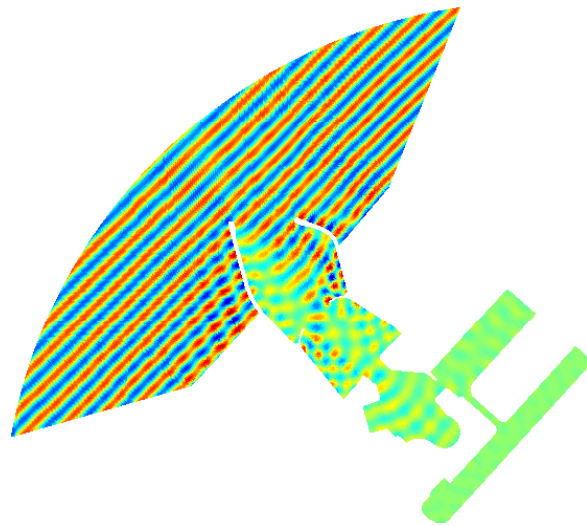Delft University of Technology

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Literature Study for the MSc Thesis
The Mild-Slope equation and its numerical implementation

by

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Preface

The graduation project of the master Applied Mathematics starts with a literature survey which spans more or less three months. After completion of the literature survey the research objectives are formed and the research part of the thesis starts. This document contains the literature study that I have performed during the last three months.

The master thesis is performed for the company Svašek Hydraulics [1], a specialist consultant in coastal, harbour and river engineering. They use advanced numerical models to determine the water dynamics, e.g. currents and waves, and sediment transport caused by water dynamics. Most of the used models are developed by the employees of Svašek Hydraulics and mainly based on the finite element method. One of these models is HARES (HArbour RESonance) which calculates the wave penetration in harbours and is especially useful in harbour and breakwater optimization studies. HARES incorporates the effects of diffraction, reflection, refraction, shoaling and directional spreading. Using HARES the natural frequency of the studied domain can be determined and therefore also the sensitivity for resonance. One of the latest additions is the inclusion of bottom friction and wave breaking, this leads to a non-linear problem. Hence more advanced solution techniques are necessary to obtain the solution. In the current software they define an outer iteration to treat the non-linearity of the problem and an inner loop which solves a linear problem. In HARES Picard iteration is used for the outer (non-linear) loop and Bi-CGSTAB preconditioned with a special form of the incomplete LU factorization for the inner loop. However the computational time gets undesired long when large domains are considered. The main aspect of the graduation project is to determine which numerical method leads to the least matrix-vector multiplications and hence a fast computation time.

Not only the different numerical methods have my interest, but also the mathematical physics behind the problem. Therefore also the derivation of the governing equations is presented in this literature study. The water dynamics in the coastal region can be described by the so called Mild-Slope equation, which includes the effects of diffraction, reflection, refraction and shoaling. The effects of bottom friction and wave breaking can be easily included by introducing an energy dissipation term. The Mild-Slope equation can be derived from the linearised small-amplitude wave equations.

Set-up of the literature study
This report is organized in the following way: In chapter 1 the linearised small-amplitude wave equations are derived, starting with the equation of motion and the equation for conservation of mass. Chapter 2 describes the combined effects of diffraction-reflection and refraction-shoaling and the corresponding equations are derived. In chapter 3 we combine the four

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[1]http://www.svasek.com/index.html

effects which leads to the Mild-Slope equation. In this chapter we also derive the corresponding boundary conditions and the equation for dissipation of wave energy due to bottom friction and wave breaking. Chapter 4 treats the numerical implementation of the Mild-Slope equation with the finite element method. The next chapter, chapter 5, contains the algorithm of the basic incomplete LU factorization and a short description of the numerical method Bi-CGSTAB derived by van der Vorst (1992). Since this method does not result in a satisfactory computational time the preconditioner shifted Laplace, Erlangga et al. (2004), and the matrix solver IDR(s), Sonneveld and van Gijzen (2008), are described in chapter 6. The shifted Laplace preconditioner and IDR(s) are tested in chapter 7 against the methods CG-S and Bi-CGSTAB. After that we have the conclusion of this literature study and the proposed research objectives and the appendix with detailed derivations.

