:

$$L(V) = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV, \tag{2.16}$$

Matrix A thus contains the discretization of the interior and the boundary points and is of the form:

$$A = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & \gamma_5 & 0 & \dots & 0 \\ a_2 & b_2 & c_3 & 0 & 0 & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & 0 & 0 & a_N & b_N & c_N \\ 0 & \dots & 0 & \epsilon_{N-4} & \epsilon_{N-3} & \epsilon_{N-2} & \epsilon_{N-1} & \epsilon_N \end{bmatrix} \tag{2.17}$$

where $a_i, b_i \; and \; c_i, \; i = 2, .., N$ depends on the interior discretization and γ_1 to γ_5 and ϵ_{N-4} to ϵ_N depend on the boundary conditions used, which we will discuss in the next section.

2.1.1 Implicit time method with the pde-boundary condition

First of all the pde boundary condition, in which the equation itself is a boundary condition, will be examined. We will investigate the behavior of the pde-boundary condition up to a second order one sided difference scheme at the boundary. First of all the finite difference schemes will be presented. Starting at the left boundary and making use of functions in the points $S_i,\ S_{i+1},\ S_{i+2},\ S_{i+3}$ the first order first derivative will be approximated by:

$$\frac{\partial V_1}{\partial S} = \alpha_1^1 V_2 + \alpha_0^1 V_1, \tag{2.18}$$

where α_1^1, α_0^1 satisfy the linear system:

$$\begin{bmatrix} 1 & 1 \\ \Delta S_1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1^1 \\ \alpha_0^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.19}$$

and the second order first derivative by:

$$\frac{\partial V_1}{\partial S} = \alpha_2^1 V_3 + \alpha_1^1 V_2 + \alpha_0^1 V_1, \tag{2.20}$$

where $\alpha_2^1, \alpha_1^1, \alpha_0^1$ satisfy:

$$\begin{bmatrix} 1 & 1 & 1 \\ (\Delta S_1 + \Delta S_2) & \Delta S_1 & 0 \\ \frac{1}{2}(\Delta S_1 + \Delta S_2)^2 & \frac{1}{2}\Delta S_1^2 & 0 \end{bmatrix} \begin{bmatrix} \alpha_2^1 \\ \alpha_1^1 \\ \alpha_0^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{2.21}$$

The first order second derivative is approximated by:

$$\frac{\partial^2 V_1}{\partial S^2} = \alpha_2^1 V_3 + \alpha_1^1 V_2 + \alpha_0^1 V_1, \tag{2.22}$$

where $\alpha_2^1, \alpha_1^1, \alpha_0^1$ satisfy:

$$\begin{bmatrix} 1 & 1 & 1 \\ (\Delta S_1 + \Delta S_2) & \Delta S_1 & 0 \\ \frac{1}{2}(\Delta S_1 + \Delta S_2)^2 & \frac{1}{2}(\Delta S_1)^2 & 0 \end{bmatrix} \begin{bmatrix} \alpha_2^1 \\ \alpha_1^1 \\ \alpha_0^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2.23}$$

and the second order second derivative by:

$$\frac{\partial^2 V_1}{\partial S^2} = \beta_3^1 V_4 + \beta_2^1 V_3 + \beta_1^1 V_2 + \beta_0^1 V_1, \tag{2.24}$$

where $\alpha_3^1, \alpha_2^1, \alpha_1^1, \alpha_0^1$ satisfy:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ (\Delta S_1 + \Delta S_2 + \Delta S_3) & (\Delta S_1 + \Delta S_2) & \Delta S_1 & 0 \\ \frac{1}{2}(\Delta S_1 + \Delta S_2 + \Delta S_3)^2 & \frac{1}{2}(\Delta S_1 + \Delta S_2)^2 & \frac{1}{2}\Delta S_1^2 & 0 \\ \frac{1}{6}(\Delta S_1 + \Delta S_2 + \Delta S_3)^3 & \frac{1}{6}(\Delta S_1 + \Delta S_2)^3 & \frac{1}{6}\Delta S_1^3 & 0 \end{bmatrix} \begin{bmatrix} \alpha_3^1 \\ \alpha_2^1 \\ \alpha_1^1 \\ \alpha_0^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{2.25}$$

The discretization at the right boundary can be done in a similar way as for the left boundary only now by taking functions in the points $S_N, S_{N-1}, S_{N-2}, S_{N-3}, S_{N-4}$

The different boundary conditions will be referred to as:

- pde1: $O(h^2)$ discretization for first and second derivative
- pde2: first derivative: $O(h^2)$, second derivative: $O(h)$
- pde3: first derivative: $O(h)$, second derivative: $O(h^2)$
- pde4: first derivative: $O(h)$, second derivative: $O(h)$
- pde5: first: $O(h)$, second derivative zero (linear boundary condition)
- pde6: first: $O(h^2)$, second derivative zero (linear boundary condition)

Once the semi-discrete system is known, the extension to the full discrete system can be made.
The semi-discrete system is given by:

$$\frac{du}{dt} + Au = 0, \tag{2.26}$$

The full-discrete equation (using an ω-scheme) is given by:

$$u^{n+1} = (I + dt\omega A)^{-1}((I - dt(1-\omega)A)u^n + dt\omega f^{n+1} + dt(1-\omega)f^n),\ \omega \in [0,1]. \tag{2.27}$$

For the stability analysis we will follow the article of *Vetzal, Forsyth and WindCliff* . A legitimate discretisation of the spatial operator $L(V)$ (2.16) has the properties that if λ_i is an eigenvalue of the matrix A then

1. Case $q \geq 0$: All of the eigenvalues must satisfy $Re(\lambda_i) \leq 0$.
2. Case $q < 0$: There is at most a single index ρ for which $Re(\lambda_\rho > 0)$.

In the case of a pde boundary condition vector $f = 0$, B then becomes $B = (I + dt\omega A)^{-1}(I - dt(1-\omega)A)$. For strict stability it is then required that:

$$||B|| \leq 1. \tag{2.28}$$

($|| \cdot ||$ is the spectral norm)

Later on it will be seen that (2.28) does not necessarily hold for the pde boundary and therefore will lead to unstabilities.
In Figures (2.2), (2.4) and (2.6) the numerical solutions, solved on a bigger and bigger grid, of a European call option with parameters $r = 0.05, \sigma = 0.5, q = 0., K = 100, T = 5$ are plotted.
In Figures (2.3), (2.5) and (2.7) the behavior of a European put option is shown for increasing S_{max}.
It can be seen that as S_{max} becomes larger that the numerical solution (solid blue line) converges toward the exact solution (dashed black line)

However we are mainly interested in the error in the point $S = K$. Table (2.1) shows the absolute error in $S = K$ for the pde boundary conditions and increasing maturity times. The number of timesteps is equal to $40 * T$, $S_{max} = 1000$ and the number of space steps is 500.

