Electromagnetically Induced Flows

in Water

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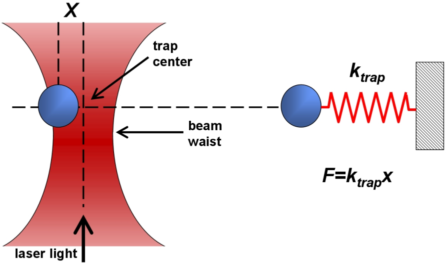
Outline

Introduction



Optical trapping consist of:

- Focussed laser beam is applied in fluid.
- Small particles are trapped by electromagnetic force.
-

Maxwell equations

The macroscopic Maxwell equations in vacuum:

$$
\begin{aligned}
-\nabla \times \mathbf{H} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} &= -\mathbf{J}, \\
\nabla \times \mathbf{E} + \mu_0 \frac{\partial \mathbf{H}}{\partial t} &= -\mathbf{K}.
\end{aligned}
$$

Compatibility relations $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{H} = 0$.

- $\mathbf{E}(\mathbf{x}, t)$, (time domain) electric field,
- $\mathbf{H}(\mathbf{x}, t)$, (time domain) magnetic field,
- $\mathbf{J}(\mathbf{x}, t)$, total electric current density,
- $\mathbf{K}(\mathbf{x}, t)$, total magnetic current density.

Electric current density

The current density consists of three terms:

- $\mathbf{J}^{\mathrm{p}}$, polarization current,
- $\mathbf{J}^{\mathrm{f}}$, free (conduction) current,
- $\mathbf{J}^{\mathrm{ext}}$, external current.

Induced current is given by $\mathbf{J}^{\mathrm{ind}} = \mathbf{J}^{\mathrm{p}} + \mathbf{J}^{\mathrm{f}}$.

The induced currents depend on $\mathbf{E}$, so $\mathbf{J}^{\mathrm{ind}} = \mathbf{J}^{\mathrm{ind}}(\mathbf{E})$.
So-called constitutive relations.

The external current is independent of $\mathbf{E}$ and $\mathbf{H}$.

Induced currents

Linear polarization, instantaneous response:

$$\mathbf{J}^{\mathsf{p}} = \varepsilon_0 \chi_{\mathsf{e}} \frac{\partial \mathbf{E}}{\partial t},$$

with χ_{e} the electric susceptibility.

Free current is proportional to the electric field (Ohm's law):

$$\mathbf{J}^{\mathsf{f}} = \sigma \mathbf{E},$$

with σ the electric conductivity.

Maxwell equations in (rigid) matter

Substitution results in:

$$
\begin{aligned}
-\nabla \times \mathbf{H} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E} &= -\mathbf{J}^{\mathrm{ext}}, \\
\nabla \times \mathbf{E} + \mu_0 \frac{\partial \mathbf{H}}{\partial t} &= 0.
\end{aligned}
$$

Electric permittivity given by:

$$\varepsilon = (1 + \chi_{\mathrm{e}})\varepsilon_0.$$

Maxwell equations in fluids

Fluid in motion: movement w.r.t. the reference frame.

Induced currents because of charges moving:

$$
\begin{aligned}
\mathbf{J}^{\text{rel}} &= -(\mu_0\varepsilon - \mu_0\varepsilon_0)\mathbf{v} \times \mathbf{H}, \\
\mathbf{K}^{\text{rel}} &= (\mu_0\varepsilon - \mu_0\varepsilon_0)\mathbf{v} \times \mathbf{E},
\end{aligned}
$$

$\mathbf{v} \ll 1$, effect will be neglected.

