On flows induced by electromagnetic fields

Literature review

Michiel de Reus

March 14, 2012

Introduction

The goal of my thesis project is to investigate what the influence of an electromagnetic field inside a fluid is. Since a fluid cannot withstand any stresses and it is known that an electromagnetic field exerts stresses on matter, it is expected that the fluid will start to flow. It are these induced flows that are the main interest of this project.
As first part of my thesis project I have completed this literature review. The first aim was to investigate the Maxwell equations, and see how they behave inside matter. Since we are interested in the stresses exerted on the fluid, it is important to determine what kind of currents there are, and as a consequence how the Maxwell stress tensor looks like. Because of the movement of the fluid, there will be feedback to the electromagnetic field itself. The aim here is to determine this feedback.
The second part consists of a study of the Navier-Stokes equations. First the general equations are stated, after which simplifications are made. Since, for now, the fluid of main interest will be water, and the flow velocities are expected to be small compared to the speed of sound, the incompressible equations are used. It is expected that heat dissipation of the electromagnetic field to the fluid will significantly contribute to the flow, so the equations need to be modified in order to allow temperature driven flows to exist.
In the final part a brief summary of the relevant equations can be found, as well as the research questions for the second stage of this project.

Contents

Chapter 1

Electromagnetic field

It is well known that the electromagnetic fields satisfy the Maxwell equations. In this chapter we will first consider the Maxwell equations in vacuum and derive conversation of energy and momentum for the electromagnetic fields. After this we will consider the so-called macroscopic Maxwell equations in matter, where polarization and magnetization play a roll.
We will first derive the equations for the fields in a stationary linear medium, and expressions for the constitutive relations. Later on we consider a non-stationary fluid, where we assume a certain velocity field is known. It turns out that the use of contra-variant formulation of the Maxwell equations simplifies the equations when we are dealing with a fluid in motion and we need to transform between different reference frames, that is the lab frame in which the fluid was originally at rest and the instantaneous rest frame. The consequence of this formulation is that all equations satisfy the framework of special relativity. At some point we can make assumptions about the non-relativistic fluid velocity to simplify the expressions.

1.1 Maxwell equations in vacuum

From [3] we have the Maxwell equations in vacuum given by

$$-\nabla \times \mathbf{B} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = -\mu_0 \mathbf{J}, \tag{1.1}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{1.2}$$

where $\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$ is the electric field, $\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$ is the magnetic field and $\mathbf{J} = \mathbf{J}(\mathbf{x}, t)$ is the (total) electric current density. The constants ε_0 and μ_0 are the permittivity of free space and permeability of free space respectively.
The current density consists of the movement of charged particles. Later on, when we consider the equations in matter, we see that it can be split in a free and an induced part. Furthermore we have an external part, which acts as source of the initial fields.
From these two equations we can derive two compatibility equations. To derive them we will use the (local) conservation of charge, which can be formulated as the continuity equation

$$\frac{\partial \rho_{\mathrm{e}}}{\partial t} = -\operatorname{div} \mathbf{J}, \tag{1.3}$$

where $\rho_{\mathrm{e}} = \rho_{\mathrm{e}}(\mathbf{x}, t)$ is the (total) electric charge density. Taking the divergence of (1.1) and (1.2) respectively, and noting that the divergence of a curl is zero, we end up with the equations

$$\varepsilon_0 \operatorname{div} \frac{\partial \mathbf{E}}{\partial t} = -\operatorname{div} \mathbf{J}, \tag{1.4}$$

$$\operatorname{div} \frac{\partial \mathbf{B}}{\partial t} = 0. \tag{1.5}$$

Using the first equation we can derive

$$\varepsilon_0 \operatorname{div} \mathbf{E} = \varepsilon_0 \int_{-\infty}^{t} \frac{\mathrm{d}}{\mathrm{d}t} \operatorname{div} \mathbf{E} \, \mathrm{d}t = \int_{-\infty}^{t} \varepsilon_0 \operatorname{div} \left(\frac{\partial \mathbf{E}}{\partial t} \right) \mathrm{d}t = -\int_{-\infty}^{t} \operatorname{div} \mathbf{J} \, \mathrm{d}t = \int_{-\infty}^{t} \frac{\partial \rho_{\mathrm{e}}}{\partial t} \, \mathrm{d}t = \rho_{\mathrm{e}},$$

where we assume by causality that there exists t_0 such that $\rho_{\mathrm{e}} = 0$ for $t \leq t_0$.
Interchanging the order of differentiation in the second equation and assuming that at some point in time there exists no magnetic field (yet), we immediately see that the divergence of the magnetic field is zero. So we get the two compatibility equations

$$\operatorname{div} \mathbf{E} = \frac{\rho_{\mathrm{e}}}{\varepsilon_0}, \tag{1.6}$$

$$\operatorname{div} \mathbf{B} = 0. \tag{1.7}$$

Note that the first equation is usually known as Gauss's law.
In subscript notation the Maxwell equations are given by

$$-\epsilon_{ijk} \partial_j B_k + \mu_0 \varepsilon_0 \partial_t E_i = -\mu_0 J_i, \tag{1.8}$$

$$\epsilon_{ijk} \partial_j E_k + \partial_t B_i = 0, \tag{1.9}$$

where ϵ_{ijk} is the Levi-Civita symbol. The compatibility relations are given by

$$\partial_i E_i = \frac{\rho_{\mathrm{e}}}{\varepsilon_0}, \tag{1.10}$$

$$\partial_i B_i = 0, \tag{1.11}$$

where a repeating index implies summation over that index. The continuity equation for electric charge will become

$$\partial_t \rho_{\mathrm{e}} = -\partial_i J_i. \tag{1.12}$$

It turns out that this notation is more convenient when considering energy and momentum conservation in the next sections.

1.1.1 Conservation of energy

By manipulating the Maxwell equations we can derive a conservation of energy statement for the electromagnetic fields. It turns out we can define a certain volumetric energy density and an energy flux density.
If we multiply (1.8) with E_i and (1.9) with B_i and add the results, we get as left-hand side

$$\begin{aligned}
&E_i\left(-\epsilon_{ijk}\partial_j B_k + \mu_0\varepsilon_0\partial_t E_i\right) + B_i\left(\epsilon_{ijk}\partial_j E_k + \partial_t B_i\right) \\
&= \mu_0\varepsilon_0 E_i\partial_t E_i + B_i\partial_t B_i - \epsilon_{ijk}E_i\partial_j B_k + \epsilon_{ijk}B_i\partial_j E_k \\
&= \partial_t\left[\frac{1}{2}\left(\mu_0\varepsilon_0 E_iE_i + B_iB_i\right)\right] - \epsilon_{ijk}E_i\partial_j B_k + \epsilon_{ijk}B_i\partial_j E_k \\
&= \partial_t\left[\frac{1}{2}\left(\mu_0\varepsilon_0 E_iE_i + B_iB_i\right)\right] + \epsilon_{ijk}E_j\partial_i B_k + \epsilon_{ijk}B_k\partial_i E_j \\
&= \partial_t\left[\frac{1}{2}\left(\mu_0\varepsilon_0 E_iE_i + B_iB_i\right)\right] + \partial_i(\epsilon_{ijk}E_jB_k),
\end{aligned}$$

where we have used $\epsilon_{ijk} = -\epsilon_{jik}$, and relabelled the dummy indices. As right-hand side we get

$$-\mu_0 E_i J_i.$$

If we now divide both sides by μ_0 and define the electromagnetic energy density as

$$u_{\text{em}} = \frac{1}{2}\left(\varepsilon_0 E_iE_i + \frac{1}{\mu_0}B_iB_i\right), \tag{1.13}$$

and the Poynting vector

$$S_i = \frac{1}{\mu_0}\epsilon_{ijk}E_jB_k, \tag{1.14}$$

we can write the equation as

$$\partial_t u_{\text{em}} = -\partial_i S_i - E_iJ_i, \tag{1.15}$$

or in vector notation

$$\frac{\partial u_{\text{em}}}{\partial t} = -\operatorname{div}\mathbf{S} - \mathbf{E}\cdot\mathbf{J}. \tag{1.16}$$

This is the conservation of electromagnetic energy in vacuum.

1.1.2 Conservation of (linear) momentum

It is well known that a charged particle in an electromagnetic field experiences a force, the Lorentz force. This force is given by

$$\mathbf{F} = q\mathbf{E} + \mathbf{v}\times\mathbf{B}, \tag{1.17}$$

where q is the electric charge of the particle and $\mathbf{v}$ its velocity.

When we consider a continuous charge density and current density, we can express the Lorentz force as a force density, given by

$$\mathbf{f} = \rho_{\mathrm{e}}\mathbf{E} + \mathbf{J}\times\mathbf{B}. \tag{1.18}$$

Switching to subscript notation the force density is given by

$$f_i = \rho_{\mathrm{e}} E_i + \epsilon_{ijk} J_j B_k. \tag{1.19}$$

We can rewrite this force density as function of the fields only. We can substitute (1.10) to eliminate ρ_{e}, and (1.8) to eliminate J_j. This results in

$$\begin{aligned} f_i &= \varepsilon_0(\partial_j E_j)E_i + \epsilon_{ijk}\left(\frac{1}{\mu_0}\epsilon_{jlm}\partial_l B_m - \varepsilon_0\partial_t E_j\right)B_k, \\ &= \varepsilon_0 E_i\partial_j E_j + \frac{1}{\mu_0}\epsilon_{ijk}\epsilon_{jlm}B_k\partial_l B_m - \varepsilon_0\epsilon_{ijk}B_k\partial_t E_j, \\ &= \varepsilon_0 E_i\partial_j E_j + \frac{1}{\mu_0}\epsilon_{jki}\epsilon_{jlm}B_k\partial_l B_m - \varepsilon_0\partial_t(\epsilon_{ijk}E_j B_k) + \varepsilon_0\epsilon_{ijk}E_j\partial_t B_k. \end{aligned} \tag{1.20}$$