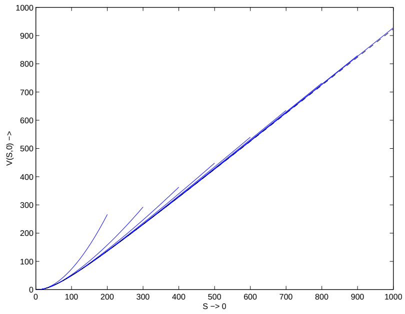

Figure 2.2: solid blue line: numerical solution with pde3, dashed black line: exact solution

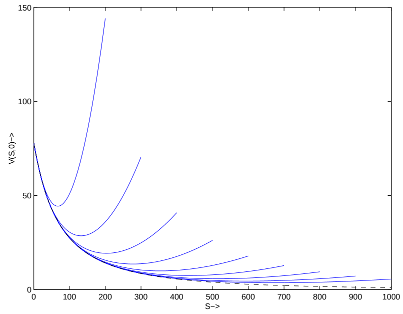

Figure 2.3: solid blue line: numerical solution with pde3, dashed black line: exact solution

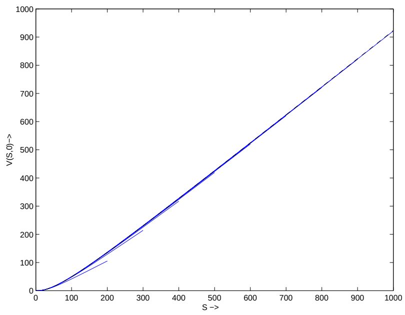

Figure 2.4: solid blue line: numerical solution with pde2, dashed black line: exact solution

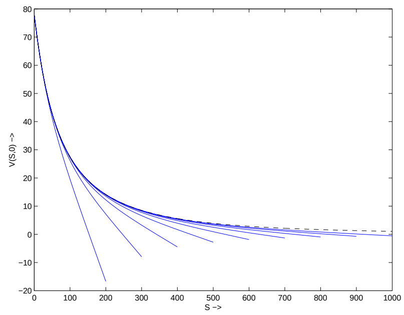

Figure 2.5: solid blue line: numerical solution with pde2, dashed black line: exact solution

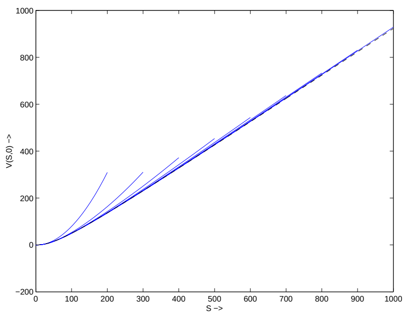

Figure 2.6: solid blue line: numerical solution with pde1, dashed black line: exact solution

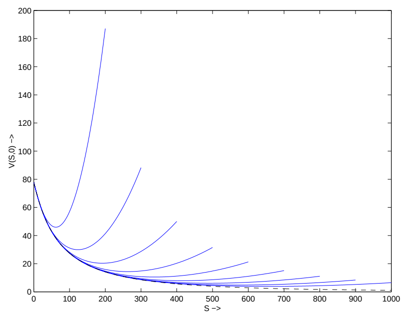

Figure 2.7: solid blue line: numerical solution with pde1, dashed black line: exact solution

In Table (2.1) it can be seen that more or less accurate results are obtained with the pde boundary conditions. However, to achieve these results the domain has to be very large, something we would like to avoid.

The differences between the exact and numerical solutions are caused by the pde-boundary-conditions. There are three main points which need to be explored in order to get more accurate results. One point where inaccuracies rise up is at the right boundary. Suppose that $r - q > 0$ then the convection term at the right boundary is positive and since the problem is solved backward in time, this means that flow is to the left. However, we take one-sided differences as though flow is to the right.
Another point of worry is the discretization matrix A. The discretization is non-legitimate and this causes positive eigenvalues, where as for stability for the ω-scheme no positive eigenvalues may occur. The non-legitimate discretization is due to the pde-boundary condition.
The last point is also related to the pde boundary conditions. One-sided differences are used here, but

T	$exact$	$pde6$	$pde5$	$pde4$	$pde3$	$pde2$
5	28.1582	28.1549	28.1549	28.1594	28.1549	28.1549
10	41.5022	41.5001	41.5001	41.5001	41.5001	41.5001
15	51.4771	51.4755	51.4755	51.4775	51.4755	51.4756
20	59.3879	59.3863	59.3863	59.3873	59.3862	59.3875
25	65.8239	65.8207	65.8207	65.8285	65.8204	65.8306
30	71.1346	71.1256	71.1256	71.1570	71.1246	71.1686
35	75.5552	75.5340	75.5340	75.6234	75.5318	75.6666
40	79.2573	79.2154	79.2154	79.4177	79.2115	79.5450
45	82.3709	82.2997	82.2997	82.6926	82.2935	83.0110
50	84.9981	84.8891	84.8891	85.5747	84.8805	86.2858

Table 2.1: $K = 100, r = 0.03, \sigma = 0.25$

was does this in fact mean.

$$\frac{\partial V_1}{\Delta S} = \frac{V_2 - V_1}{\Delta S} = \frac{V_2 - V_0}{2\Delta S} + \frac{\Delta S}{2}\frac{V_2 - 2V_1 + V_0}{(\Delta S)^2}, \tag{2.29}$$

where V_0 is a virtual point. Actually some more diffusion is added to the problem at the left boundary and at the right boundary diffusion is subtracted. This may also have influence on the solution. The second order first derivative at the left boundary can also be rewritten

$$\frac{\partial V_1}{\Delta S} = \frac{V_2 - V_1}{\Delta S} = \frac{V_2 - V_0}{2\Delta S} + \frac{(\Delta S)^2}{2}\frac{-V_3 + 3V_2 - 3V_1 - V_0}{(\Delta S)^3}, \tag{2.30}$$

the last term is a numerical approximation to minus the third derivative. So in this case extra dispersion is added to the problem.
Taking a look at the first order second derivative yields

$$\frac{\partial^2 V_1}{\partial S^2} = \frac{V_3 - 2V_2 + V_1}{(\Delta S)^2} = \frac{V_2 - 2V_1 + V_0}{(\Delta S)^2} + (\Delta S)\frac{V_3 - 3V_2 + 3V_1 - V_0}{(\Delta S)^3}, \tag{2.31}$$

where the last term is a numerical approximation to the third derivative. So also here extra dispersion is added to the problem.
Adding extra diffusion or dispersion to the problem may also cause inaccurate results. Therefore we try to set up a solver which does not depend on the boundary condition. This approach of solving with an explicit method on a tree-structured grid will be discussed in the next section.