Contents

Geometry

Figure 1: The geometry in a vertical section

Used symbols

$a(x, y, z)$	Amplitude; maximal deviation from the mean $z = 0$
c	Wave celerity $c = \frac{\omega}{k_0}$
c_g	Group velocity $c_g = nc$
D_b	Dissipation of wave energy due to wave breaking
D_f	Dissipation of wave energy due to bottom friction
E	Mechanical wave energy
E_k	Kinematic wave energy
E_p	Potential wave energy
$\boldsymbol{F}$	Total of forces per unit mass acting on a fluid element $\boldsymbol{F} = (F_1, F_2, F_3)^T$
$\boldsymbol{g}$	Gravitational acceleration $\boldsymbol{g} = (0, 0, -g)^T$
$h(x, y)$	Water height from the ocean bottom to the still water level
$\bar{h}$	Average water depth
H	Wave height $H = 2a$
k_0	Wave number when no energy dissipation is present $k_0 = \frac{2\pi}{L}$
k	Wave number when energy dissipation is present
l_0	Free surface characteristic length $l_0 = g/\omega^2$
L	Wave length $L = \frac{2\pi}{k_0}$
n_0	Fraction of mechanical wave energy in a wave that is transmitted forward each wave period
$\boldsymbol{p}(\boldsymbol{x}, t)$	Pressure vector $\boldsymbol{p} = (p_x, p_y, p_z)^T$
P_a	Environmental pressure
P	Wave power
R	Reflection coefficient
T	Wave period $T = \frac{2\pi}{\omega}$
$\boldsymbol{v}(\boldsymbol{x}, t)$	Velocity vector $\boldsymbol{v} = (v_1, v_2, v_3)^T = (u, v, w)^T$
W	Energy source term for the dissipation of wave energy
W_f	Energy source term due to bottom friction
W_b	Energy source term due to wave breaking
$\boldsymbol{x}$	Spatial coordinates $\boldsymbol{x} = (x, y, z)^T$
δ_{ik}	The Kronecker δ function
$\zeta(x, y, t)$	Height measured from the still water level $z = 0$
η	Viscosity coefficient $\eta > 0$
θ	Angle between an incidental wave and the normal vector $\mathbf{n}$
ν	Kinematic viscosity $\nu = \eta/\rho$
ρ	Water density
σ	The mean slope
σ_{ik}	Stress tensor of the fluid
σ'_{ik}	Viscous stress tensor
Φ	Velocity potential $\boldsymbol{v} = \nabla\Phi$
Ψ	Force potential $\boldsymbol{F} = -\nabla\Psi$
$\boldsymbol{\Omega}(\boldsymbol{x}, t)$	Vorticity vector defined by $\boldsymbol{\Omega} = \nabla \times \boldsymbol{v}$
ω	Wave frequency

Chapter 1

Surface gravity waves and their governing equations

In this chapter the equations that are used to describe the surface gravity waves are derived. We start with the derivation of the Navier-Stokes equation and its boundary conditions. These boundary conditions are non-linear in the unknowns, therefore the linearisation of these equations is treated. To be able to say more about the motion of the waves the solution of the two-dimensional system of equations is derived. And the end of this chapter this solution is used to say something about wave characteristics like the wave velocity and the wave length.

Ocean waves can be classified by their restoring force, e.g. compressibility, gravity and earth rotation. In harbours and near the shore the water surface waves are the ones most prominent and their main restoring force is gravity. The waves in this category are called the surface gravity waves. These waves have a time scale (1 - 25 s for wind waves and swells, 10 min - 2 h for tsunamis) such that compressibility and surface tension at one extreme and earth rotation at the other are of little importance (Mei, 1989).