If we use the identity

$$\epsilon_{jki}\epsilon_{jlm} = \delta_{kl}\delta_{im} - \delta_{km}\delta_{il},$$

we can write

$$\begin{aligned} \epsilon_{jki}\epsilon_{jlm}B_k\partial_l B_m &= \partial_l(\epsilon_{jki}\epsilon_{jlm}B_k B_m) - B_m\partial_l(\epsilon_{jki}\epsilon_{jlm}B_k), \\ &= \partial_l((\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})B_k B_m) - B_m\partial_l((\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})B_k), \\ &= \partial_l(\delta_{kl}\delta_{im}B_k B_m) - \partial_l(\delta_{km}\delta_{il}B_k B_m) - B_m\partial_l(\delta_{kl}\delta_{im}B_k) - B_m\partial_l(\delta_{km}\delta_{il}B_k), \\ &= \partial_k(B_k B_i) - \partial_i(B_k B_k) - B_i\partial_k B_k + B_k\partial_i B_k, \\ &= B_k\partial_k B_i + B_i\partial_k B_k - \partial_i(B_k B_k) - B_i\partial_k B_k + \frac{1}{2}\partial_i(B_k B_k), \\ &= B_k\partial_k B_i - \frac{1}{2}\partial_i(B_k B_k). \end{aligned}$$

Substituting in (1.20) and using (1.14) results in

$$f_i = \varepsilon_0 E_i\partial_j E_j + \frac{1}{\mu_0}B_k\partial_k B_i - \frac{1}{2\mu_0}\partial_i(B_k B_k) - \mu_0\varepsilon_0\partial_t S_i + \varepsilon_0\epsilon_{ijk}E_j\partial_t B_k.$$

The last term on the right-hand side can be rewritten using (1.9). This results in

$$\begin{aligned} \varepsilon_0\epsilon_{ijk}E_j\partial_t B_k &= \varepsilon_0\epsilon_{ijk}E_j(-\epsilon_{klm}\partial_l E_m), \\ &= \varepsilon_0\epsilon_{kji}\epsilon_{klm}E_j\partial_l E_m, \\ &= \varepsilon_0 E_j\partial_j E_i - \frac{1}{2}\varepsilon_0\partial_i(E_j E_j), \end{aligned}$$

where we have used the identity we have just derived in terms of B. Substituting this expression, adding a term

$$\frac{1}{\mu_0} B_i \partial_j B_j = 0,$$

and rearranging and relabelling the terms results in the expression

$$f_i = \varepsilon_0 \left(E_i \partial_j E_j + E_j \partial_j E_i \right) + \frac{1}{\mu_0} \left(B_i \partial_j B_j + B_j \partial_j B_i \right) - \frac{1}{2} \partial_i \left(\varepsilon_0 E_j E_j + \frac{1}{\mu_0} B_j B_j \right) - \mu_0 \varepsilon_0 \partial_t S_i. \quad (1.21)$$

We want to write this force density as divergence of some stress tensor. This can be accomplished by defining the symmetric Maxwell stress tensor T, with components

$$T_{ij} = \varepsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E_k E_k \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B_k B_k \right), \quad (1.22)$$

with respect to the standard basis. We see that for the divergence of the stress tensor we have

$$\begin{aligned}
\partial_j T_{ij} &= \partial_j \left(\varepsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E_k E_k \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B_k B_k \right) \right), \\
&= \varepsilon_0 \left(E_i \partial_j E_j + E_j \partial_j E_i - \frac{1}{2} \delta_{ij} \partial_j E_k E_k \right) + \frac{1}{\mu_0} \left(B_i \partial_j B_j + B_j \partial_i B_i - \frac{1}{2} \delta_{ij} \partial_j B_k B_k \right), \\
&= \varepsilon_0 \left(E_i \partial_j E_j + E_j \partial_j E_i - \frac{1}{2} \partial_i E_j E_j \right) + \frac{1}{\mu_0} \left(B_i \partial_j B_j + B_j \partial_i B_i - \frac{1}{2} \partial_i B_j B_j \right).
\end{aligned}$$

Comparison with (1.21) results in

$$f_i = \partial_j T_{ij} - \mu_0 \varepsilon_0 \partial_t S_i, \quad (1.23)$$

the conservation of electromagnetic momentum. Note f_i can be interpreted as the time derivative of the momentum density. Then $\partial_j T_{ij}$ is the momentum density flux in the different directions, and $\mu_0 \varepsilon_0 S_i$ the momentum density.
In vector notation we would write the Maxwell stress tensor as

$$\overleftrightarrow{\mathbf{T}} = \varepsilon_0 \mathbf{E} \otimes \mathbf{E} + \frac{1}{\mu_0} \mathbf{B} \otimes \mathbf{B} - \frac{1}{2} \left(\varepsilon_0 \|\mathbf{E}\|^2 + \frac{1}{\mu_0} \|\mathbf{B}\|^2 \right) \overleftrightarrow{\mathbf{I}},$$

with $\overleftrightarrow{\mathbf{I}}$ the identity tensor. Then the conservation of momentum can be expressed as

$$\mathbf{f} = \operatorname{div} \overleftrightarrow{\mathbf{T}} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{S}}{\partial t}. \quad (1.24)$$

1.2 Maxwell equations in matter

When we consider electromagnetic fields in matter, it is convenient to work with a different set of equations. Matter reacts in a certain way to electromagnetic fields. The reaction is a combination of polarization and magnetization. We are not interested in the microscopic properties of these reactions, but will consider them from a macroscopic point of view.

1.2.1 Polarization and magnetization

For the analysis of the effect of polarization and magnetization we follow [3] and define two new quantities, the volumetric polarization density $\mathbf{P}$ and the volumetric magnetization density $\mathbf{M}$.
A polarization density gives rise to a bounded volume charge and a bounded surface charge. The bounded volume charge density is given by

$$\rho_{\mathrm{b}} = -\operatorname{div}\mathbf{P}, \tag{1.25}$$

and the bounded surface charge density is given by

$$\sigma_{\mathrm{b}} = \mathbf{P}\cdot\hat{\mathbf{n}}, \tag{1.26}$$

where $\hat{\mathbf{n}}$ is the outward pointing normal unit vector. The surface is the boundary of the medium in which polarization takes place. Furthermore, changing polarization gives rise to a polarization current, because of the bound charge moving around. The corresponding polarization current density is given by

$$\mathbf{J}_{\mathrm{p}} = \frac{\partial \mathbf{P}}{\partial t}. \tag{1.27}$$

A magnetization density gives rise to a bounded volume current and a bounded surface current. The bounded volume current density is given by

$$\mathbf{J}_{\mathrm{b}} = \nabla\times\mathbf{M}, \tag{1.28}$$

and the bounded surface current density is given by

$$\mathbf{K}_{\mathrm{b}} = \mathbf{M}\times\hat{\mathbf{n}}, \tag{1.29}$$

where again $\hat{\mathbf{n}}$ is the outward pointing normal unit vector. The surface is the boundary of the medium in which magnetization takes place.
Together with the polarization density and magnetization density, we can define two new fields, the electric displacement field

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}, \tag{1.30}$$

and the auxiliary magnetic field

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}. \tag{1.31}$$

In order to determine the electromagnetic field we have to know how the matter will react on applied fields, that is we need to know the constitutive relations.

Using the Maxwell equations for vacuum, we will derive a new equivalent equation, in terms of the fields $\mathbf{D}$ and $\mathbf{H}$. The (total) current density can now be split in three parts, the current densities induced by the polarization and magnetization respectively and the free current density. That is we have

$$
\begin{aligned}
\mathbf{J} &= \mathbf{J}^{\mathrm{p}} + \mathbf{J}^{\mathrm{m}} + \mathbf{J}^{\mathrm{f}} \\
&= \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M} + \mathbf{J}^{\mathrm{f}}.
\end{aligned}
$$

Likewise the charge density can be written as the sum of the charge density caused by the polarization and a free charge density. We have

$$
\begin{aligned}
\rho_e &= \rho_p + \rho_f \\
&= -\operatorname{div} \mathbf{P} + \rho_f.
\end{aligned}
$$

Substituting (1.30) and (1.31) into (1.1) and rearranging terms results in the so-called Maxwell equations in matter,

$$
-\nabla \times \mathbf{H} + \frac{\partial \mathbf{D}}{\partial t} = -\mathbf{J}^{\mathrm{f}}, \tag{1.32}
$$

$$
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{1.33}
$$

with compatibility equations

$$
\operatorname{div} \mathbf{D} = \rho_{\mathrm{f}}, \tag{1.34}
$$

$$
\operatorname{div} \mathbf{B} = 0. \tag{1.35}
$$

To be able to solve this system we need so-called constitutive relations for the displacement field, the auxiliary field and the free current density. These will be discussed in the next section.

1.2.2 Constitutive relations for fluids

The simplest way to model how matter reacts to applied electromagnetic fields, assumes linear polarization and magnetization. For now we assume the fields have a low frequency, so that the medium can be regarded non-dispersive. For linear materials we have

$$
\mathbf{P} = \varepsilon_0 \chi_{\mathrm{e}} \mathbf{E} \tag{1.36}
$$

$$
\mathbf{M} = \chi_{\mathrm{m}} \mathbf{H}, \tag{1.37}
$$

with the constants χ_{e} and χ_{m} which are called the electric susceptibility and magnetic susceptibility respectively. Substituting these relations in (1.30) and (1.31) results in the constitutive relations

$$
\mathbf{D} = \varepsilon \mathbf{E} \tag{1.38}
$$

$$
\mathbf{B} = \mu \mathbf{H}, \tag{1.39}
$$

where $\varepsilon = \varepsilon_0 (1 + \chi_{\mathrm{e}})$ is the electric permittivity and $\mu = \mu_0 (1 + \chi_{\mathrm{m}})$ the magnetic permeability.

For conducting media we assume Ohm's law applies, which says that

$$\mathbf{J}^{\mathrm{f}} = \sigma \mathbf{E}, \tag{1.40}$$

where the constant σ is called the electric conductivity.