2.1.2 Explicit time methods solved on a tree structured grid

Solving the partial differential equation explicitly on a tree mesh means that we solve the equation back to one space point as illustrated in figure (2.8)

For the first derivative in point 1 we need functions in the points 2,3,4, but certainly not in point 0. The solution is thus not influenced by the boundary conditions. The method applied to solve the Black-Scholes equation on a tree-structured grid is:

- Take an equidistant grid from S_{min} to S_{max}

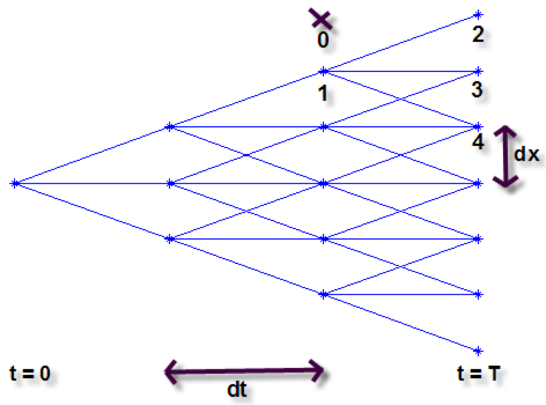

Figure 2.8: Eigenfunction

- For the convective part, upstream discretization is used, because the central scheme cannot be used in combination with an explicit method.
- Explicit Euler forward is used for the time integration and the number of timestep is smaller then $\frac{K-S_{min}}{\Delta S}$. The problem is not solved back to one point, but toward the smallest interval which still contains the point K (instead of K we can also take another point of interest)
- Interpolation in the point K gives the desired result.
- To make the method stable the criterion $|1-\lambda_i dt| < 1$ has to be satisfied, with λ_i eigenvalues of the discretization matrix A. It seems very hard to fulfill this criterion, which is a drawback for this method.

Using Gerschgorin an upper- and lowerbound for the eigenvalues can be found. Defining

$$\alpha_i = -\frac{1}{2}(\frac{\beta_i}{\Delta S} - |\frac{\beta_i}{\Delta S}|) + \frac{f_i}{\Delta S^2},\ i = 2, .., N \tag{2.32}$$

$$\gamma_i = \frac{1}{2}(\frac{\beta_i}{\Delta S} + |\frac{\beta_i}{\Delta S}|) + \frac{f_i}{\Delta S^2}\ i = 2, .., N \tag{2.33}$$

with

$$\beta_i = (r-q)S_i, \tag{2.34}$$

$$f_i = \frac{1}{2}\sigma^2 S_i^2. \tag{2.35}$$

Then the following inequality has to be fulfilled to get a stable method

$$-2(\alpha_i + \gamma_i) - r > \frac{2}{\Delta t}. \tag{2.36}$$

Substituting (2.32) in (2.36) and assuming $r - q > 0$ (which usually is the case) gives

$$\frac{(r-q)S_i}{\Delta S} + \frac{\sigma^2 S_i^2}{\Delta S^2} - r < -\frac{1}{\Delta t}. \tag{2.37}$$

It will be shown that restriction (2.37) is hard to satisfy. This will be due to the use of the Explicit Euler method and that is why other methods will be applied such as Runge-Kutta-Chebyshev methods (but this is one of the future goals).
The most ideal case would be if the point of interest is the center of the computational domain (as a consequence $\Delta t = \frac{-T}{Q} = \frac{-T}{2N}$, where Q is the number of timesteps and N the number of spacesteps). All other cases are worse and it will be shown that it is already hard to satisfy the restriction for the most ideal case.

$$\frac{(r-q)S_i}{\Delta S} + \frac{\sigma^2 S_i^2}{\Delta S^2} - r \leq -\frac{1}{\Delta t}, \tag{2.38}$$
$$\frac{(r-q)S_i}{\Delta S} + \frac{\sigma^2 S_i^2}{\Delta S^2} - r \leq \frac{2N}{T}. \tag{2.39}$$
$$\tag{2.40}$$

$S_i = S_{min} + (i-1)\Delta S$ and $\Delta S = \frac{S_{max}-S_{min}}{N}$.
If the above equation holds for $N+1$ then it holds for all $i = 1, .., N+1$.

$$\frac{(r-q)(S_{min}+N\Delta S)}{\Delta S} + \frac{\sigma^2 (S_{min}+N\Delta S)^2}{\Delta S^2} - r \leq \frac{2N}{T}$$
$$\frac{(r-q)S_{min}}{\Delta S} + (r-q)N + \frac{\sigma^2 S_{min}}{\Delta S^2} + \frac{2\sigma^2 S_{min} N}{\Delta S} + \sigma^2 N^2 - r \leq \frac{2N}{T}$$
$$\frac{(r-q)S_{min}N}{S_{max}-S_{min}} + (r-q)N + \frac{\sigma^2 S_{min} N^2}{(S_{max}-S_{min})^2} + \frac{2\sigma^2 S_{min} N}{S_{max}-S_{min}} + \sigma^2 N^2 - r - \frac{2N}{T} \leq 0$$
$$(\frac{2\sigma^2 S_{min}}{S_{max} S_{min}} + \sigma^2 + \frac{\sigma^2 S_{min} N^2}{(S_{max}-S_{min})^2})N^2 + (\frac{(r-q)S_{min}}{S_{max}-S_{min}} + (r-q) - \frac{2}{T})N - r \leq 0$$

This inequality can be solved and it gives an upperbound for the number of space steps N such that the method is certainly stable:

$$N \leq -\frac{1}{2}a + \sqrt{(\frac{1}{2}a)^2 - b} \tag{2.41}$$
$$a = \frac{r - q - \frac{2}{T} + \frac{(r-q)S_{min}}{(S_{max}-S_{min})}}{\frac{2\sigma^2 S_{min}}{S_{max}-S_{min}} + \sigma^2 + \frac{\sigma^2 S_{min}}{(S_{max}-S_{min})^2}} \tag{2.42}$$
$$b = \frac{-r}{\frac{2\sigma^2 S_{min}}{S_{max}-S_{min}} + \sigma^2 + \frac{\sigma^2 S_{min}}{(S_{max}-S_{min})^2}} \tag{2.43}$$

This upperbound is a very sharp upperbound. So the number of spacesteps can be taken somewhat bigger, but not much.
For a European call option with $r = 0.03, q = 0, \sigma = 0.25$ the upperbound (2.41) can be plotted as a function of S_{min} and S_{max} (see Figure (2.9)).