In this section the governing equations for surface gravity waves will be derived. For the derivation of these equation of wave motion the following assumptions, as described in the Shore Protection Manual (1984), are made;

1. The fluid is homogeneous and incompressible; hence the density ρ can be considered constant.

2. The surface tension can be neglected, for increasing wave length the surface tension becomes less important.

3. The Coriolis effect, due to the rotation of the earth, can be neglected.

4. Pressure at the free surface is uniform and constant.

1.1 The Navier-Stokes equations

For the derivation of the well-known Navier-Stokes equations the approach by Dingemans (1997) is followed. It starts with the equation for conservation of mass and the equation of motion. The equation for conservation of mass reads

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{1.1.1}$$

and with the assumption that the fluid is homogeneous and incompressible equation (1.1.1) reduces to

$$\nabla \cdot \boldsymbol{v} = 0. \tag{1.1.2}$$

The general form of the equation of motion for an incompressible fluid is denoted by

$$\rho \frac{dv_i}{dt} = -\frac{\partial \sigma_{ik}}{\partial x_k} + \rho F_i, \tag{1.1.3}$$

where $\frac{d}{dt}$, the total derivative. is given by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla.$$

The term $\boldsymbol{F} = (F_1, F_2, F_3)^T$ represents the total of forces per unit mass acting on a fluid element. Since we only consider surface gravity waves the force $\boldsymbol{F}$ is given by

$$\boldsymbol{F} = \boldsymbol{g} = (0, 0, -g)^T. \tag{1.1.4}$$

σ_{ik} denotes the stress tensor for a real fluid with

$$\sigma_{ik} = -p\delta_{ik} + \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) = -p\delta_{ik} + \sigma'_{ik},$$

where $\eta > 0$ is the viscosity coefficient and δ_{ik} the Kronecker δ. The stress tensor represents the irreversible flow of momentum due to the viscosity of the fluid. Equation (1.1.2) states that $\partial v_j / \partial x_j = 0$ and since the order of partial differentiation can be reversed the following equation is obtained

$$\rho \frac{dv_i}{dt} = -\frac{\partial p}{\partial x_i} + \eta \frac{\partial^2 v_i}{\partial x_k^2} + \rho F_i,$$

or in vectorial form

$$\rho \frac{d\boldsymbol{v}}{dt} = -\nabla p + \eta \Delta \boldsymbol{v} + \rho \boldsymbol{F}. \tag{1.1.5}$$

Due to the form of the force $\boldsymbol{F}$, equation (1.1.4), it is derivable from a potential Ψ, this gives

$$\boldsymbol{F} = -\nabla \Psi = -\nabla(-\boldsymbol{g} \cdot \boldsymbol{x}),$$

with $\boldsymbol{g} \cdot \boldsymbol{x} = -gz$. Substitution of the force potential into equation (1.1.5) gives

$$\begin{aligned} \rho \frac{d\boldsymbol{v}}{dt} &= -\nabla p + \eta \Delta \boldsymbol{v} - \rho \nabla(gz), \\ &= -\nabla (p + \rho g z) + \eta \Delta \boldsymbol{v}. \end{aligned} \tag{1.1.6}$$

Equations (1.1.2) and (1.1.6) are also known as the Navier-Stokes equations. Dividing equation (1.1.6) by ρ and introducing $\nu = \eta/\rho$, the kinematic viscosity, gives

$$\frac{d\boldsymbol{v}}{dt} = -\nabla\left(\frac{\boldsymbol{p}}{\rho} + gz\right) + \nu\Delta\boldsymbol{v},$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\boldsymbol{v} = -\nabla\left(\frac{\boldsymbol{p}}{\rho} + gz\right) + \nu\Delta\boldsymbol{v}. \tag{1.1.7}$$

On the right-hand side of equation (1.1.7) the viscosity coefficient ν is present. To investigate whether this term has a strong influence on the behaviour of surface gravity waves the curl is taken on both sides of equation (1.1.7). This results in

$$\nabla\times\left(\frac{\partial\boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\right) = \nabla\times\left[-\nabla\left(\frac{\boldsymbol{p}}{\rho} + gz\right) + \nu\Delta\boldsymbol{v}\right].$$