Energy equation (1)

Define the electromagnetic energy density as

$$u_{\mathrm{em}} = \frac{1}{2}\left(\varepsilon_0\|\mathbf{E}\|^2 + \mu_0\|\mathbf{H}\|^2\right),$$

and the Poynting vector as

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}.$$

Then from the Maxwell equations one can derive:

$$\frac{\partial}{\partial t} u_{\mathrm{em}} = -\nabla \cdot \mathbf{S} - \mathbf{E} \cdot \mathbf{J} - \mathbf{H} \cdot \mathbf{K}.$$

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Energy equation (2)

For linear polarization we had:

$$\mathbf{J} = (\varepsilon - \varepsilon_0)\frac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E}.$$

This results in:

$$\frac{\partial}{\partial t} u_{\text{em}} = -\nabla \cdot \mathbf{S} - \frac{1}{2}(\varepsilon - \varepsilon_0)\frac{\partial}{\partial t}\|\mathbf{E}\|^2 - \sigma\|\mathbf{E}\|^2.$$

The polarization term in general oscillates.

The conduction term is strictly negative, so acts as an energy sink.

Momentum equation

Likewise we can derive the momentum equation:

$$\mathbf{f} = -\frac{\partial \mathbf{S}}{\partial t} + \nabla \cdot \overleftrightarrow{\mathbf{T}}.$$

Here $\overleftrightarrow{\mathbf{T}}$ is the Maxwell stress tensor given by

$$\overleftrightarrow{\mathbf{T}} = \mu_0 \mathbf{H} \otimes \mathbf{H} + \varepsilon_0 \mathbf{E} \otimes \mathbf{E} - \frac{1}{2}\mu_0 \|\mathbf{H}\|^2 \mathbb{I} - \frac{1}{2}\varepsilon_0 \|\mathbf{E}\|^2 \mathbb{I}.$$

The force density is given by:

$$\mathbf{f} = (\nabla \cdot \mathbf{E})\mathbf{E} + (\nabla \cdot \mathbf{H})\mathbf{H} + \mu_0 \mathbf{J} \times \mathbf{H} - \varepsilon_0 \mathbf{K} \times \mathbf{E}.$$

Monochromatic source

External current density of only one frequency:

$$\mathbf{J}^{\text{ext}}(\mathbf{x}, t) = \tilde{\mathbf{J}}^{\text{ext}}(\mathbf{x}) \cos(\omega t + \delta).$$

Resulting fields are also time-harmonic:

$$\begin{aligned} \mathbf{E}(\mathbf{x}, t) &= \tilde{\mathbf{E}}(\mathbf{x}) \cos(\omega t + \delta_1), \\ \mathbf{H}(\mathbf{x}, t) &= \tilde{\mathbf{H}}(\mathbf{x}) \cos(\omega t + \delta_2), \end{aligned}$$

with a possible phase-shift.

Complex fields (1)

Notice that we can write:

$$\mathbf{J}^{\text{ext}}(\mathbf{x}, t) = \text{Re}\left\{\hat{\mathbf{J}}^{\text{ext}}(\mathbf{x})e^{-\imath\omega t}\right\}.$$

Like-wise for the fields:

$$\begin{aligned}\mathbf{E}(\mathbf{x}, t) &= \text{Re}\left\{\hat{\mathbf{E}}(\mathbf{x})e^{-\imath\omega t}\right\},\\ \mathbf{H}(\mathbf{x}, t) &= \text{Re}\left\{\hat{\mathbf{H}}(\mathbf{x})e^{-\imath\omega t}\right\}.\end{aligned}$$

The phase-shifts are absorbed in $\hat{\mathbf{J}}^{\text{ext}}$, $\hat{\mathbf{E}}$ and $\hat{\mathbf{H}}$.

Complex fields (2)

Time derivatives become complex multiplications:

$$\frac{\partial}{\partial t}\text{Re}\left\{\hat{\mathbf{E}}(\mathbf{x})e^{-\imath\omega t}\right\} = \text{Re}\left\{-\imath\omega\hat{\mathbf{E}}(\mathbf{x})e^{-\imath\omega t}\right\}.$$

Substitution in the Maxwell equations results in:

$$\begin{aligned}
\text{Re}\left\{\left[-\nabla\times\hat{\mathbf{H}}(\mathbf{x}) - \imath\omega\varepsilon_0\hat{\mathbf{E}}(\mathbf{x})\right]e^{-\imath\omega t}\right\} &= \text{Re}\left\{-\hat{\mathbf{J}}(\mathbf{x})e^{-\imath\omega t}\right\},\\
\text{Re}\left\{\left[\nabla\times\hat{\mathbf{E}}(\mathbf{x}) - \imath\omega\mu_0\hat{\mathbf{H}}(\mathbf{x})\right]e^{-\imath\omega t}\right\} &= 0,
\end{aligned}$$

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Complex polarization (1)

Linear polarization, $\mathbf{J}^{\mathrm{p}} = \varepsilon_0 \chi_{\mathrm{e}} \frac{\partial \mathbf{E}}{\partial t}$ is non-physical.