From Figure (2.9) it follows that it does not matter how big the interval is taken, the maximal N will always be 6, which is a strict restraint and will lead to inaccurate results.
This example shows that Euler forward is not a very good time-discretization method in this case and there is a need for better suited time-discretization methods (It is suggested that the Runge-Kutta-Chebyshev method might solve our problem. So it will be one of the future research topics to investigate this).

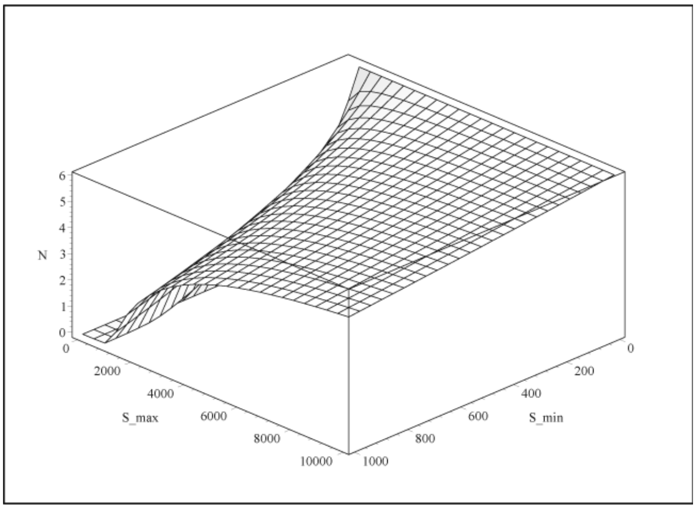

Figure 2.9: upperbound for number of space steps

2.2 Interest rate model

The next model that will be examined is the one-dimensional Hull-White model. The partial differential equation for this model is given by

$$\frac{\partial V}{\partial t} + (\Theta(t) - ar)\frac{\partial V}{\partial r} + \frac{1}{2}\sigma_r^2\frac{\partial^2 V}{\partial r^2} - rV = 0 \tag{2.44}$$

First we will derive an exact solution for a zero coupon bond under this model, which will be used as a benchmark for the numerical solution. A problem for the numerical computation of the zero coupon bond under the Hull-White model is that the boundary conditions are not known. To deal with this problem we will use an explicit method on a tree structure, or an implicit method with the whole equation as a boundary condition to solve the problem. These methods will be compared and results will be given. The approaches may give inaccurate results, which are caused by the $' - rV'$ term, since r can either be positive or negative and the solution grows exponentially fast for small r. Therefore we like the scale out this term, which can be done by making use of the exact solution of the zero coupon bond. In the next sections we will discuss all the details mentioned above.

2.2.1 Exact solution for the *zero coupon bond* under the *Hull-White* interest rate model

In this section an exact solution for a simple zero coupon bond under the Hull-White interest rate model is derived. The exact solution will be used as a reference for the numerical results and later on this exact solution will be used for the transformation to scale out the $' - rV'$ term.
A simple zero coupon bond is a contract for which the holder pays a certain premium at the beginning of the period and receives 1 at the end of that period. The pricing equation for the zero coupon bond under the Hull-White interest rate model is given by (2.44).
We assume that the solution of (2.44) can be written in the form

$$V(r,t) = e^{A(t)+rB(t)}, \tag{2.45}$$

with final condition

$$V(r,T) = 1.$$

Substitution of (2.45) in (2.44) gives

$$\begin{aligned}&(A'(t) + rB'(t))V(r,t) + (\Theta(t) - a(t)r)B(t)V(r,t),\\&+\frac{1}{2}\sigma_r^2(t)B^2(t)V(r,t) - rV(r,t) = 0.\end{aligned} \tag{2.46}$$

Rearranging terms gives

$$V(r,t)([A'(t) + \Theta(t)B(t) + \frac{1}{2}\sigma_r^2(t)B^2(t)] + r[B'(t) - aB(t) - 1]) = 0,$$

and since this must hold for any $V(r,t)$ it follows that

$$[A'(t) + \Theta(t)B(t) + \frac{1}{2}\sigma_r^2(t)B^2(t)] + r[B'(t) - aB(t) - 1] = 0.$$

This equation is valid for all r and so both terms between brackets have to be zero, which leads to the system with two unknowns $A(t)$ and $B(t)$

$$A'(t) + \Theta(t)B(t) + \frac{1}{2}\sigma_r^2(t)B^2(t) = 0, \tag{2.47}$$

$$B'(t) - a(t)B(t) - 1 = 0. \tag{2.48}$$

The final condition for $V(r,T)$ is given by (2.46). Since neither $A(t)$ nor $B(t)$ are functions of r it follows that $A(t)$ and $B(t)$ are both zero on T.

$$A(T) = 0, \tag{2.49}$$

$$B(T) = 0. \tag{2.50}$$

Using an appropriate integrating factor, the solution of (2.48) is given by

$$B(t) = \frac{-\int_t^T e^{\int_u^T a(s)ds} du}{e^{\int_t^T a(s)ds}}. \tag{2.51}$$

Equation (2.47) can be solved by using $-\int_t^T A'(t)dt = A(t) - A(T) = A(t)$ (the last equivalence holds due to the endcondition (2.49): $A(T) = 0$). So $A(t)$ can be computed by rearranging (2.47) to $A'(t) = -\Theta(t)B(t) - \frac{1}{2}\sigma_r^2 B^2(t)$ and integrating from T to t.

$$A(t) = -\int_t^T [-\Theta(s)B(s) - \frac{1}{2}\sigma_r^2(s)B^2(s)]ds. \tag{2.52}$$

(2.53)

The exact solution of the zero coupon bond under the Hull-White model (2.44) is given by

$$\begin{aligned} V(r,t) &= e^{A(t)+rB(t)}, && (2.54)\\ B(t) &= \frac{-\int_t^T e^{\int_u^T a(s)ds}du}{e^{\int_t^T a(s)ds}}, && (2.55)\\ A(t) &= -\int_t^T [-\Theta(s)B(s) - \frac{1}{2}\sigma_r^2(s)B^2(s)]ds. && (2.56)\end{aligned}$$