It is interesting to note that these constitutive relations are only valid in reference frames where the medium is in rest. For non-fluid media this will usually be no problem, but when considering fluids, different material parts in general have different velocities. We will need to find the expressions for the constitutive relations in the stationary lab frame. That is we need to transform the constitutive relations from the instantaneous rest frame, to the lab frame. Following [4] we will use the four-vector framework of special relativity to determine these transformations. The notation and different tensors are defined in Appendix A.
To determine the correct constitutive relations, we will determine tensor equations, which in the instantaneous rest frame of a certain fluid particle reduce to the ordinary constitutive relation for linear media. Since by definition all the tensors obey the transformation rules, these equations will be correct in all (inertial) reference frames, in particular in the lab frame.
From (A.2) we notice that when the regular velocity is zero, the velocity four-vector is equal to

$$V^{\mu} = \begin{pmatrix} c & 0 & 0 & 0 \end{pmatrix},$$

suggesting that the correct form for the polarization is

$$D^{\mu\nu} V_{\nu} = c^2 \varepsilon F^{\mu\nu} V_{\nu}.$$

In vector notation this is equal to

$$\begin{pmatrix} 0 & c\mathbf{D} \\ -c\mathbf{D} & \cdot \times \mathbf{H} \end{pmatrix} \begin{pmatrix} -\gamma c \\ \gamma \mathbf{v} \end{pmatrix} = c^2 \varepsilon \begin{pmatrix} 0 & \frac{1}{c}\mathbf{E} \\ -\frac{1}{c}\mathbf{E} & \cdot \times \mathbf{B} \end{pmatrix} \begin{pmatrix} -\gamma c \\ \gamma \mathbf{v} \end{pmatrix}.$$

Writing out the two equations results in

$$\gamma c \mathbf{D} \cdot \mathbf{v} = c^2 \varepsilon \frac{1}{c} \gamma \mathbf{E} \cdot \mathbf{v}, \tag{1.41}$$

$$\gamma c^2 \mathbf{D} + \gamma \mathbf{v} \times \mathbf{H} = c^2 \varepsilon \gamma \mathbf{E} + c^2 \varepsilon \gamma \mathbf{v} \times \mathbf{B}. \tag{1.42}$$

Clearly this reduces to linear polarization if we substitute $\mathbf{v} = 0$. Likewise for the magnetization we have a similar expression in terms of the dual field tensors,

$$\mu H^{\nu\mu} V_{\nu} = G^{\nu\mu} V_{\nu},$$

In vector notation this is equal to

$$\mu \begin{pmatrix} 0 & \mathbf{H} \\ -\mathbf{H} & -c(\cdot \times \mathbf{D}) \end{pmatrix} \begin{pmatrix} -\gamma c \\ \gamma \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{B} \\ -\mathbf{B} & -\frac{1}{c}(\cdot \times \mathbf{E}) \end{pmatrix} \begin{pmatrix} -\gamma c \\ \gamma \mathbf{v} \end{pmatrix}.$$

Writing out the equations results in

$$\gamma \mu \mathbf{H} \cdot \mathbf{v} = \gamma \mathbf{B} \cdot \mathbf{v}, \tag{1.43}$$

$$\gamma \mu c \mathbf{H} - \gamma \mu c \mathbf{v} \times \mathbf{D} = \gamma c \mathbf{B} - \gamma \frac{1}{c} \mathbf{v} \times \mathbf{E}. \tag{1.44}$$

Again we see that the equations reduce to the expected equations when $\mathbf{v} = 0$ is substituted. Rewriting (1.42) and (1.44) results in the constitutive relations

$$\mathbf{D} = \varepsilon\mathbf{E} + \varepsilon\mathbf{v} \times \mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{H}, \tag{1.45}$$

$$\mathbf{B} = \mu\mathbf{H} - \mu\mathbf{v} \times \mathbf{D} + \frac{1}{c^2}\mathbf{v} \times \mathbf{E}. \tag{1.46}$$

Rewriting (1.42) and (1.44) results in the corresponding compatibility relations

$$\mathbf{D} \cdot \mathbf{v} = \varepsilon\mathbf{E} \cdot \mathbf{v}, \tag{1.47}$$

$$\mathbf{B} \cdot \mathbf{v} = \mu\mathbf{H} \cdot \mathbf{v}. \tag{1.48}$$

We see that (1.45) and (1.46) are coupled. They both depend on three fields. If we assume the fluid velocity is not too large, we can simplify the expressions by neglecting higher order terms. At this point we will only consider the first correction term and neglect all terms much smaller than $\mu_0\varepsilon_0 = \frac{1}{c^2}$. Substituting (1.45) and (1.46) into each other results in

$$\begin{aligned}
\mathbf{D} &= \varepsilon\mathbf{E} + \varepsilon\mathbf{v} \times \left(\mu\mathbf{H} - \mu\mathbf{v} \times \mathbf{D} + \frac{1}{c^2}\mathbf{v} \times \mathbf{E}\right) - \frac{1}{c^2}\mathbf{v} \times \mathbf{H} \\
&= \varepsilon\mathbf{E} + (\mu\varepsilon - \mu_0\varepsilon_0)\,\mathbf{v} \times \mathbf{H} - \mu\varepsilon\mathbf{v} \times (\mathbf{v} \times \mathbf{D}) + \varepsilon\mu_0\varepsilon_0\mathbf{v} \times (\mathbf{v} \times \mathbf{E})\,, \\
&\approx \varepsilon\mathbf{E} + (\mu\varepsilon - \mu_0\varepsilon_0)\,\mathbf{v} \times \mathbf{H} - \mu\varepsilon\mathbf{v} \times (\mathbf{v} \times \mathbf{D})\,, \\
&= \varepsilon\mathbf{E} + (\mu\varepsilon - \mu_0\varepsilon_0)\,\mathbf{v} \times \mathbf{H} - \mu\varepsilon\mathbf{v} \times \left(\mathbf{v} \times \left(\varepsilon\mathbf{E} + \varepsilon\mathbf{v} \times \mathbf{B} - \frac{1}{c^2}\mathbf{v} \times \mathbf{H}\right)\right), \\
&\approx \varepsilon\mathbf{E} + (\mu\varepsilon - \mu_0\varepsilon_0)\,\mathbf{v} \times \mathbf{H}.
\end{aligned}$$

Likewise we get the same result for $\mathbf{B}$,

$$\mathbf{B} \approx \mu\mathbf{H} - (\mu\varepsilon - \mu_0\varepsilon_0)\mathbf{v} \times \mathbf{E}.$$

At this point our assumption seems somehow arbitrary. From the numerical results obtained later on we will see whether they are valid for the situations we will consider.

Finally we need the free current density. In non-conducting media this will of course be zero. When we have positive electric conductivity, we again construct a tensor equation that reduces to the correct equation for $\mathbf{v} = 0$. It is clear that the correct expression is given by

$$\sigma F^{\mu\nu} V_\nu = (J_\mathrm{f})^\mu.$$

In vector notation this is equal to

$$\sigma \begin{pmatrix} 0 & \frac{1}{c}\mathbf{E} \\ -\frac{1}{c}\mathbf{E} & \cdot \times \mathbf{B} \end{pmatrix} \begin{pmatrix} -\gamma c \\ \gamma\mathbf{v} \end{pmatrix} = \begin{pmatrix} c\rho_\mathrm{f} \\ \mathbf{J}^\mathrm{f} \end{pmatrix}$$

Working out these equations results in

$$
\begin{aligned}
\gamma \frac{1}{c} \sigma \mathbf{E} \cdot \mathbf{v} &= c\rho_{\mathrm{f}}, \\
\gamma \sigma \mathbf{E} + \gamma \sigma \mathbf{v} \times \mathbf{B} &= \mathbf{J}^{\mathrm{f}}.
\end{aligned}
$$

Rewriting results in the in the equations

$$
\begin{aligned}
\rho_{\mathrm{f}} &= \gamma \frac{1}{c^2} \sigma \mathbf{E} \cdot \mathbf{v}, \\
\mathbf{J}^{\mathrm{f}} &= \gamma \sigma \mathbf{E} + \gamma \sigma \mathbf{v} \times \mathbf{B}.
\end{aligned}
$$

We see that for $\mathbf{v} = 0$ this reduces to $\mathbf{J}^{\mathrm{f}} = \sigma \mathbf{E}$ and $\rho_{\mathrm{f}} = 0$. For low velocities we have $\gamma \approx 1$, and $\frac{\|\mathbf{v}\|}{c^2} \approx 0$, so we can write

$$
\begin{aligned}
\rho_{\mathrm{f}} &\approx 0, \\
\mathbf{J}^{\mathrm{f}} &\approx \sigma \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right).
\end{aligned}
$$

Finally we want to know the induced current in terms of the **H** field instead of the **B** field. If we substitute for **B** and only keep the largest term we get as result

$$
\mathbf{J}^{\mathrm{f}} = \sigma \left(\mathbf{E} + \mu \mathbf{v} \times \mathbf{H} \right).
$$

So to conclude we have the constitutive relations

$$
\begin{aligned}
\mathbf{D} &= \varepsilon \mathbf{E} + \left(\mu\varepsilon - \mu_0 \varepsilon_0 \right) \mathbf{v} \times \mathbf{H}, && (1.49) \\
\mathbf{B} &= \mu \mathbf{H} - (\mu\varepsilon - \mu_0 \varepsilon_0) \mathbf{v} \times \mathbf{E}, && (1.50) \\
\mathbf{J}^{\mathrm{f}} &= \sigma \left(\mathbf{E} + \mu \mathbf{v} \times \mathbf{H} \right), && (1.51) \\
\rho_{\mathrm{f}} &= 0. && (1.52)
\end{aligned}
$$

where we have changed the approximations to equalities. Later on we will justify this by quantitative estimations of the approximation errors.