Using that the curl of the gradient ($\nabla\times(\nabla\boldsymbol{a})$) and the divergence of the curl ($\nabla\cdot(\nabla\times\boldsymbol{a})$) are always equal to zero for any vector $\boldsymbol{a}$ and some forward calculations (see Appendix, A.1) gives

$$\frac{d\boldsymbol{\Omega}}{dt} = (\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{v} + \nu\Delta\boldsymbol{\Omega} \qquad \text{with } \boldsymbol{\Omega} = \nabla\times\boldsymbol{v}, \tag{1.1.8}$$

where $\boldsymbol{\Omega}(\boldsymbol{x},t)$ denotes the vorticity vector. Equation (1.1.8) physically means that the rate of change in the vorticity is due to stretching and twisting of vortex lines (first term on the right-hand side) and to viscous diffusion (second term on the right-hand side). For surface water waves a good approximation is to consider the fluid as inviscid, equation (1.1.8) reduces to

$$\frac{d\boldsymbol{\Omega}}{dt} = (\boldsymbol{\Omega}\cdot\nabla)\boldsymbol{v}. \tag{1.1.9}$$

Dingemans (1997) describes that short water waves usually are considered as a perturbation of the state of rest. When a fluid is at rest the vorticity $\boldsymbol{\Omega}$ is zero, hence the fluid is irrotational. Only the presence of viscosity can generate vorticity in the velocity field. The effect of viscosity will only be felt at the boundary of the sea bottom, therefore it can only generate vorticity there. The vorticity can diffuse inwards, but this takes only place when the motion is in only one direction. Since wave motion is considered as oscillatory motion, the direction of the flow changes often. Hence the inward diffusion remains restricted to the bottom boundary layer with thickness of order $\mathcal{O}(\sqrt{2\nu/\omega})$ (Stokes length, Dingemans (1997)) where ω is the frequency of the motion. This means that the major part of the fluid for short waves can be considered as irrotational. It is important to keep in mind that in regions of wave breaking the zero vorticity assumption does not hold.

In the derivation above we made two additional assumptions about the motion of surface gravity waves, namely;

1. The fluid is inviscid, i.e. $\nu \equiv 0$.

2. The major part of the fluid can be considered as irrotational, i.e. $\nabla\times\boldsymbol{v} \equiv 0$.

The resulting Navier-Stokes equations without the presence of viscosity, also known as the Euler equations, are given by

$$\nabla\cdot\boldsymbol{v} = 0, \tag{1.1.10}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \right) \boldsymbol{v} = -\nabla \left(\frac{\boldsymbol{p}}{\rho} + gz \right). \tag{1.1.11}$$

The assumption of irrotational flow results in the expression of the velocity $\boldsymbol{v}$ as the gradient of a scalar potential

$$\boldsymbol{v} = \nabla \Phi. \tag{1.1.12}$$

For the equation of conservation of mass, equation (1.1.10), substitution of the velocity potential leads to the Laplace equation;

$$\Delta \Phi = 0. \tag{1.1.13}$$

1.2 Derivation of the Bernoulli equation

To be able to determine the pressure in a surface gravity wave the Bernoulli equation is needed. Here the derivation of the Bernoulli equation is given as is described in Dingemans (1997).

Rewriting equation (1.1.11) gives,

$$\frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{2} \nabla \boldsymbol{v}^2 = -\nabla \left(\frac{\boldsymbol{p}}{\rho} + gz \right), \qquad \text{with } \nabla \boldsymbol{v}^2 = \nabla (\boldsymbol{v} \cdot \boldsymbol{v}). \tag{1.2.1}$$

Substituting the velocity potential into equation (1.2.1) results in

$$\nabla \left(\frac{\partial \Phi}{\partial t} + \frac{1}{2} (\nabla \Phi)^2 \right) = -\nabla \left(\frac{\boldsymbol{p}}{\rho} + gz \right). \tag{1.2.2}$$

Integration of equation (1.2.2) with respect of the space variables gives

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} (\nabla \Phi)^2 = -\frac{\boldsymbol{p}}{\rho} - gz + C(t).$$

Rewriting this equation and omitting the term $C(t)$ results in the Bernoulli equation

$$-\frac{\boldsymbol{p}}{\rho} = \frac{\partial \Phi}{\partial t} + \frac{1}{2} (\nabla \Phi)^2 + gz. \tag{1.2.3}$$

As can be seen from equation (1.2.3) the total pressure $\boldsymbol{p}$ is determined by a hydrostatic and hydrodynamic contribution. In section 1.3 the Bernoulli equation will also be used to derive a boundary condition at the free surface.