2.2.2 Numerical solution for the $zero\ coupon\ bond$ under the $Hull-White$ interest rate model

The Hull-White interest model is given by (2.44). The spatial operator $L(V)$ is defined by:

$$L(V) = (\Theta(t) - ar)\frac{\partial V}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial r^2} - rV. \tag{2.57}$$

At first we concentrate on the discretization of the spatial operator. The convection term can either by positive or negative changing sign at $\Theta(t) - ar = 0$ and therefore we use upstream discretization for the convection part. The problem is solved backward in time and therefore the upstream discretization is given by:

$$\beta\frac{\partial u}{\partial r} = \begin{cases} \beta\frac{u_{i+1}-u_i}{\Delta r_i} & \beta > 0\\ \beta\frac{u_i-u_{i-1}}{\Delta r_{i-1}} & \beta < 0\end{cases} \tag{2.58}$$

For the diffusion part we will use the three point stencil and since a uniform grid is used ($\Delta r_n = \Delta r_{n-1} = \Delta r$)
We define

$$\begin{aligned} \alpha_i &= -\frac{1}{2}(\frac{\beta_i}{\Delta r} - |\frac{\beta_i}{\Delta r}|) + \frac{f_i}{\Delta r^2},\ i = 2,..,N && (2.59)\\ \gamma_i &= \frac{1}{2}(\frac{\beta_i}{\Delta r} + |\frac{\beta_i}{\Delta r}|) + \frac{f_i}{\Delta r^2}\ i = 2,..,N && (2.60)\end{aligned}$$

with

$$\begin{aligned} \beta_i &= \Theta(t) - ar_i,\ i = 2,..,N && (2.61)\\ f_i &= 0.5\sigma^2,\ i = 2,..,N && (2.62)\end{aligned}$$

As suggested by Hull-White we choose $\Delta r = \sqrt{n\sigma^2\Delta t},\ n \in N$. This gives the following discretization matrix for the spatial operator (2.57)

$$A = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & 0 & 0 & 0\\ \alpha_2 & -\alpha_2-\beta_2-r_2 & \beta_2 & 0 & 0 & 0 & 0\\ 0 & \alpha_3 & -\alpha_3-\beta_3-r_3 & \beta_3 & 0 & 0 & 0\\ 0 & 0 & \ddots & \ddots & \ddots & 0 & 0\\ 0 & 0 & 0 & \alpha_{N-1} & -\alpha_{N-1}-\beta_{N-1}-r_{N-1} & \beta_{N-1} & 0\\ 0 & 0 & 0 & 0 & \alpha_N & -\alpha_N-\beta_N-r_N & \beta_N\\ 0 & 0 & 0 & \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4\end{bmatrix} \tag{2.63}$$

(where γ_1 till γ_4 and ϵ_1 till ϵ_4 are depending on the type of discretization used at the boundary) It can directly be seen that, since r can either be positive or negative, diagonal dominance is not guaranteed. To what extent this will cause trouble will be examined in the next sections, where we investigate the use of an explicit time method where we solve on a tree structure and the implicit time method, where we discretize the whole equation as a boundary condition.

2.2.3 Implicit Time method with pde boundary conditions

The implicit method we use is the ω-scheme with $\omega = 0.5$ (actually this is not an implicit scheme, but an implicit-explicit scheme), which is certainly stable as long as the eigenvalues of the spatial operator are all smaller or equal than zero. Since the discretization matrix (2.63) is not diagonally dominant we can not estimate the eigenvalues with Gerschgorin to be smaller or equal than zero. As a consequence we can not conclude that the ω-scheme will always be stable. However, taking a look at the eigenvalues of the discretization matrix A shows that these eigenvalues are less then zero as long as a is not too small.
Another point of care are the boundary conditions. In the previous chapter these conditions were analysed for the Black-Scholes equation and one observation was the fact that at the right boundary the flow was to the left while we take one-sided differences at this boundary as though flow is to the right. This was a possible explanation for the strange behavior at this boundary.
Taking a look at the spatial operator (2.57) one can see that the flow at the boundaries will be in the same direction as the one-sided differences, taken at that boundary, as long as the following two condition are fulfilled

$$r_{min} < \frac{\Theta}{a}, \tag{2.64}$$
$$r_{max} > \frac{\Theta}{a}. \tag{2.65}$$

The first inequality will always be satisfied, since Θ and a are both positive, but the second inequality gives an upperbound for the spatial step that can be taken.

$$\begin{aligned} r_{max} &> \frac{\Theta}{a}, \\ Q\Delta r &> \frac{\Theta}{a}, \\ -Q^2 n \Delta t \sigma^2 &> \frac{\Theta^2}{a^2}, \\ n &> \frac{\Theta^2}{a^2 \sigma^2 T Q}, \end{aligned}$$

therefore n will be taken equal to $\lceil \frac{\Theta^2}{a^2\sigma^2 TQ} \rceil$ in this case

2.2.4 Numerical results (zero coupon bond)

By means of a number of examples we will investigate the accuracy of this method. We again consider six types of boundary conditions

- pde1: $O(h^2)$ discretization for first and second derivative
- pde2: first derivative: $O(h^2)$ second derivative: $O(h)$
- pde3: first derivative: $O(h)$ second derivative: $O(h^2)$

- pde4: first derivative: $O(h)$ second derivative: $O(h)$
- pde5: first derivative: $O(h)$ second derivative: 0
- pde6: first derivative: $O(h^2)$ second derivative: 0

In Figures (2.10) to (2.13) the quotient of the exact and numerical solution is plotted. The used parameters are : $a = 0.05,\ \Theta = 0.025,\ \sigma = 0.01$.