1.2.3 Modified Maxwell equations in fluids

We will now substitute all the constitutive relations in the field equations to derive the final equations governing the electromagnetic field inside a (possibly) moving fluid. If we substitute the relations (1.49) and (1.51) into equation (1.32) we get

$$
-\nabla \times \mathbf{H} + \frac{\partial}{\partial t} \left[\varepsilon \mathbf{E} + \left(\mu\varepsilon - \mu_0 \varepsilon_0 \right) \mathbf{v} \times \mathbf{H} \right] = -\sigma \left(\mathbf{E} + \mu \mathbf{v} \times \mathbf{H} \right).
$$

Rewriting results in

$$
-\nabla \times \mathbf{H} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = -(\varepsilon - \varepsilon_0) \frac{\partial \mathbf{E}}{\partial t} - \left(\mu\varepsilon - \mu_0 \varepsilon_0 \right) \frac{\partial}{\partial t} \left(\mathbf{v} \times \mathbf{H} \right) - \sigma \left(\mathbf{E} + \mu \mathbf{v} \times \mathbf{H} \right).
$$

Likewise, if we substitute (1.50) in (1.33) we get

$$\nabla \times \mathbf{E} + \frac{\partial}{\partial t}\left(\mu \mathbf{H} - (\mu\varepsilon - \mu_0\varepsilon_0)\mathbf{v} \times \mathbf{E}\right) = 0,$$

which results in

$$\nabla \times \mathbf{E} + \mu_0 \frac{\partial \mathbf{H}}{\partial t} = -\left(\mu - \mu_0\right)\frac{\partial \mathbf{H}}{\partial t} + (\mu\varepsilon - \mu_0\varepsilon_0)\frac{\partial}{\partial t}\left(\mathbf{v} \times \mathbf{E}\right).$$

We now define the induced electric and magnetic current densities. They are minus the right-hand sides of the modified Maxwell equations we have just derived. So we have

$$-\mathbf{J}^{\mathrm{ind}} = -(\varepsilon - \varepsilon_0)\frac{\partial \mathbf{E}}{\partial t} - (\mu\varepsilon - \mu_0\varepsilon_0)\frac{\partial}{\partial t}\left(\mathbf{v} \times \mathbf{H}\right) - \sigma\left(\mathbf{E} + \mu\mathbf{v} \times \mathbf{H}\right), \tag{1.53}$$

$$-\mathbf{K}^{\mathrm{ind}} = -\left(\mu - \mu_0\right)\frac{\partial \mathbf{H}}{\partial t} + (\mu\varepsilon - \mu_0\varepsilon_0)\frac{\partial}{\partial t}\left(\mathbf{v} \times \mathbf{E}\right). \tag{1.54}$$

In addition to the induced currents there will be external currents. These are controlled currents, independent of the fields. It are these currents that deliver the field's energy and momentum in the first place. The final Maxwell equations governing the fields are given by

$$-\nabla \times \mathbf{H} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = -\mathbf{J}^{\mathrm{ind}} - \mathbf{J}^{\mathrm{ext}}, \tag{1.55}$$

$$\nabla \times \mathbf{E} + \mu_0 \frac{\partial \mathbf{H}}{\partial t} = -\mathbf{K}^{\mathrm{ind}} - \mathbf{K}^{\mathrm{ext}}. \tag{1.56}$$

We will also derive the corresponding compatibility relations. Taking the divergence of 1.55, results in (1.4) so we get the corresponding compatibility equation

$$\operatorname{div}\mathbf{E} = \frac{\rho_{\mathrm{e}}}{\varepsilon_0}.$$

Although magnetic monopoles do not exist, we have seen that in matter magnetic currents can be induced, and we need (induced) magnetic charge to keep the framework complete. Stipulating local conservation of magnetic charge, we define

$$\rho_{\mathrm{m}} = -\operatorname{div}\mathbf{K}, \tag{1.57}$$

which is a continuity equation like (1.3). Now taking the divergence of (1.56) results in

$$\mu_0 \operatorname{div}\frac{\partial \mathbf{H}}{\partial t} = -\operatorname{div}\mathbf{K}. \tag{1.58}$$

Following the exact same derivation for the first equation and assuming, by causality, that $\mathbf{K} = 0$ for $t \leq t_0$ for certain t_0, we arrive at the compatibility relation

$$\operatorname{div}\mathbf{H} = \frac{\rho_{\mathrm{m}}}{\mu_0}. \tag{1.59}$$

1.2.4 Boundary conditions

In order to determine the electromagnetic field in a certain region of space, we need appropriate boundary conditions.

We have boundaries where the material properties are not continuous or there are surface currents. We assume the boundaries are fixed in space and time.

The derivation of the boundary conditions can be found for example in [3]. They follow directly from the Maxwell equations. Here we will only state the results.

At a boundary we can write the electric and magnetic field as the sum of a orthogonal and tangential component,

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_{\text{ort}} + \mathbf{E}_{\text{tan}}, \\ \mathbf{H} &= \mathbf{H}_{\text{ort}} + \mathbf{H}_{\text{tan}}. \end{aligned}$$

Furthermore at a boundary we have two separate regions A and B, each on one side of the boundary. We denote by $\hat{\mathbf{n}}$ the unit normal vector pointing from region B to A. On the boundary itself we have possibly electric and magnetic charge densities σ_{e} and σ_{m}, and current densities $\mathbf{J}_{\text{surf}}$ and $\mathbf{K}_{\text{surf}}$. The four boundary conditions for the field components are then given by

$$\mathbf{E}_{A,\text{tan}} - \mathbf{E}_{B,\text{tan}} = \mathbf{K}_{\text{surf}} \times \hat{\mathbf{n}}, \tag{1.60}$$

$$\mathbf{E}_{A,\text{ort}} - \mathbf{E}_{B,\text{ort}} = \frac{\sigma_{\text{e}}}{\varepsilon_0}, \tag{1.61}$$

$$\mathbf{H}_{A,\text{tan}} - \mathbf{H}_{B,\text{tan}} = \mathbf{J}_{\text{surf}} \times \hat{\mathbf{n}}, \tag{1.62}$$

$$\mathbf{H}_{A,\text{ort}} - \mathbf{H}_{B,\text{ort}} = \frac{\sigma_{\text{m}}}{\mu_0}. \tag{1.63}$$

These four boundary conditions together with the Maxwell equations and the charge and current densities completely determine the fields.

Of course in cases with certain symmetries we can consider different boundary conditions, so that we can reduce the geometry.

1.3 Energy and momentum inside a fluid

We have derived the field equations inside a (possibly moving) fluid. We will now derive equations for the energy and momentum conservation. We proceed as in the vacuum case.

1.3.1 Energy equation

For simplicity we will use subscript notation here. If we multiply (1.55) with E_i and (1.56) with H_i and add the results, we get as left-hand side

$$\begin{aligned}
&E_i\left(-\epsilon_{ijk}\partial_j H_k + \varepsilon_0\partial_t E_i\right) + H_i\left(\epsilon_{ijk}\partial_j E_k + \mu_0\partial_t H_i\right) \\
&= \varepsilon_0 E_i\partial_t E_i + \mu_0 H_i\partial_t H_i - \epsilon_{ijk}E_i\partial_j H_k + \epsilon_{ijk}H_i\partial_j E_k \\
&= \partial_t\left[\frac{1}{2}\left(\varepsilon_0 E_i E_i + \mu_0 H_i H_i\right)\right] - \epsilon_{ijk}E_i\partial_j H_k + \epsilon_{ijk}H_i\partial_j E_k \\
&= \partial_t\left[\frac{1}{2}\left(\varepsilon_0 E_i E_i + \mu_0 H_i H_i\right)\right] + \epsilon_{ijk}E_j\partial_i H_k + \epsilon_{ijk}H_k\partial_i E_j \\
&= \partial_t\left[\frac{1}{2}\left(\varepsilon_0 E_i E_i + \mu_0 H_i H_i\right)\right] + \partial_i(\epsilon_{ijk}E_j H_k),
\end{aligned}$$

where we have used $\epsilon_{ijk} = -\epsilon_{jik}$, and relabelled the dummy indices. As right-hand side we get

$$-E_i J_i - H_i K_i.$$

If we now define the energy density as

$$u_{\text{em}} = \frac{1}{2}\left(\varepsilon_0 E_i E_i + \mu_0 H_i H_i\right), \tag{1.64}$$

and the Poynting vector

$$S_i = \epsilon_{ijk}E_j H_k, \tag{1.65}$$

we can write the equation as

$$\partial_t u_{\text{em}} = -\partial_i S_i - E_i J_i - H_i K_i, \tag{1.66}$$

or in vector notation

$$\frac{\partial u_{\text{em}}}{\partial t} = -\operatorname{div}\mathbf{S} - \mathbf{E}\cdot\mathbf{J} - \mathbf{H}\cdot\mathbf{K}. \tag{1.67}$$

This is the conservation of electromagnetic energy in matter. Although the expression does not explicitly depend on the velocity $\mathbf{v}$, the currents do, so the energy density also depends on the fluid velocity.

1.3.2 Momentum equation

We can also derive the momentum equation, from which the force density follows. Multiplying (1.55) by μ_0 and taking the cross product with $\mathbf{H}$ results in

$$-\mu_0\epsilon_{ijk}\epsilon_{klm}H_i\partial_l H_m + \mu_0\varepsilon_0\epsilon_{ijk}H_j\partial_t E_k = -\mu_0\epsilon_{ijk}H_j J_k.$$

Rewriting using the identity

$$\epsilon_{ijk}\epsilon_{klm}H_i\partial_l H_m = -H_k\partial_k H_i + \frac{1}{2}\partial_i(H_k H_k),$$

results in

$$\mu_0 H_k \partial_k H_i - \frac{1}{2}\mu_0 \partial_i (H_k H_k) + \mu_0 \varepsilon_0 \epsilon_{ijk} H_j \partial_t E_k = \mu_0 \epsilon_{ijk} J_j H_k.$$

Using (1.59) we have

$$\begin{aligned} \mu_0 H_k \partial_k H_i &= \mu_0 \partial_k (H_k H_i) - \mu_0 H_i \partial_k H_k, \\ &= \mu_0 \partial_k (H_k H_i) - \rho_{\mathrm{m}} H_i. \end{aligned}$$

Rearranging terms and relabelling results in the equation

$$\partial_j \left[\mu_0 H_j H_i - \frac{1}{2}\mu_0 \delta_{ij} H_j H_j \right] - \mu_0 \varepsilon_0 \epsilon_{ijk} (\partial_t E_j) H_k = \rho_{\mathrm{m}} H_i + \mu_0 \epsilon_{ijk} J_j H_k. \tag{1.68}$$

Likewise multiplying (1.56) by ε_0 and taking the cross product with $\mathbf{E}$ results in

$$\varepsilon_0 \epsilon_{ijk} \epsilon_{klm} E_i \partial_l E_m + \mu_0 \varepsilon_0 \epsilon_{ijk} E_j \partial_t H_k = -\varepsilon_0 \epsilon_{ijk} E_j K_k,$$

which we can rewrite to

$$-\varepsilon_0 E_k \partial_k E_i + \frac{1}{2}\varepsilon_0 \partial_i (E_k E_k) + \mu_0 \varepsilon_0 \epsilon_{ijk} E_j \partial_t H_k = \varepsilon_0 \epsilon_{ijk} K_j E_k.$$