Better form, temporal dependency, localized in space:

$$\mathbf{J}^{\mathrm{p}} = \varepsilon_0 \int_0^{\infty} f(\tau) \frac{\partial \mathbf{E}}{\partial t}(\mathbf{x}, t - \tau) \, \mathrm{d}\tau.$$

The complex form is given by:

$$\mathbf{J}^{\mathrm{p}} = \mathrm{Re} \left\{ -\imath \omega \varepsilon_0 \left[\int_0^{\infty} f(\tau) e^{\imath \omega \tau} \, \mathrm{d}\tau \right] \hat{\mathbf{E}}(\mathbf{x}) e^{-\imath \omega t} \right\}.$$

Complex polarization (2)

The complex permittivity is defined as $\hat{\varepsilon} = \varepsilon' + \imath\varepsilon''$.

The real part gives the polarization current:

$$\varepsilon'(\omega) = \varepsilon_0 \int_0^\infty f(\tau) e^{\imath\omega\tau}\, \mathrm{d}\tau.$$

The imaginary part gives the free current:

$$\varepsilon''(\omega) = \frac{\sigma(\omega)}{\omega}.$$

Complex Maxwell equations (1)

The (complex) induced current is then given by:

$$\hat{\mathbf{J}}^{\mathsf{ind}} = -\imath\omega(\hat{\varepsilon} - \varepsilon_0)\hat{\mathbf{E}}.$$

Substitution in the Maxwell equations results in:

$$\begin{aligned}
\mathrm{Re}\left\{\left[-\nabla \times \hat{\mathbf{H}}(\mathbf{x}) - \imath\omega\hat{\varepsilon}\hat{\mathbf{E}}(\mathbf{x})\right] e^{-\imath\omega t}\right\} &= \mathrm{Re}\left\{-\hat{\mathbf{J}}^{\mathsf{ext}}(\mathbf{x})e^{-\imath\omega t}\right\}, \\
\mathrm{Re}\left\{\left[\nabla \times \hat{\mathbf{E}}(\mathbf{x}) - \imath\omega\mu_0\hat{\mathbf{H}}(\mathbf{x})\right] e^{-\imath\omega t}\right\} &= 0.
\end{aligned}$$

The two separate induced currents form one complex term.

Complex Maxwell equations (2)

Finally, drop the real parts, and strip off $e^{-\imath\omega t}$.

The resulting complex Maxwell equations are:

$$\begin{aligned} -\nabla \times \hat{\mathbf{H}}(\mathbf{x}) - \imath\omega\hat{\varepsilon}\hat{\mathbf{E}}(\mathbf{x}) &= -\hat{\mathbf{J}}^{\text{ext}}(\mathbf{x}), \\ \nabla \times \hat{\mathbf{E}}(\mathbf{x}) - \imath\omega\mu_0\hat{\mathbf{H}}(\mathbf{x}) &= 0. \end{aligned}$$

The compatibility relations are:

$$\nabla \cdot \hat{\mathbf{H}} = 0 \text{ and } \nabla \cdot \hat{\mathbf{E}} = 0.$$

The equations no longer depend on the time variable t.

Time Averaging (1)

The frequencies are extremely large compared to all the movement in the liquid:

$f = 2.5 \cdot 10^{14}$ Hz for $\lambda_0 = 1.2\,\mu$m.

The quantities are time-averaged:

$$\langle f(\mathbf{x}, t)\rangle = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \mathrm{Re}\left\{\hat{f}(\mathbf{x})e^{-\imath\omega t}\right\}\,\mathrm{d}t.$$

$f(\mathbf{x}, t)$ can be replaced by other quantities.