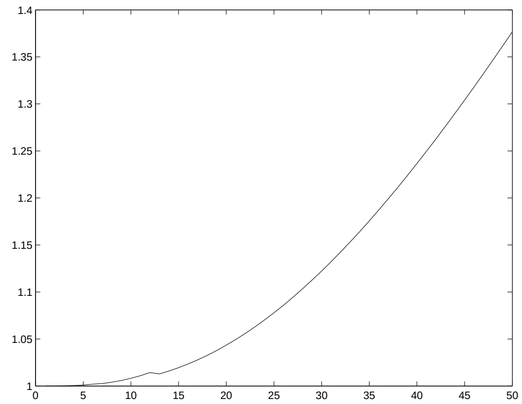

Figure 2.10: quotient exact/ numerical solution for the zero coupon bond with pde1

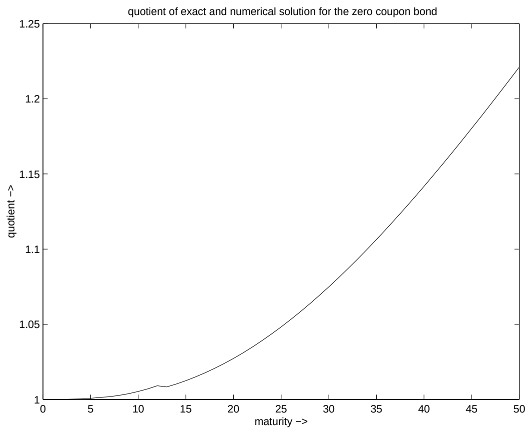

Figure 2.11: quotient exact/ numerical solution for the zero coupon bond with pde2

2.2.5 Numerical results (caplets)

A Caplet is a particular type of European option whose underlying is the curve of interest rates. It can also be seen as a call option on the short rate. The Caplet is a contract which protects the buyer from paying to much interest in the future. At time T_f the forward interest rate for the period T_S to T_e is defined. If this rate becomes to high the caplet will guarantee a payoff, making up for the high forward interest rate one has to pay. If future interest rate is low then nothing will happen, but one also will not pay to much interest in the future. The Payoff at T_f for a Caplet $Caplet(t, T_f, T_S, T_e, \tau, K)$ is given by

$$Caplet(T_f, T_f, T_S, T_e, \tau, K) = (P(T_f, T_S) - (1 + \tau K)P(T_f, T_e))^+, \tag{2.66}$$

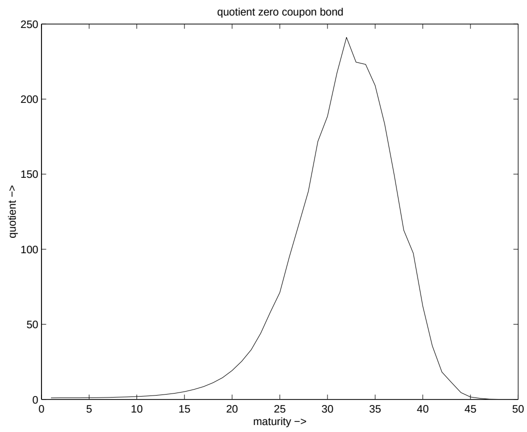

Figure 2.12: quotient exact/ numerical solution for the zero coupon bond with pde3

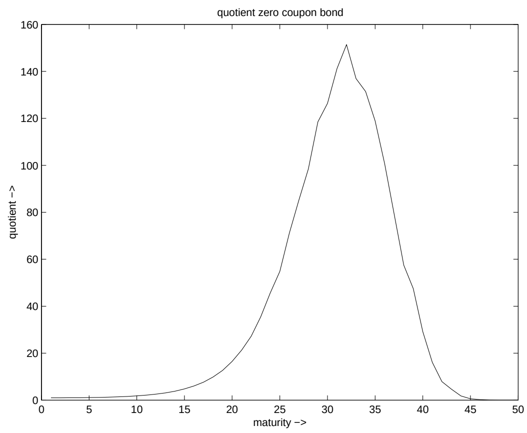

Figure 2.13: quotient exact/ numerical solution for the zero coupon bond with pde4

where $P(t,T)$ is the solution of the zero coupon bond and τ the day count fraction.
In [5] the exact solution for the Caplet is derived and this exact solution will be used as a benchmark to validate the numerical results. The problem is solved implicitly with the pde boundary condition pde3 . Exact and numerical solutions for the Caplet with parameters: $\Theta = 0.25,\ a = 0.05,\ \sigma = 0.01,\ K = 0.06,\ T_f = 5,\ T_S = 5,\ T_e = 6$ are plotted in figure (2.14) and (2.15).

In (2.16) and (2.17) the exact and numerical value of a Caplet with parameters: $\Theta = 0.025,\ a = 0.05,\ \sigma = 0.01,\ K = 0.06,\ T_f = 5,\ T_S = 5,\ T_e = 6$ are plotted.

In table (2.2) we compare the exact and numerical values of some Caplets.

2.2.6 Solving on a tree structured mesh with Euler forward

The results for the implicit method with the natural boundary condition are not satisfactory and therefore we try an explicit approach. We will first discuss the stability conditions for this method.
A is the discretization matrix of the spatial operator (2.57). The eigenvalues λ_i of A and the eigenvalues Λ_i of the full-discretization matrix $B(= (I - dtA))$ are related by