Using (1.58) we have

$$\begin{aligned} \varepsilon_0 E_k \partial_k E_i &= \varepsilon_0 \partial_k (E_k E_i) - \varepsilon_0 E_i \partial_k E_k, \\ &= \varepsilon_0 \partial_k (E_k E_i) - \rho_{\mathrm{e}} E_i. \end{aligned}$$

Rearranging terms and relabelling results in the equation

$$-\partial_j \left[\varepsilon_0 E_j E_i - \frac{1}{2}\varepsilon_0 \delta_{ij} E_j E_j \right] + \mu_0 \varepsilon_0 \epsilon_{ijk} E_j \partial_t H_k = -\rho_{\mathrm{e}} E_i + \varepsilon_0 \epsilon_{ijk} K_j E_k. \tag{1.69}$$

Defining the stress tensor T_{ij} by

$$T_{ij} = \mu_0 (H_j H_i) + \varepsilon_0 (E_j E_i) - \frac{1}{2}\mu_0 \delta_{ij} (H_j H_j) - \frac{1}{2}\varepsilon_0 \delta_{ij} (E_j E_j), \tag{1.70}$$

subtracting (1.69 from (1.68) and using (1.65) we can write

$$f_i = \partial_j T_{ij} - \partial_t S_i, \tag{1.71}$$

where f_i is the force density given by

$$f_i = \rho_{\mathrm{e}} E_i + \rho_{\mathrm{m}} H_i + \mu_0 \epsilon_{ijk} J_j H_k - \varepsilon_0 \epsilon_{ijk} K_j E_k. \tag{1.72}$$

In vector notation this force density is equal to

$$\mathbf{f} = \rho_{\mathrm{e}} \mathbf{E} + \rho_{\mathrm{m}} \mathbf{H} + \mu_0 \mathbf{J} \times \mathbf{H} - \varepsilon_0 \mathbf{K} \times \mathbf{E}. \tag{1.73}$$

1.4 Frequency domain

Up until now we have only considered the fields and equations in the time domain. Using Fourier transforms in the time variable we can switch to the frequency domain. This transformation works well with the Maxwell equations because they are linear in the field components. In the frequency domain we can investigate the interesting phenomenon of the electric permittivity being a function of the frequency. As we will see this leads to distortion of a wave and dissipation.

1.4.1 Fourier transform

We define the Fourier transform of a quantity $y(\mathbf{x}, t)$ as

$$\hat{y}(\mathbf{x}, \omega) = \int_{-\infty}^{+\infty} y(\mathbf{x}, t) e^{-i\omega t}\, \mathrm{d}t.$$

We denote this integral operator as $\mathcal{F}$, so we can write

$$\hat{y}(\mathbf{x}, \omega) = \mathcal{F}\left[y(\mathbf{x}, t)\right].$$

For this expression to make sense the integral has to convergence, so we have certain restrictions on the function $y(\mathbf{x}, t)$. At this moment we will not dig into the details of this convergence issue, but since we are working with physical fields we expect them to be smooth and bounded. In terms of the field components we get the transforms

$$\hat{E}_i(\mathbf{x}, \omega) = \int_{-\infty}^{+\infty} E_i(\mathbf{x}, t) e^{-i\omega t}\, \mathrm{d}t, \tag{1.74}$$

$$\hat{H}_i(\mathbf{x}, \omega) = \int_{-\infty}^{+\infty} H_i(\mathbf{x}, t) e^{-i\omega t}\, \mathrm{d}t, \tag{1.75}$$

$$\hat{D}_i(\mathbf{x}, \omega) = \int_{-\infty}^{+\infty} D_i(\mathbf{x}, t) e^{-i\omega t}\, \mathrm{d}t, \tag{1.76}$$

$$\hat{B}_i(\mathbf{x}, \omega) = \int_{-\infty}^{+\infty} B_i(\mathbf{x}, t) e^{-i\omega t}\, \mathrm{d}t, \tag{1.77}$$

and corresponding inverse Fourier transforms

$$E_i(\mathbf{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{E}_i(\mathbf{x}, \omega) e^{i\omega t}\, \mathrm{d}\omega, \tag{1.78}$$

$$H_i(\mathbf{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{H}_i(\mathbf{x}, \omega) e^{i\omega t}\, \mathrm{d}\omega, \tag{1.79}$$

$$D_i(\mathbf{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{D}_i(\mathbf{x}, \omega) e^{i\omega t}\, \mathrm{d}\omega, \tag{1.80}$$

$$B_i(\mathbf{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{B}_i(\mathbf{x}, \omega) e^{i\omega t}\, \mathrm{d}\omega. \tag{1.81}$$

Likewise we can determine the Fourier transforms of other quantities such as the electric and magnetic current densities.

One important property of the Fourier transformation that we will use is that

$$\mathcal{F}\left[\frac{\partial^n y}{\partial t^n}\right] = (i\omega)^n \hat{y}, \tag{1.82}$$

so it converts derivatives to ordinary multiplication, that is, it transforms a differential equation into an algebraic equation. If we take the Fourier transformation on both sides of the Maxwell equations (1.55) and (1.56), we get

$$-\varepsilon_{ijk}\partial_j \hat{H}_k + i\omega\varepsilon_0 \hat{E}_i = -\hat{J}_i^{\text{ind}} - \hat{J}_i^{\text{ext}}, \tag{1.83}$$

$$\varepsilon_{ijk}\partial_j \hat{E}_k + i\omega\mu_0 \hat{H}_i = -\hat{K}_i^{\text{ind}} - \hat{K}_i^{\text{ext}}. \tag{1.84}$$

We see that there are only spatial derivatives left. In certain situations this form is easier to solve, although if we need the field components for further analysis we of course need to take the inverse transform.

1.5 Dispersion of the permittivity

Up until now we have assumed that matter reacts the same to any applied electromagnetic field, regardless how fast the fields change in time. One of the consequences of this property is that polarization and conduction occur instantaneously, that is, given a certain change in the electromagnetic fields, the matter instantaneously rearranges itself to retain the linear relationships $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{J}\text{f} = \sigma\mathbf{E}$. Of course such an instantaneous reaction cannot occur on physical grounds, so a more complex mechanism has to govern the polarization. Following [4] we assume the polarization is not only a function of the present electric field strength but of all values in the past. In general this means we can write

$$D_i(\mathbf{x}, t) = \varepsilon_0 E_i(\mathbf{x}, t) + \varepsilon_0 \int_0^\infty f(\tau) E_i(\mathbf{x}, t - \tau)\, \mathrm{d}\tau. \tag{1.85}$$

This means that the polarization is not local any more in the temporal variable. In the space variables we assume it still to be local. The precise way in which the medium reacts depends on our choice of $f(\tau)$. Note that for the special choice

$$f(\tau) = \chi_{\text{e}}\delta(\tau),$$

we get the original model with instantaneous response back.

To further analyse the dispersion relation, we consider the frequency domain, by taking the Fourier transform of the fields. If we substitute (1.80) and (1.78) into (1.85) and interchange the order of integration we get as result

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{D}_i(\mathbf{x},\omega)e^{i\omega t}\,\mathrm{d}\omega = \\
&= \varepsilon_0\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{E}_i(\mathbf{x},\omega)e^{i\omega t}\,\mathrm{d}\omega + \varepsilon_0\int_0^{\infty} f(\tau)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{E}_i(\mathbf{x},\omega)e^{i\omega(t-\tau)}\,\mathrm{d}\omega\right]\mathrm{d}\tau, \\
&= \varepsilon_0\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{E}_i(\mathbf{x},\omega)e^{i\omega t}\left[1+\int_0^{\infty} f(\tau)e^{-i\omega\tau}\,\mathrm{d}\tau\right]\mathrm{d}\omega, \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\varepsilon_0\hat{E}_i(\mathbf{x},\omega)\left[1+\int_0^{\infty} f(\tau)e^{-i\omega\tau}\,\mathrm{d}\tau\right]e^{i\omega t}\,\mathrm{d}\omega,
\end{aligned}
$$

so we see that for the complex field vectors we have the relation

$$\hat{D}_i(\mathbf{x},\omega) = \hat{\varepsilon}(\omega)\hat{E}_i(\mathbf{x},\omega), \tag{1.86}$$

where we have defined the complex electric permittivity by

$$\hat{\varepsilon}(\omega) = \varepsilon_0\left[1+\int_0^{\infty} f(\tau)e^{-i\omega\tau}\,\mathrm{d}\tau\right]. \tag{1.87}$$

We see that when we assume the permittivity is local in the space coordinates we get a linear complex permittivity relation. Through the function $f(\tau)$ this relation depends on the specific media. We will not dig in the microscopic properties of matter to determine $f(\tau)$, instead we will take the values of $\hat{\varepsilon}(\omega)$ for the media of our interest from the literature.

It is interesting to see how we can relate the real permittivity and conductivity to the complex permittivity. Consider (1.55) together with (1.53) and assume $\mathbf{v} = 0$. Then we

$$-\epsilon_{ljk}\partial_j H_k + \varepsilon_0\partial_t E_l = -(\varepsilon-\varepsilon_0)\partial_t E_l - \sigma E_l - J_l^{\mathrm{ext}}.$$

Rewriting and taking the Fourier transform w.r.t. t results in

$$-\epsilon_{ljk}\partial_l\hat{H}_k + i\omega\left(\varepsilon - i\frac{\sigma}{\omega}\right)\hat{E}_l = -\hat{J}_l^{\mathrm{ext}}.$$

Comparing with (1.86) we can identify

$$\hat{\varepsilon}(\omega) = \varepsilon'(\omega) - i\varepsilon''(\omega) = \varepsilon(\omega) - i\frac{\sigma(\omega)}{\omega},$$

where we made the frequency dependency explicit. We see that the imaginary part of the complex permittivity can be associated with conductivity. This suggests intuitively that it is associated with energy dissipation from the electromagnetic fields to the medium.
If we now consider general $\mathbf{v}$, and assume $\mu = \mu_0$, we have the equations

$$
\begin{aligned}
-\varepsilon_{ljk}\partial_j\hat{H}_k + i\omega\hat{\varepsilon}\hat{E}_l &= -i\omega\mu_0\left(\hat{\varepsilon}-\varepsilon_0\right)\epsilon_{ljk}v_j\hat{H}_k - \hat{J}_l^{\mathrm{ext}}, && (1.88)\\
\varepsilon_{ljk}\partial_j\hat{E}_k + i\omega\mu_0\hat{H}_l &= i\omega\mu_0(\varepsilon-\varepsilon_0)\epsilon_{ljk}v_j\hat{E}_k - \hat{K}_l^{\mathrm{ext}}. && (1.89)
\end{aligned}
$$

Notice that at the right-hand of (1.89) we only need the real part of the permittivity. We see that a non-zero velocity field, can result in a loss or gain in energy, through the complex permittivity.