Time Averaging (2)

The results for different quantities are:

$$
\begin{aligned}
\langle \mathbf{E}(\mathbf{x}, t) \rangle &= 0, \\
\langle \mathbf{E}(\mathbf{x}, t) \cdot \mathbf{E}(\mathbf{x}, t) \rangle &= \frac{1}{2} \mathrm{Re} \left\{ \hat{\mathbf{E}}(\mathbf{x}) \cdot \hat{\mathbf{E}}^*(\mathbf{x}) \right\}, \\
\langle \mathbf{S}(\mathbf{x}, t) \rangle = \langle \mathbf{E}(\mathbf{x}, t) \times \mathbf{H}(\mathbf{x}, t) \rangle &= \frac{1}{2} \mathrm{Re} \left\{ \hat{\mathbf{E}}(\mathbf{x}) \times \hat{\mathbf{H}}^*(\mathbf{x}) \right\}, \\
\left\langle \frac{\partial}{\partial t} \left[\mathbf{E}(\mathbf{x}, t) \cdot \mathbf{E}(\mathbf{x}, t) \right] \right\rangle &= 0.
\end{aligned}
$$

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Time-averaged energy equation (1)

The time-averaged energy equation is given by:

$$\left\langle \frac{\partial}{\partial t} u_{\mathrm{em}} \right\rangle = -\nabla \cdot \langle \mathbf{S} \rangle - \langle \mathbf{E} \cdot \mathbf{J} \rangle .$$

Using the time-harmonic representation, we have:

$$\hat{\mathbf{J}} = -\imath\omega(\hat{\varepsilon} - \varepsilon_0)\hat{\mathbf{E}}.$$

Also we define the complex Poynting vector

$$\hat{\mathbf{S}} = \frac{1}{2}\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*.$$

Time-averaged energy equation (2)

Substitution of the time-averaged expressions results in:

$$0 = -\nabla \cdot \mathrm{Re}\left\{\hat{\mathbf{S}}\right\} - \frac{1}{2}\omega\varepsilon''\hat{\mathbf{E}} \cdot \hat{\mathbf{E}}^*.$$

We have $\mathrm{Re}\left\{\hat{\mathbf{S}}\right\}$ the time-averaged energy flux.

The second term shows the time-averaged dissipation:

$$q_{\mathrm{em}} = \frac{1}{2}\omega\varepsilon''\hat{\mathbf{E}} \cdot \hat{\mathbf{E}}^*.$$

The electromagnetic energy is transformed to heat.

Time-averaged Lorentz force (1)

The time-averaged Lorentz force is given by

$$\langle \mathbf{f} \rangle = \mu_0 \langle \mathbf{J} \times \mathbf{H} \rangle.$$

Using again the relation $\hat{\mathbf{J}} = -\imath\omega(\hat{\varepsilon} - \varepsilon_0)\hat{\mathbf{E}}$, the time-averaged result is

$$\begin{aligned} \langle \mathbf{f} \rangle &= \frac{1}{2}\mu_0 \mathrm{Re}\left\{-\imath\omega(\hat{\varepsilon} - \varepsilon_0)\hat{\mathbf{E}} \times \hat{\mathbf{H}}\right\}, \\ &= \omega\mu_0 \left[\varepsilon'' \mathrm{Re}\left\{\hat{\mathbf{S}}\right\} + (\varepsilon' - \varepsilon_0)\mathrm{Im}\left\{\hat{\mathbf{S}}\right\}\right]. \end{aligned}$$

GB: properties

Properties of the Gaussian beam model:

- Widely used to model laser beams, analytical solution available,
- Transforms easily through lens systems, only the parameters change,
- Gaussian decay (e^{-cr^2}) in axial direction,
- Only an approximation, does not satisfy the Maxwell equations.

GB: derivation (1)

The Maxwell equations are rewritten in terms of a vector potential $\hat{\mathbf{A}}$.
The potential is assumed to have the form:

$$\hat{\mathbf{A}}(x, y, z) = \psi(x, y) e^{\imath \hat{k} z} \hat{\mathbf{y}}.$$

With:

- z the propagation direction,
- y the polarization direction,
- $\hat{k}$ the (complex) wave number, $\hat{\varepsilon} = \frac{\varepsilon_0 c^2}{\omega^2} \hat{k}^2$.