$$\Lambda_i = 1 - \Delta\lambda_i \tag{2.67}$$

For stability it must hold that

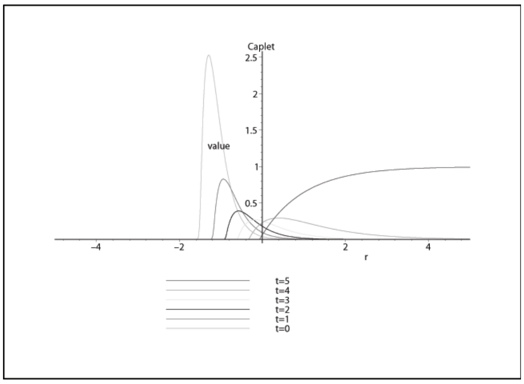

Figure 2.14: exact solution

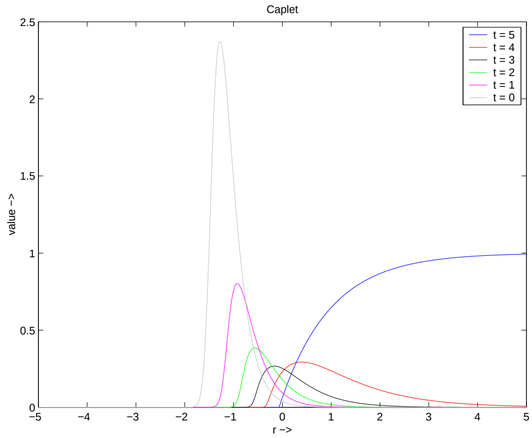

Figure 2.15: numerical solution

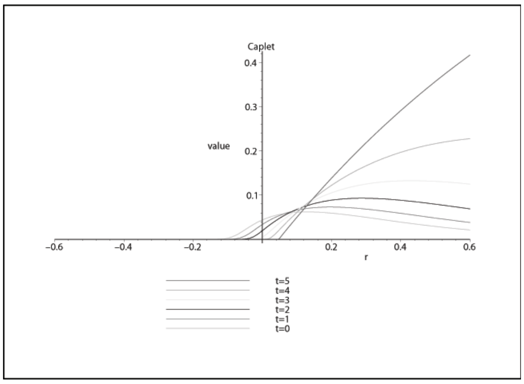

Figure 2.16: exact solution

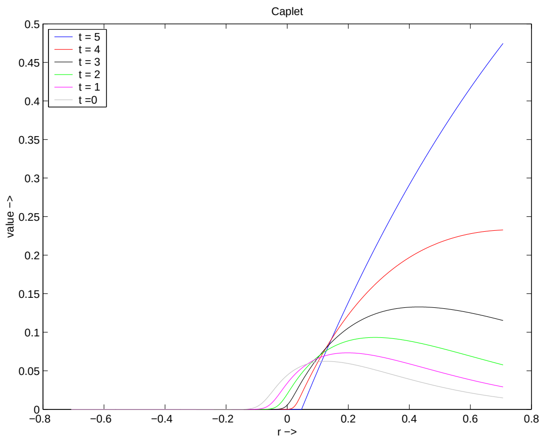

Figure 2.17: numerical solution

T_f	T_S	T_e	Θ	$exact$	$numerical$	σ	a	n
5	5	6	0.025	0.0440	0.0438	0.01	0.05	3
5	5	10	0.025	0.2958	0.2958	0.01	0.05	3
10	10	11	0.025	0.0459	0.0460	0.01	0.05	2
10	10	15	0.025	0.2016	0.2030	0.01	0.05	2
15	15	16	0.025	0.0207	0.0209	0.01	0.05	1
15	15	20	0.025	0.0770	0.0777	0.01	0.05	1
20	20	21	0.025	0.0062	0.0062	0.01	0.05	1
20	20	25	0.025	0.0210	0.0209	0.01	0.05	1
30	30	31	0.025	0.0003	0.0003	0.01	0.05	1

Table 2.2: number of time steps is 200

$$|\Lambda_i| \le 1$$

which is the case if

$$\frac{2}{\Delta t} \le \lambda_i \le 0 \qquad (2.68)$$

T_f	T_S	T_e	Θ	$exact$	$numerical$	σ	a	
5	5	6	0.25	0.0382	0.0388	0.01	0.05	50
5	5	10	0.25	0.0562	0.0574	0.01	0.05	50
10	10	11	0.25	$2 \cdot 10^{-5}$	$2.3 \cdot 10^{-5}$	0.01	0.05	25
10	10	15	0.25	$2.4\ \dot{1}0^{-5}$	$2.7 \cdot 10^{-5}$	0.01	0.05	25
15	15	16	0.25	$2.1 \cdot 10^{-10}$	$3.0 \cdot 10^{-10}$	0.01	0.05	17
15	15	20	0.25	$2.3 \cdot 10^{-10}$	$3.2 \cdot 10^{-10}$	0.01	0.05	17

Table 2.3: number of time steps = 1000

(N.B. Δt is negative)
Discretization of (2.57) may lead to problems due to the term $' - rV'$, since r can either be positive and negative. To avoid such problems equation (2.57) will be transformed to a convection-diffusion kind of problem. This means that due to the transformation the $' - rV'$ term will drop out.

Transforming the 1-dimensional Hull-White equation For the transformation of (2.44) we take $V = V_{sol}\widetilde{V}$, where V_{sol} is the solution for the zero-coupon bond under the Hull-White model and $\widetilde{V}$ is the new variable which we should solve. Replacing V with $V_{sol}\widetilde{V}$ in (2.44) yields

$$\begin{aligned} & V_{sol}\frac{\partial \widetilde{V}}{\partial t} + \widetilde{V}\frac{\partial V_{sol}}{\partial t} + (\Theta(t) - ar)(V_{sol}\frac{\partial \widetilde{V}}{\partial r} + \widetilde{V}\frac{\partial V_{sol}}{\partial r}) \\ & + \frac{1}{2}\sigma_r^2(V_{sol}\frac{\partial^2 \widetilde{V}}{\partial r^2} + 2\frac{\partial V_{sol}}{\partial r}\frac{\partial \widetilde{V}}{\partial r} + \widetilde{V}\frac{\partial^2 V_{sol}}{\partial r^2}) \\ & - rV_{sol}\widetilde{V} = 0. \end{aligned}$$

Rearranging terms gives

$$\begin{aligned} & V_{sol}\frac{\partial \widetilde{V}}{\partial t} + (\Theta(t) - ar + \sigma_r^2\frac{\partial V_{sol}}{\partial r})\frac{\partial \widetilde{V}}{\partial r} + \frac{1}{2}\sigma_r^2 V_{sol}\frac{\partial^2 \widetilde{V}}{\partial r^2} \\ & + \widetilde{V}(\frac{\partial V_{sol}}{\partial t} + (\Theta - a(t)r)\frac{\partial V_{sol}}{\partial r} + \frac{\partial^2 V_{sol}}{\partial r^2} - rV_{sol}) = 0. \end{aligned} \tag{2.69}$$

And since V_{sol} is the zero-coupon bond, which satisfies

$$\frac{\partial V_{sol}}{\partial t} + (\Theta(t) - ar)\frac{\partial V_{sol}}{\partial r} + \frac{\partial^2 V_{sol}}{\partial r^2} - rV_{sol} = 0. \tag{2.70}$$

Equation (2.69) becomes

$$V_{sol}\frac{\partial \widetilde{V}}{\partial t} + (\Theta(t) - ar + \sigma_r^2\frac{\partial V_{sol}}{\partial r})\frac{\partial \widetilde{V}}{\partial r} + \frac{1}{2}\sigma_r^2 V_{sol}\frac{\partial^2 \widetilde{V}}{\partial r^2} = 0. \tag{2.71}$$

The solution for the zero coupon bond is given by

$$V_{sol} = e^{A(t)+rB(t)}$$

and substituting this in (2.70) leads to the convection-diffusion equation

$$\frac{\partial \widetilde{V}}{\partial t} + (\Theta(t) - ar + \sigma_r^2 B(t))\frac{\partial \widetilde{V}}{\partial r} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 \widetilde{V}}{\partial r^2} = 0. \tag{2.72}$$