1.6 Time-averaged quantities

Considering the fields in the frequency domain, we see them as the superposition of all the modes with frequencies $\omega \in (-\infty, +\infty)$. If we want to determine specific properties of the field, such as the value of the Poynting vector or the Lorentz force density for a specific location $\mathbf{x}$, we usually are not interested in the highly oscillatory behaviour, but in the time-averaged values of such quantities. In this section we will derive some results using time-averaging in the frequency domain, although these results are not derived firmly, and are only established on an intuitive basis. When used in a later stage of this research project, more thorough arguments are needed.

If we consider the Lorentz force as divergence of the stress tensor, we have from (1.71) the equality

$$f_l = \partial_j T_{lj} - \partial_t S_l,$$

with the stress tensor given by (1.70),

$$T_{lj} = \mu_0(H_j H_l) + \varepsilon_0(E_j E_l) - \frac{1}{2}\mu_0 \delta_{lj}(H_j H_j) - \frac{1}{2}\varepsilon_0 \delta_{lj}(E_j E_j),$$

where we change the label i to l because i will now be the imaginary unit. To determine the time-averaged force per frequency, we need the time-averaged values of $E_j E_i$ and $H_j H_i$. Using (1.78) and the fact that E_j is real we can write

$$\begin{aligned} E_l(\mathbf{x}, t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{Re}\left\{\hat{E}_l(\mathbf{x}, \omega)e^{i\omega t}\right\} \mathrm{d}\omega, \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{2}\left[\hat{E}_l(\mathbf{x}, \omega)e^{i\omega t} + \bar{E}_l(\mathbf{x}, \omega)e^{-i\omega t}\right] \mathrm{d}\omega, \end{aligned}$$

so we see that the field frequency density is equal to

$$\frac{1}{4\pi}\left[\hat{E}_l(\mathbf{x}, \omega)e^{i\omega t} + \bar{E}_l(\mathbf{x}, \omega)e^{-i\omega t}\right], \tag{1.90}$$

which is periodic in t with period $T = \frac{2\pi}{\omega}$. Likewise for the auxilary magnetic field we have frequency density

$$\frac{1}{4\pi}\left[\hat{H}_l(\mathbf{x}, \omega)e^{i\omega t} + \bar{H}_l(\mathbf{x}, \omega)e^{-i\omega t}\right]. \tag{1.91}$$

The time-averaged of a periodic function $f(t)$ is equal to

$$\langle f(t)\rangle = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\,\mathrm{d}t.$$

If we want to determine the time average of $E_l E_j$, we multiply the corresponding frequency densities, and integrate over a period. This results in

$$
\begin{aligned}
\langle E_l E_j \rangle &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{1}{16\pi^2}\left(\hat{E}_l(\mathbf{x},\omega)e^{i\omega t} + \bar{E}_l(\mathbf{x},\omega)e^{-i\omega t}\right)\cdot\left(\hat{E}_j(\mathbf{x},\omega)e^{i\omega t} + \bar{E}_j(\mathbf{x},\omega)e^{-i\omega t}\right)\,\mathrm{d}t,\\
&= \frac{1}{16\pi^2}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left(\hat{E}_l(\mathbf{x},\omega)\bar{E}_j(\mathbf{x},\omega) + \bar{E}_l(\mathbf{x},\omega)\hat{E}_j(\mathbf{x},\omega)\right)\,\mathrm{d}t\, +\\
&\quad \frac{1}{16\pi^2}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left(\hat{E}_l(\mathbf{x},\omega)\hat{E}_j(\mathbf{x},\omega)e^{2i\omega t} + \hat{E}_l(\mathbf{x},\omega)\bar{E}_j(\mathbf{x},\omega)e^{-2i\omega t}\right)\,\mathrm{d}t,\\
&= \frac{1}{16\pi^2}\left(\hat{E}_l(\mathbf{x},\omega)\bar{E}_j(\mathbf{x},\omega) + \bar{E}_l(\mathbf{x},\omega)\hat{E}_j(\mathbf{x},\omega)\right),\\
&= \frac{1}{8\pi^2}\mathrm{Re}\left\{\hat{E}_l(\mathbf{x},\omega)\bar{E}_j(\mathbf{x},\omega)\right\}.
\end{aligned}
$$

Doing the same for all the other terms results in a time-averaged stress tensor

$$
\langle T_{lj}\rangle = \frac{1}{8\pi^2}\left[\mu_0\mathrm{Re}\left\{\hat{H}_l\bar{H}_j\right\} + \varepsilon_0\mathrm{Re}\left\{\hat{E}_l\bar{E}_j\right\} - \frac{1}{2}\delta_{lj}\left(\mu_0|\hat{H}_l|^2 - \varepsilon_0|\hat{E}_l|^2\right)\right],
$$

where we dropped the dependency on $\mathbf{x}$ and ω. If we determine the time-average of $\partial_t S_l$, the result is zero. If we apply differentiation w.r.t. t to the product of (1.90) and (1.91) we end up with

$$
\begin{aligned}
\langle \partial_t S_l\rangle &= \frac{1}{T}\frac{1}{16\pi}\epsilon_{ljk}\int_{-\frac{T}{2}}^{\frac{T}{2}}\partial_t\left[\left(\hat{E}_j e^{i\omega t} + \bar{E}_j e^{-i\omega t}\right)\left(\hat{H}_k e^{i\omega t} + \bar{H}_k e^{-i\omega t}\right)\right]\,\mathrm{d}t,\\
&= \frac{1}{T}\frac{1}{16\pi}\epsilon_{ljk}\left(\hat{E}_j e^{i\omega t} + \bar{E}_j e^{-i\omega t}\right)\left(\hat{H}_k e^{i\omega t} + \bar{H}_k e^{-i\omega t}\right)\Big|_{-\frac{T}{2}}^{\frac{T}{2}},\\
&= \frac{1}{T}\frac{1}{16\pi}\epsilon_{ljk}\left(\hat{E}_j\hat{H}_k e^{2i\omega t} + \hat{E}_j\bar{H}_k + \bar{E}_j\hat{H}_k + \bar{E}_j\bar{H}_k e^{-2i\omega t}\right)\Big|_{-\frac{T}{2}}^{\frac{T}{2}},\\
&= 0,
\end{aligned}
$$

where the last step follows because all terms are periodic in t and vanish. So we now see that for the time-averaged Lorentz force we have

$$
\langle f_l\rangle = \partial_j\langle T_{lj}\rangle = \frac{1}{8\pi^2}\partial_j\left[\mu_0\mathrm{Re}\left\{\hat{H}_l\bar{H}_j\right\} + \varepsilon_0\mathrm{Re}\left\{\hat{E}_l\bar{E}_j\right\} - \frac{1}{2}\delta_{lj}\left(\mu_0|\hat{H}_l|^2 + \varepsilon_0|\hat{E}_l|^2\right)\right].
$$

This expression can be used to determine the force density on the fluid, averaged over time. Since the electromagnetic field oscillates on a time-scale much smaller than we expect in our flow problems, this approach makes sense. Of course we will still need to determine whether this approach is valid, using for example numerical experiments.

Chapter 2

Navier-Stokes equations

In this chapter we will outline the different equations governing the flow of a fluid. All the equations follow from the different conservation laws. In particular we use the conservation of mass, linear momentum, angular momentum and energy.
We have $\mathbf{v}$ the velocity of the fluid, which in general is a function of position and time, so $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$. We will now determine the equations for $\mathbf{v}$ in terms of the material properties and other variables.

2.1 Conservation of mass

For the fluid in motion we assume that mass is (locally) conserved. From [1] we have the continuity equation given by

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{2.1}$$

where $\rho(\mathbf{x}, t)$ is the mass density and $\mathbf{u}(\mathbf{x}, t)$ the velocity of the fluid. In subscript notation this becomes

$$\partial_t \rho + \partial_j (\rho u_j) = 0. \tag{2.2}$$

Expanding the differentiation results in

$$\partial_t \rho + \rho \partial_j u_j + u_j \partial_j \rho = 0. \tag{2.3}$$

This equation can be simplified using certain assumptions on the flow, such as incompressibility. If we introduce the operator

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \tag{2.4}$$

called the material derivative, we can write the continuity equation as

$$\frac{1}{\rho} \frac{\mathrm{D}\rho}{\mathrm{D}t} = -\operatorname{div} \mathbf{u}. \tag{2.5}$$

The material derivative is the derivative along the path of a fluid particle. It can be applied to field variables, that is variables that are functions of $\mathbf{u}$ and t. In subscript notation we will write D_t, so we can write

$$\frac{1}{\rho}\mathrm{D}_t\rho = -\partial_i v_i. \tag{2.6}$$

We see that if the material derivative of the mass density is zero, the divergence of the velocity vanishes.