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GB: derivation (2)

ψ is expanded around

$$\hat{s}^2 = \left(\frac{\lambda_0}{2\pi w_0 \hat{n}}\right)^2 .$$

The zero order term is used:

$$\psi_0 = \frac{w_0}{w(z)} \exp\left(\imath \hat{k} \frac{\rho^2}{2R(z)} - \frac{\rho^2}{w(z)^2} - \imath \frac{\alpha}{k} \frac{\rho^2}{w(z)^2} - \imath\zeta(z)\right) .$$

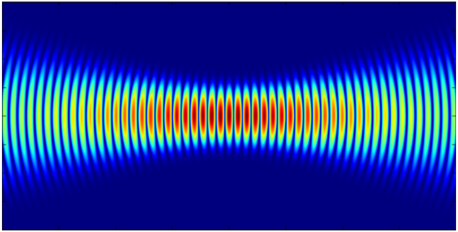
ρ is the axial distance $\sqrt{x^2 + y^2}$.

Expected accuracy of order $\mathcal{O}(|\hat{s}|^2)$.

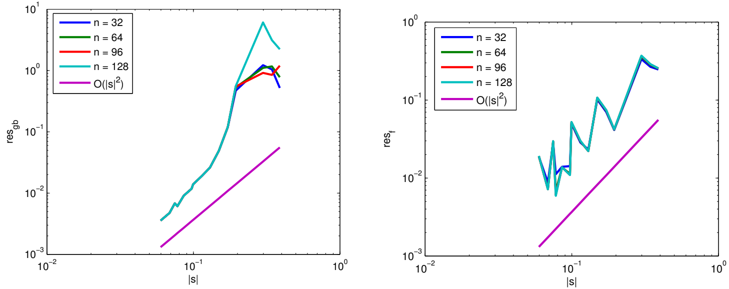
GB: electric field

Electric field has two components:

$$
\begin{aligned}
E_{\text{pol}} &= E_0 \exp(\imath \hat{k} z)\psi_0, \\
E_{\text{prop}} &= E_0 \exp\left(\imath \hat{k} z\right) \psi_0 \left[-\frac{1}{R(z)} - \frac{2\imath}{\hat{k} w(z)^2} + \frac{2\alpha}{k \hat{k} w(z)^2} \right] y,
\end{aligned}
$$



GB: accuracy



Figuur: Residual of the Maxwell equations and moment equation.

Accurate to order $\mathcal{O}(|\hat{s}|^2)$. We have $0.15 < |\hat{s}| < 0.4$.

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Gaussian dipole array

The goal is to:

- Create a source that does satisfy the Maxwell equations,
- that resembles a Gaussian beam (focussing, exponential radial decay).

The idea is to combine a lot of simple, exact solutions for perfect dipoles.

Perfect dipole (1)

The perfect dipole is a simple point source.

The external current density is given by:

$$\hat{\mathbf{J}}^{\text{ext}}(\mathbf{x}, t) = \delta(\mathbf{x} - \mathbf{x}_{\text{s}}) I_0 \cos(\omega t - \varphi)\hat{\mathbf{d}},$$

with

- $\delta(\mathbf{x} - \mathbf{x}_{\text{s}})$ localizes the source at $\mathbf{x}_{\text{s}}$.
- $\hat{\mathbf{d}}$ is the orientation of the point source.

Using Green's functions we can derive an analytic expression for the electric and magnetic field for the dipole.