1.3 Boundary conditions for the Laplace equation $\Delta \Phi$

In this section the boundary conditions for the Laplace equation $\Delta \Phi = 0$ are derived. As can be seen in figure 1 there are two types of boundaries, i.e. the air-water interface and the bottom of the ocean. At the free surface there are two unknowns, namely the velocity potential Φ and the elevation of the free surface ζ. Therefore two conditions are needed. At the ocean bottom the only unknown is the velocity potential Φ, only one boundary condition

will lead to an unique solution. The derivation of the boundary conditions as described by Mei (1989) is followed. Mei assumes that along the boundaries the fluid only moves tangentially.

Boundary conditions for the free surface

For the free surface the following function is introduced

$$F(x, y, z, t) = z - \zeta(x, y, t), \tag{1.3.1}$$

it is easily seen that at the free surface, when $z = \zeta$, it holds that $F = 0$. A point $\boldsymbol{x}$ on the moving surface has velocity $\boldsymbol{q}$, after a short time dt the free surface is described by

$$F(\boldsymbol{x} + \boldsymbol{q}dt, t + dt),$$

and the Taylor expansion around $F(\boldsymbol{x}, t)$ is given by

$$F(\boldsymbol{x} + \boldsymbol{q}dt, t + dt) = F(\boldsymbol{x}, t) + \boldsymbol{q} \cdot \nabla F \; dt + \frac{\partial F}{\partial t}dt + \mathcal{O}(dt)^2,$$
$$= F(\boldsymbol{x}, t) + \left(\frac{\partial F}{\partial t} + \boldsymbol{q} \cdot \nabla F\right) dt + \mathcal{O}(dt)^2.$$

The equation above should also be zero when $z = \zeta$. The first term on the left hand side equals zero by definition as $z = \zeta$, hence it follows that the second term also should be zero for $z = \zeta$. The assumption of tangential motion requires that $\boldsymbol{v} \cdot \nabla F = \boldsymbol{q} \cdot \nabla F$, this implies

$$\frac{\partial F}{\partial t} + \boldsymbol{v} \cdot \nabla F = 0 \quad \text{for } z = \zeta. \tag{1.3.2}$$

Substitution of the velocity potential, equation (1.1.12), and the function for the free surface, equation (1.3.1), into equation (1.3.2) gives

$$-\frac{\partial \zeta}{\partial t} - \frac{\partial \Phi}{\partial x}\frac{\partial \zeta}{\partial x} - \frac{\partial \Phi}{\partial y}\frac{\partial \zeta}{\partial y} + \frac{\partial \Phi}{\partial z} = 0 \quad \text{for } z = \zeta,$$

rewriting results in

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \Phi}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial \zeta}{\partial y} = \frac{\partial \Phi}{\partial z} \quad \text{for } z = \zeta. \tag{1.3.3}$$

Equation (1.3.3) is denoted as the kinematic boundary condition. Which corresponds to the assumption that the fluid moves only tangentially to the surface. The second boundary condition can be obtained with the Bernoulli equation, equation (1.2.3). The assumption of a uniform and constant pressure P_a, the atmospheric pressure, at the free surface is substituted in the Bernoulli equation, this gives

$$-\frac{P_a}{\rho} = \frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla \Phi)^2 + g\zeta \quad \text{for } z = \zeta.$$

This boundary condition is known as the dynamical boundary condition at the free surface.

Boundary condition for the ocean bottom

At the ocean bottom it holds that, see figure 1, $z = -h(x, y)$. Equation (1.3.1) transforms for the sea bottom into

$$F(x, y, z) = z + h(x, y),$$

with $F = 0$ as $z = -h$. For the boundary at the ocean bottom the same condition as stated in equation (1.3.2) must hold, since here the movement is also tangentially. This results in the kinematic boundary condition

$$-\frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial \Phi}{\partial y} \frac{\partial h}{\partial y} \quad \text{for } z = -h. \tag{1.3.4}$$

Summary of the boundary conditions

The boundary conditions that are derived in this section for surface gravity waves are

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \Phi}{\partial x} \frac{\partial \zeta}{\partial x} + \frac{\partial \Phi}{\partial y} \frac{\partial \zeta}{\partial y} = \frac{\partial \Phi}{\partial z} \qquad ; \quad z = \zeta,$$