2.2 Conservation of (linear) momentum

Using conservation of (linear) momentum and following the derivation from [1], we arrive at the general expression for conservation of linear momentum,

$$\rho\mathrm{D}_t v_i = \partial_j T_{ij} + f_i^{\mathrm{b}}, \tag{2.7}$$

where $f_i^{\mathrm{b}}(\mathbf{x}, t)$ is the volumetric body force density and T_{ij} the stress tensor.
Writing out the material derivative results in

$$\rho\left(\partial_t v_i + v_j\partial_j v_i\right) = \partial_j T_{ij} + f_i^{\mathrm{b}}. \tag{2.8}$$

We have to determine the relation between the stress tensor and the other quantities that describe the flow of the fluid. First of all we write the stress tensor as the sum of an isotropic part and a non isotropic part

$$T_{ij} = -p\delta_{ij} + T'_{ij}, \tag{2.9}$$

with p the mechanical pressure and T'_{ij} the non-isotropic part of the stress tensor or the deviatoric stress tensor. If we consider the fluid to be a Newtonian fluid, the deviatoric stress tensor can be written as

$$T'_{ij} = 2\mu\left(e_{ij} - \frac{1}{3}\Delta\delta_{ij}\right), \tag{2.10}$$

where μ is the viscosity, e_{ij} is given by

$$e_{ij} = \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right),$$

and $\Delta = e_{kk} = \partial_k v_k$, with summation over k implied. In general the viscosity is a function of the temperature.
Substitution of (2.10)and (2.9) in (2.8) results in

$$\begin{aligned}
\rho\left(\partial_t v_i + v_j\partial_j v_i\right) &= \partial_j\left[-p\delta_{ij} + 2\mu\left(e_{ij} - \frac{1}{3}\Delta\delta_{ij}\right)\right] + f_i^{\mathrm{b}}, \\
&= -\partial_i p + 2\partial_j(\mu e_{ij}) - \frac{2}{3}\partial_i(\mu\Delta) + f_i^{\mathrm{b}}, \\
&= -\partial_i p + \partial_j\left(\mu\left(\partial_j v_i + \partial_i v_j\right)\right) - \frac{2}{3}\partial_i(\mu\partial_j v_j) + f_i^{\mathrm{b}}.
\end{aligned}$$

This is the most general form for Newtonian fluids. If we assume temperature differences are small, then the viscosity is homogeneous and can be taken out of the derivatives. This results in the simpler form

$$\rho\left(\partial_t v_i + v_j \partial_j v_i\right) = -\partial_i p + \mu \partial_j^2 v_i + \frac{1}{3}\mu \partial_i(\partial_j v_j) + f_i^{\mathrm{b}}, \tag{2.11}$$

where we have interchanged the order of differentiation to combine two terms.
A final important simplification is to consider the flow incompressible. This means that the material derivative of the mass density of the fluid is zero. From (2.6) it follows directly that

$$\partial_i v_i = 0.$$

If we substitute this in (2.11) we have as result the incompressible Navier-Stokes equations with constant viscosity,

$$\rho\left(\partial_t v_i + v_j \partial_j v_i\right) = -\partial_i p + \mu \partial_j^2 v_i + f_i^{\mathrm{b}}. \tag{2.12}$$

We can write this slightly different if we use

$$v_j \partial_j v_i = \partial_j(v_j v_i) - v_i \partial_j v_j = \partial_j(v_j v_i).$$

The result then is

$$\rho\left(\partial_t v_i + \partial_j(v_j v_i)\right) = -\partial_i p + \mu \partial_j^2 v_i + f_i^{\mathrm{b}}. \tag{2.13}$$

In general the Navier-Stokes equation together with the continuity equation are four equations in five unknowns, v_i, ρ and p, so our system is not complete yet. In the incompressible case, the equation $\mathrm{D}_t \rho = 0$ is this final equation. When we consider compressible flow we need to look at the energy equation and the thermodynamic relation.

2.3 Conservation of energy

When the fluid contains heat sources, and heat transfer plays a role we have to take conservation of energy into account. On a particular control volume S there are two forces acting. The work done by the body force is

$$\int_S v_i f_i^{\mathrm{b}} \,\mathrm{d}V.$$

The work done by the surface forces are given by

$$\int_{\partial S} v_i T_{ij} \hat{n}_j \,\mathrm{d}A = \int_S \partial_j (v_i T_{ij}) \,\mathrm{d}S,$$

where we have used the divergence theorem. Equating them, the change of internal energy by the forces is equal to

$$\partial_j(v_i T_{ij}) + v_i f_i^{\mathrm{b}} = v_i\left(\partial_j T_{ij} + f_i^{\mathrm{b}}\right) + T_{ij}\partial_j v_i.$$

To derive the corresponding equation, we define $E = E(\mathbf{x}, t)$ the (total) energy density. Consider a (time dependent) volume $\mathcal{V}(t)$. Changes in the total energy density can occur because of forces doing work, and the production and flow of heat. The forces involved are the body force, with density f_i^{b}, which acts over the volume $\mathcal{V}(t)$ and the force resulting from the stress tensor which acts over the boundary $\partial\mathcal{V}(t)$. The work done by a force f_i is given by $W = f_i v_i$. The force resulting from the stress tensor over the surface of $\mathcal{V}(t)$ is given by $T_i = T_{ij}\hat{n}_j$, where $\hat{n}_j$ is the outward pointing normal vector. We can then write for the conservation of energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}(t)} E\,\mathrm{d}V = \int_{\mathcal{V}(t)} f_i^{\mathrm{b}} v_i\,\mathrm{d}V + \int_{\partial\mathcal{V}(t)} T_i v_i\,\mathrm{d}A + Q, \tag{2.14}$$

where Q is the sum of the heat production and heat flux, which we can write as

$$Q = \int_{\mathcal{V}(t)} q\,\mathrm{d}V - \int_{\partial\mathcal{V}(t)} q_i^* \hat{n}_i\,\mathrm{d}A,$$

with q the heat source density and q_i^* the heat flux. The heat flux is usually given by Fourier's law,

$$q_i^* = -k\partial_i T,$$

with T the temperature and $k \geq 0$ the heat conduction coefficient. In general $k = k(T)$, but in practice the temperature differences are small enough so we can consider k constant.
From the transport equation we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}(t)} E\,\mathrm{d}V = \int_{\mathcal{V}(t)} \left(\partial_t E + \partial_i(E v_i)\right)\,\mathrm{d}V.$$

If in (2.14) we transform the area integrals over $\partial\mathcal{V}(t)$ to volume integrals over $\mathcal{V}(t)$ by use of the divergence theorem, all terms are volume integrals over the arbitrary volume $\mathcal{V}(t)$ and we can conclude the integrands must be equal. So the conservation of energy in differential form states

$$\begin{aligned}\partial_t E + \partial_i(E v_i) &= f_i^{\mathrm{b}} v_i + \partial_j(T_{ij} v_i) + q - \partial_i q_i^*,\\ &= f_i^{\mathrm{b}} v_i + \partial_j(T_{ij} v_i) + k\partial_i^2 T + q,\end{aligned}$$

where we consider k constant. Using (2.7) we can write

$$f_i^{\mathrm{b}} v_i + \partial_j(T_{ij} v_i) = f_i^{\mathrm{b}} v_i + v_i\partial_j T_{ij} + T_{ij}\partial_j v_i = \rho v_i \mathrm{D}_t v_i + T_{ij}\partial_j v_i,$$

so we arrive at the equation

$$\partial_t E + \partial_i(E v_i) = \rho v_i \mathrm{D}_t v_i + T_{ij}\partial_j v_i + k\partial_i^2 T + q. \tag{2.15}$$

This is the conservation of energy equation, where still have to substitute the stress tensor. Note that we can write the energy density as the sum of the internal energy and the kinetic energy for each particle,

$$E = \rho\left(e + \frac{1}{2}v_i v_i\right), \tag{2.16}$$

with e the specific internal energy (per unit mass).

2.4 Thermodynamic relations

Up until now we have three major equations, the continuity equation, the result of conservation of mass, the Navier-Stokes equation (this is a vector equation, so we can consider this three separate equation, one for each component), the result of conservation of momentum and the energy equation, the result of the conservation of energy. The number of unknown quantities is give, we have the velocity $\mathbf{u}$ (technically this are three unknowns), the mass density ρ, the pressure p, the temperature T and the internal energy e. This means we need two more equations, that give the relation between the state variables ρ, p and T and the specific internal energy e. In general we need an equation of state

$$p = p(\rho, T), \tag{2.17}$$

and the relation

$$e = e(\rho, T). \tag{2.18}$$

The functional form of those equations depend on the specific fluid that is considered and the assumptions that are made. From these two relations we can determine $p = p(e, \rho)$ and $T = T(e, \rho)$. If we assume that the temperature differences in the fluid are relatively small, we can use the specific heat capacity. We have to distinguish between the specific heat capacity at constant volume and constant pressure. Since the compressibility of water is relatively small we will use the specific heat capacity at constant volume, and because the temperature varies only little we consider it a constant, despite its dependence on the state variables such as temperature and pressure. This way we get as relation for the internal energy

$$e = c_p T, \tag{2.19}$$

where c_p is the specific heat at constant pressure, for a certain reference temperature.

2.5 Overview of equations

We have derived a number of equations that governing the flow of a fluid. In general we have the differential equations

$$\begin{cases} \partial_t \rho + \partial_i(\rho v_i) &= 0, \\ \rho\left(\partial_t v_i + v_j \partial_j v_i\right) &= -\partial_i p + \mu \partial_j^2 v_i + \frac{1}{3}\mu \partial_i(\partial_j v_j) + f_i^{\mathrm{b}}, \\ \partial_t E + \partial_i(E v_i) &= \rho v_i \mathrm{D}_t v_i + T_{ij}\partial_j v_i + k\partial_i^2 T + q, \end{cases}$$

where E is given by (2.16) and T_{ij} by (2.9), together with the relations $p = p(e, \rho)$ and $T = T(e, \rho)$. We still need to specify f_i^{b} and q, the (body) force density and heat source. As we will see, in our application, these quantities follow from the Maxwell equations, where f_i^{b} is the force exerted by means of the Maxwell stress tensor, and q is the heat source resulting from the dissipation of the electromagnetic field.
For our analysis, we will assume the flow is incompressible. In this case the first equation reduces to $\partial_i u_i = 0$.
Furthermore, we assume the temperature to vary only slightly. In the momentum equation we will assume that the density depends on the temperature, so that temperature differences can induce

flow. In order to allow this, we follow [5] and write the density and the temperature as the difference between a reference value and a small deviation, $\rho = \rho_0 + \rho'$ and $T = T_0 + T'$. Since ρ' is small, we can write it as

$$\rho' = \left(\frac{\partial \rho_0}{\partial T}\right)_p T' = -\rho_0 \beta T',$$

with β the thermal expansion coefficient. Using this expression we have for the density the expression

$$\rho = \rho_0 + \rho' = \rho_0 \left(1 - \beta(T - T_0)\right). \tag{2.20}$$

Finally we need an equation in the temperature T. In the energy equation we assume the density to be constant, and we use $e = c_p T$. The left-hand side of the energy equation can now be written as

$$\begin{aligned}
\partial_t E + \partial_i (E v_i) &= \partial_t E + v_i \partial_i E + E + \partial_i v_i, \\
&= \mathrm{D}_t E, \\
&= \rho \mathrm{D}_t e + \rho \mathrm{D}_t \left(\frac{1}{2} v_i v_i\right), \\
&= \rho c_p \mathrm{D}_t T + \rho v_i \mathrm{D}_t v_i,
\end{aligned}$$

where in the last line we applied the chain rule. Cancelling equal terms, we arrive at the temperature equation

$$\rho c_p \left(\partial_t T + v_i \partial_i T\right) = T_{ij} \partial_j v_i + k \partial_i^2 T + q. \tag{2.21}$$