Perfect dipole (2)

Fields resulting from the dipole:

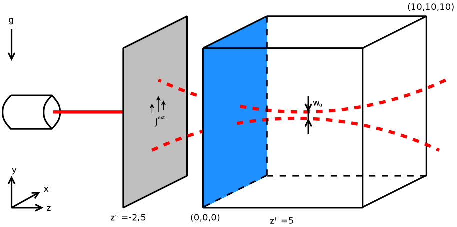
$$\begin{aligned}
\hat{\mathbf{E}} &= \imath\omega\mu_0 I_0 g(\mathbf{r}) \left[\hat{\mathbf{d}} - (\hat{\mathbf{d}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} + \left(\frac{1}{\hat{k}^2 r^2} - \frac{\imath}{\hat{k} r} \right) \left(3(\hat{\mathbf{d}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \hat{\mathbf{d}} \right) \right], \\
\hat{\mathbf{H}} &= I_0 g(\mathbf{r}) \left(\imath\hat{k} - \frac{1}{r} \right) \hat{\mathbf{r}} \times \hat{\mathbf{d}}.
\end{aligned}$$

The Green's function is given by $g(\mathbf{r}) = \frac{1}{4\pi r} e^{\imath\hat{k}r}$, $\mathbf{r} = \mathbf{x} - \mathbf{x}_s$.

External current density

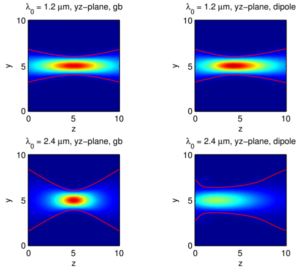
The external current density is prescribed such that:

- It is zero everywhere outside the source plane,
- The source plane is divided in a rectangular grid,
- For each grid point $\hat{\mathbf{J}}^{\text{ext}}$ is proportional to $\hat{\mathbf{E}}_{\text{gb}}$.



The field in the domain is calculated using the Green's function.

Resulting field



This methods results in:

- Similar fields for small wavelengths.
- Deviations for larger wavelengths: larger $|\hat{s}|$ value.
- Constant I_0: Numerically normalized incoming power.

Basis of the NS equations.

The equations are based on:

- Conservation of mass and incompressibility: continuity equation, $\nabla \cdot \mathbf{v} = 0$.
- Conservation of momentum: Navier-Stokes equations, including f_{em}.
- Boussinesq approximation: Gravity driven buoyancy effects in NS equations.
- Conservation of energy: temperature equation with source term q_{em}.

Incompressible Navier Stokes equations.

For the simulations we have used the equations:

$$
\begin{aligned}
\partial_i v_i &= 0, \\
\rho_0 v_j \partial_j v_i &= -\partial_i p' + gy\delta_{i2}\partial_2\rho + \mu\partial_j^2 v_i + f_i^{\mathrm{em}}, \\
\rho_0 c_{\mathrm{p}} v_i \partial_i T &= k\partial_i^2 T + q_{\mathrm{em}},
\end{aligned}
$$

Steady-state equations, with

- $p' = p + \rho g y$ the modified pressure,
- $\rho = \rho_0 (1 - \beta(T - T_0))$.
- $\mathbf{f}^{\mathrm{em}}$ the electromagnetic force density,
- q_{em} the electromagnetic heat dissipation.

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Solving algorithm: SIMPLE (1)

Semi-Implicit Method for Pressure Linked Equations.

Not that simple:

- Pressure and velocity are coupled,
- Non-linear system.

Each equation is solved separately. This is iterated until solution is reached.

Differentiation is discretized using central differences, Integrations are discretized using a midpoint rule.

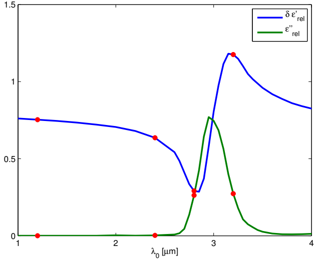
Solving algorithm: SIMPLE (2)

The iterative procedure is given by:

- $\mathbf{u}^{i+1}$ is solved by using p'^{i} and $\mathbf{u}^{i}$ (to handle the non-linearity),
- T^{i+1} is solved using $\mathbf{u}^{i+1}$,
- ρ^{i+1} is solved using T^{i+1},
- p'^{i+1} is solved using $\mathbf{u}^{i+1}$ and ρ^{i+1}.
- $\mathbf{u}^{i+1}$ is corrected using p'^{i+1} to satisfy the continuity equation.

Repeated until tolerance is met.