Using Gerschgorin it can be shown that when upstream discretization for convection and the standard three point method for diffusion are used, which are used here, the right part of (2.68) is always fulfilled. As proposed by Hull and White we take $\Delta r = \sqrt{n\sigma^2 \Delta t}$. They also suggest to take $n = 3$, for the speed of convergence, but in this case n (together with Δt) has to be chosen in such a way that the left part of (2.68) is fulfilled. The left-hand-side of (2.68) is satisfied if

$$|\frac{\Theta(t) - ar + \sigma^2 B(t)}{\Delta r}| + \frac{\sigma^2}{\Delta r} \leq \frac{-1}{\Delta t}, \ \forall r, \ B(t), \ a, \ \Theta, \ \sigma \tag{2.73}$$

This inequality is derived as follows.
The method is stable if (2.68) is satisfied. The right part of this equation is always satisfied, so we only have to worry about the left part, which says that λ_i has to be greater then $\frac{2}{\Delta t}$.
The eigenvalues of matrix A can be estimated with Gerschgorin given a lower bound for the eigenvalues:

$$\lambda_{lowerbound} \geq -2(\frac{\Theta(t) - ar + \sigma^2 B(t)}{\Delta r} + \frac{\sigma^2}{\Delta r^2})$$

and as long as $\lambda_{lowerbound}$ will be above $\frac{2}{\Delta t}$ the method will be stable. This is exactly what equation (2.73) states. From (2.73) an upper- and lowerbound for n can be derived.
Suppose that $\Theta(t) - ar + \sigma^2 B(t) \geq 0$. This is the case if

$$r \leq \frac{-\sigma^2 B(t) - \Theta(t)}{-a}, \tag{2.74}$$

$$\tag{2.75}$$

The worst case appears as $r = r_{max}$ and $r_{max} = Q\Delta r = Q\sqrt{n\Delta t \sigma^2}$. So it can be concluded that $\Theta(t) - ar + \sigma^2 B(t) \geq 0$ if

$$Q \leq \frac{(\frac{-\sigma^2 B(t) - \Theta(t)}{-a})^2}{n\sigma^2 T}. \tag{2.76}$$

Since it is now known that $\Theta(t) - ar + \sigma^2 B(t) \geq 0$ an upperbound for n can be deduced. Starting from (2.73)it can be shown that

$$\frac{\Theta(t) - ar + \sigma^2 B(t)}{\Delta r} + \frac{\sigma^2}{\Delta r} \leq \frac{-1}{\Delta t}, \tag{2.77}$$

$$\frac{-\sqrt{n}(\Theta(t) - ar + \sigma^2 B(t))\frac{T}{Q}}{\sqrt{\frac{T}{Q}\sigma^2}} - 1 + n \geq 0, \tag{2.78}$$

Solving this equation for n yields

$$n \leq \sqrt{-\frac{1}{2}b + \sqrt{(\frac{1}{2}b)^2 - c}} \tag{2.79}$$

$$b = \frac{-(\Theta(t) - ar + \sigma^2 B(t))\frac{T}{Q}}{\sqrt{\frac{T}{Q}\sigma^2}} \tag{2.80}$$

$$c = -1 \tag{2.81}$$

(2.76) is more easily satisfied then the criterion for the Black-Scholes equation.

2.2.7 Numerical results (Caplet)

(N.B. the zero coupon is priced exactly under the transformed equation, because the begin solution does not change in time. As a consequence, the transformed solution is equal to the exact solution.
The transformed equation has also been solved with an implicit method and the pde-boundary conditions. This, however, gives exactly the same results as the non-transformed equation for the zero coupon bond and the Caplets.)
In table (2.4) the exact and numerical solution of a Caplet under the transformed Hull-White equation are compared.

T_f	T_S	T_e	Θ	$exact$	$numerical$	σ	a	n
5	5	6	0.025	0.0440	0.0444	0.01	0.05	50
5	5	10	0.025	0.2958	0.2939	0.01	0.05	50
10	10	11	0.025	0.0459	0.0461	0.01	0.05	50
10	10	15	0.025	0.2016	0.2008	0.01	0.05	50
15	15	16	0.025	0.0207	0.0207	0.01	0.05	50
15	15	20	0.025	0.0770	0.0760	0.01	0.05	50
20	20	21	0.025	0.0062	0.0060	0.01	0.05	50
20	20	25	0.025	0.0210	0.0201	0.01	0.05	50
30	30	31	0.025	0.0003	0.0002	0.01	0.05	50

Table 2.4: number of time steps is 200

Chapter 3

Conclusion

In chapter (2) the models for the equity underlying and the interest rate were treated. It was seen that the Black-Scholes formula could be solved with an implicit method and the pde-boundary condition (discretizing the whole equation using one-sided differences), but the computational domain had to be very big (Another point of worry was that stability can need be guaranteed and that the flow at the right boundary was opposite to the one-sided differences taken). Another approach was to solve the formula on a tree structured grid with an explicit method. The problem here is that the central stencil cannot be used and the upstream discretization used is less accurate.
Solving the Hull-White problem with an implicit method also had the disadvantage that we could not guarantee stability. However, it seemed that the method was stable, although we did not proof this. The stability could not be guaranteed because of the $' - rV'$ term and that was why a transformation was performed. After the transformation the equation seemed also solvable with the explicit method on a tree structured-grid. The criterion so satisfy stability is, however, still complex and for the moment we achieved stability by simply changing the right parameters.
Furthermore the two models were examined and the problems for each model were explained.

It can be concluded that solving a partial differential equation without known boundaries can be done in two ways. One is to solve with an implicit method and pde-boundary condition (although when has to be very careful). The other is by solving explicit on a tree method. Both methods are worth some more investigation.

Chapter 4

Future goals

In this thesis two approaches were treated to solve partial differential equation without known boundary condition. The approach of solving with an implicit method using the pde-boundary conditions need to be applied with care, but is worth investigating if this approach can be used for the solver of the three-factor model. For the approach with the explicit method solved on a tree-structured grid first of all the restrictions for the time step need to be found to make the method stable. Since upwind discretization is used for the convection part the results can become inaccurate, this can be solved by discretizing the equation in a different way. However Euler forward has some strict stability conditions and we are going to try another time-discretization method, namely the Runge-Kutta-Chebyshev, to see if this method is more suitable for our problem.
The final goal is to implement a three dimensional solver for the three-factor Heston / Hull-White model.

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