So the final set of equations we will start our analysis with is given by

$$\begin{cases}
\partial_i v_i &= 0, \\
\rho \left(\partial_t v_i + v_j \partial_j v_i\right) &= -\partial_i p + \mu \partial_j^2 v_i + f_i^{\mathrm{b}}, \\
\rho c_p \left(\partial_t T + v_i \partial_i T\right) &= T_{ij} \partial_j v_i + k \partial_i^2 T + q,
\end{cases}$$

where ρ is given by (2.20).

2.6 Boundary conditions

So solve the equations stated earlier we need boundary conditions for the velocity and temperature. Information about the boundary conditions can be found in [2]. For the velocity at a solid boundary there are two common possibilities. In the inviscid case we assume the velocity is parallel to the boundary, that is, no fluid particle penetrates the boundary,

$$\mathbf{v} \cdot \hat{\mathbf{n}} = 0,$$

with $\hat{\mathbf{n}}$ the unit normal vector. In case of a viscous fluid, we assume the no-slip boundary condition, which says that the fluid 'sticks' to the boundary and its velocity is zero,

$$\mathbf{v} = 0.$$

For the temperature we can either assume the boundary is perfectly insulated, or that a certain heat flux is present. The latter is implemented by using a reference temperature and a certain heat conductivity coefficient.

For boundaries where fluid is allowed to cross we need different boundary conditions. The most common situation we will consider is a domain with an inflow and an outflow boundary. At the inflow we prescribe a certain velocity and temperature. At the outflow we assume the normal gradient of these quantities is zero.

Notice that we did not specify the pressure. The value of the pressure will usually follow from the equations. In case that the velocity at an inflow is not known we can prescribe the pressure. The velocity at the boundary then follows from the equations.

Chapter 3

Research questions

We start the analysis with the assumption that there is no feedback from the moving fluid to the electromagnetic fields. This means that we assume that the coupling between the Maxwell equations and the Navier-Stokes equations is one-way. We can solve the Maxwell equations, either in the time or frequency domain, using the material properties. From the resulting fields the force density can be determined. In case we are working in the frequency domain, we determine the time-averaged value of the force density. Furthermore if we assume the material has a positive conductivity, we determine the dissipation.
After determining the electromagnetic fields, we solve the Navier-Stokes equations, using the force density and dissipation term determined earlier in the stage.
Analytical solutions can only be determined in the most trivial case, one dimensional. The incompressible Navier-Stokes equations will lead to zero velocity in this case. For higher dimensional problems with more interesting geometries and dispersion we will need numerical methods.
The first step will be to investigate which software packages are suitable for these particular equations. Here we will both look at time-domain and frequency-domain methods. Interesting open source packages are OpenFOAM for the Navier-Stokes equations and Meep for the time-domain Maxwell equations. Also the commercial package COMSOL will be investigated. The advantage of the open source packages is that there is more flexibility. With these packages we will first investigate several simple cases, in particular the ones with analytical solutions, to validate the numerical methods used.
It is important part to investigate how accurate it is to neglect the effect of the flow on the electromagnetic fields. Since the complete coupled system is likely to hard to solve simultaneously, an iterative process is most likely to be suitable. We will first determine the electromagnetic fields and the corresponding force density and dissipation, considering the velocity field constant. From this we can we can determine the new velocity field, which in turn leads to new solution for the electromagnetic fields. In this process it is important to determine appropriate time stepping, so that the computations are still feasible, but the significant phenomena are not suppressed.
The next step will be to consider more complex geometries and electromagnetic sources. This will hopefully lead to more interesting flow phenomena. Our fluid of main interest is water. For water there are two regions in the electromagnetic spectrum that are of interest for us. It is well known that the absorption in the microwave region is large, which we expect to lead to relatively large temperature induced flow. Another interesting source is that of lasers. Although water is highly

transparent at the frequency of common lasers, because of their ability to deliver high power it is still an interesting source of heat.

Appendix A

Four-vector notation

In order to determine the right constitutive relations, and energy and momentum equations, we will use the four-vector notation from special relativity. Since the Maxwell equations are Lorentz invariant, we can get compact notation in this way. Once we have the correct expressions for the constitutive relations and energy and momentum equations, we will make assumptions about the fluid velocity being non-relativistic, in order to simplify things
We change to the coordinates $x^0 = ct$ and $x^i = x_i$ for $i = 1, 2, 3$. We use the notations of [3] and [6]. The partial derivatives are now given by

$$\partial_\mu = \begin{pmatrix} \frac{1}{c}\partial_t & \partial_1 & \partial_2 & \partial_3 \end{pmatrix}.$$

We will use the so-called $(- + ++)$ convention, meaning that the metric tensor is given by

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A.1}$$

We will need the four-vectors for the velocity and the electric and magnetic current densities. For the velocity we have

$$V^\mu = \begin{pmatrix} \gamma c & \gamma v_1 & \gamma v_2 & \gamma v_3 \end{pmatrix}, \tag{A.2}$$

where γ is the famous gamma factor, given by

$$\gamma = \frac{1}{\sqrt{1 - \frac{\|v\|^2}{c^2}}}.$$

For ordinary velocities we have $\gamma \approx 1$, and we will use this simplification later on. We define the electric current density four-vector

$$J^\mu = \begin{pmatrix} c\rho_\mathrm{e} & J_1 & J_2 & J_3 \end{pmatrix}, \tag{A.3}$$

and the magnetic current density four-vector

$$K^\mu = \begin{pmatrix} c\rho_\text{m} & K_1 & K_2 & K_3 \end{pmatrix}, \tag{A.4}$$

where J_i and K_i are the components of the normal current densities. If we separate the vector in a time and space part, we can write the space part in vector notation. We then get

$$V^\mu = \begin{pmatrix} \gamma c & \gamma \mathbf{v} \end{pmatrix}, \; J^\mu = \begin{pmatrix} c\rho_\text{e} & \mathbf{J} \end{pmatrix} \text{ and } K^\mu = \begin{pmatrix} c\rho_\text{m} & \mathbf{K} \end{pmatrix}.$$

We can now define the field tensor $F^{\nu\mu}$ as

$$F^{\mu\nu} = \begin{pmatrix} 0 & \frac{E_1}{c} & \frac{E_2}{c} & \frac{E_3}{c} \\ -\frac{E_1}{c} & 0 & \mu_0 H_3 & -\mu_0 H_2 \\ -\frac{E_2}{c} & -\mu_0 H_3 & 0 & \mu_0 H_1 \\ -\frac{E_3}{c} & \mu_0 H_2 & -\mu_0 H_1 & 0 \end{pmatrix}, \tag{A.5}$$

and the dual field tensor $G^{\nu\mu}$ as

$$G^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F^{\rho\sigma} = \begin{pmatrix} 0 & \mu_0 H_1 & \mu_0 H_2 & \mu_0 H_3 \\ -\mu_0 H_1 & 0 & -\frac{E_3}{c} & \frac{E_2}{c} \\ -\mu_0 H_2 & \frac{E_3}{c} & 0 & -\frac{E_1}{c} \\ -\mu_0 H_3 & -\frac{E_2}{c} & \frac{E_1}{c} & 0 \end{pmatrix}. \tag{A.6}$$

In vector notation these tensors are equal to

$$F^{\nu\mu} = \begin{pmatrix} 0 & \frac{1}{c}\mathbf{E} \\ -\frac{1}{c}\mathbf{E} & \cdot \times (\mu_0 \mathbf{H}) \end{pmatrix}$$

and

$$G^{\nu\mu} = \begin{pmatrix} 0 & \mu_0 \mathbf{H} \\ -\mu_0 \mathbf{H} & \cdot \times (\frac{1}{c}\mathbf{E}) \end{pmatrix},$$

where $\cdot \times \mathbf{E}$ means that vector with which is multiplied is inserted in the curl operator.
The Maxwell equations can now be stated as

$$\partial_\nu F^{\mu\nu} = J^\mu, \tag{A.7}$$
$$\partial_\nu G^{\mu\nu} = K^\mu. \tag{A.8}$$

Conversation of electric and magnetic charge are expressed as

$$\partial_\mu J^\mu = 0, \tag{A.9}$$
$$\partial_\mu K^\mu = 0. \tag{A.10}$$

Further expression can be derived for the Lorentz force and Poynting theorem as the gradient of the energy-stress tensor. At this point we will not need these results since we already derived them directly from the Maxwell equations.

Bibliography

[1] G. K. Batchelor. *An Introduction to Fluid Dynamics.* Cambridge University Press, cambridge mathematical library edition, 2000.

[2] J. H. Ferziger and M. Perić. *Computational Methods for Fluid Dynamics.* Springer, 3rd edition, 2002.

[3] David J. Griffiths. *Introduction to Electrodynamics.* Prentice Hall, 3rd edition, 1999.

[4] L. D. Landau and E. M. Lifshitz. *Electrodynamics of Continuous Media.* Pergamon Press, 2nd edition, 1984.

[5] L. D. Landau and E. M. Lifshitz. *Fluid Mechanics.* Pergamon Press, 2nd edition, 1984.

[6] Jack Vanderlinde. *Classical Electromagnetic Theory.* Kluwer Academic Publishers, 2nd edition, 2004.