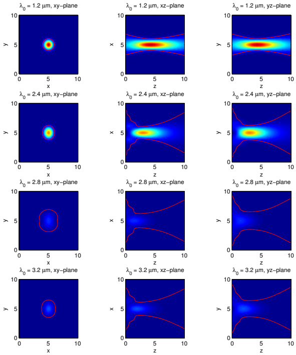
Frequencies in water



Four wavelengths are selected,

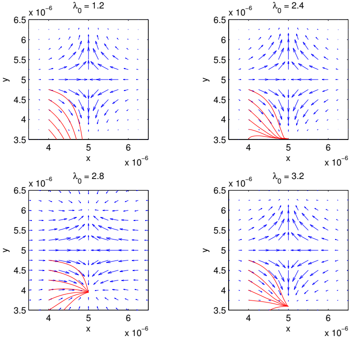
- One small one, $1.2\mu m$, no losses,
- One larger one, $2.4\mu m$, almost no losses,
- Two larger ones, $2.8\mu m$ and $3.2\mu m$, with larger losses.

Intensities



Figuur: Intensities for the four different wavelenghts, using the dipole array method.

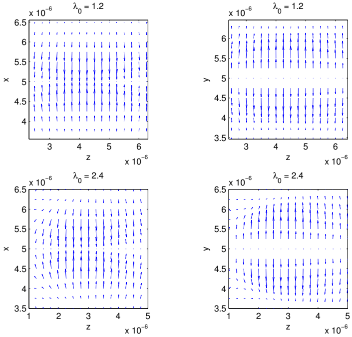
Force density (1)



Figuur: Force density in plane perpendicular to propagation.

- Polarization direction: Away from the centre,
- Perpendicular direction: Towards the centre

Force density (2)



Figuur: Force density for $1.2\mu m$ and $2.4\mu m$, z direction.

- No component in propagation direction.

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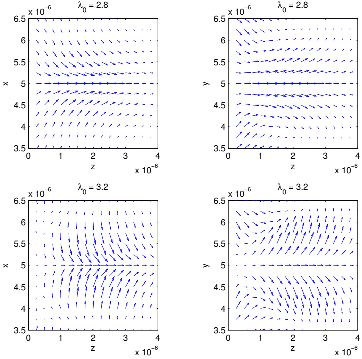
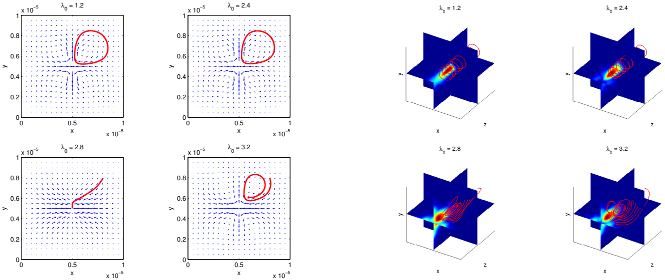
Force density (3)



Figuur: Force density for 2.8μm and 3.2μm, z direction.

- Relatively large component in propagation direction.

Resulting velocity



Velocity in the *xy*-plane, at the point of focus.
Streamlines indicate circular motion for small wavelengths.

Temperature rise

For $P_0 = 1mW$, maximum temperature rise is given by:

wavelength	Energy absorbed	max. ΔT
1.2 μm	0.001	0.047
2.4 μm	0.053	1.906
2.8 μm	0.998	37.653
3.2 μm	0.110	42.654

For $P_0 = 50mW$, maximum temperature rise is given by:

wavelength	max. ΔT
1.2 μm	2.0
2.4 μm	95.2

The Boussinesq approximation is only valid for small increases.

Conclusion

Some conclusions drawn from the simulations:

- The dipole array is an adequate alternative for the GB,
- No losses: Force density leads to loops,
- Losses: Complicated flows, large component in propagation direction.
- Temperature increase is too large, Boussinesq not valid,
- Use of compressible Navier-Stokes to capture total flow.

Further research

More work needs to be done at:

- Establishing whether the correct force is used.
- Use compressible NS to capture effect due to heat.
- Temperature dependent viscosity.
- Experimental set-up, to capture characteristic flow.
- Investigate possible back-action by fluid on fields.

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