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} (\nabla \Phi)^2 + g\zeta = -\frac{P_a}{\rho} \qquad ; \quad z = \zeta,$$

$$\frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial \Phi}{\partial y} \frac{\partial h}{\partial y} = 0 \qquad ; \quad z = -h.$$

1.4   Linearisation of boundary conditions

The equations that are obtained so far are

$$\Delta \Phi = 0 \qquad\qquad ; \quad -h(\boldsymbol{x}) \leq z \leq \zeta(\boldsymbol{x}, t) \tag{1.4.1}$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \Phi}{\partial x} \frac{\partial \zeta}{\partial x} + \frac{\partial \Phi}{\partial y} \frac{\partial \zeta}{\partial y} = \frac{\partial \Phi}{\partial z} \qquad ; \quad z = \zeta(\boldsymbol{x}, t) \tag{1.4.2}$$

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2} (\nabla \Phi)^2 + g\zeta = -\frac{P_a}{\rho} \qquad ; \quad z = \zeta(\boldsymbol{x}, t) \tag{1.4.3}$$

$$\frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial x} \frac{\partial h}{\partial x} + \frac{\partial \Phi}{\partial y} \frac{\partial h}{\partial y} = 0 \qquad ; \quad z = -h(\boldsymbol{x}) \tag{1.4.4}$$

It is easily seen that these equations are non-linear in the unknown velocity potential Φ and the free-surface elevation ζ. Finding a solution for non-linear equations is in general very difficult, therefore a linearisation of equations (1.4.1) - (1.4.4) is made based on the description made by Mei (1989).

The scale for Φ is chosen as $a\omega L/2\pi$, with $a = |\zeta|$ the amplitude of the wave, ω the wave frequency and L the wave length, and for $\boldsymbol{x}$ the scale is given by $L/2\pi$. With this choice the velocity $\boldsymbol{v}$ will be scaled with $a\omega$. The dimensionless variables, denoted with primes, are determined by

$$\Phi = \frac{a\omega L}{2\pi} \Phi', \quad (x, y, z, h) = \frac{L}{2\pi}(x', y', z', h'), \quad t = \frac{1}{\omega} t' \quad \text{and} \quad \zeta = a\zeta'.$$

Inserting these scaled variables into equations (1.4.1) - (1.4.4) gives the following set of dimensionless equations (see Appendix, A.2)

$$\left(\frac{\partial^2}{\partial(x')^2} + \frac{\partial^2}{\partial(y')^2} + \frac{\partial^2}{\partial(z')^2}\right)\Phi = \Delta'\Phi' = 0 \qquad ; \quad -h' \leq z' \leq \epsilon\zeta' \qquad (1.4.5)$$

$$\frac{\partial\Phi'}{\partial z'} = \frac{\partial\zeta'}{\partial t'} + \epsilon\left(\frac{\partial\Phi'}{\partial x'}\frac{\partial\zeta'}{\partial x'} + \frac{\partial\Phi'}{\partial y'}\frac{\partial\zeta'}{\partial y'}\right) \qquad ; \quad z' = \epsilon\zeta' \qquad (1.4.6)$$

$$\frac{\partial\Phi'}{\partial t'} + \frac{1}{2}\epsilon(\nabla'\Phi')^2 + \left(\frac{2\pi g}{\omega^2 L}\right)\zeta' = -P_a' \qquad ; \quad z' = \epsilon\zeta' \qquad (1.4.7)$$

$$\frac{\partial\Phi'}{\partial z'} + \frac{\partial\Phi'}{\partial x'}\frac{\partial h'}{\partial x'} + \frac{\partial\Phi'}{\partial y'}\frac{\partial h'}{\partial y'} = 0 \qquad ; \quad z' = -h' \qquad (1.4.8)$$

where $\epsilon = \frac{2\pi a}{L}$ = wave slope and $P_a' = \frac{2\pi P_a}{\rho a\omega^2 L}$.

In the dimensionless equations the term ϵ of the wave slope appears, when the wave slope is small ($\epsilon \ll 1$) the waves are called small-amplitude waves. Equations (1.4.6) and (1.4.7) are still non-linear in the unknown variables Φ' and ζ'. The unknown free surface elevation ζ' differs only an amount of $\mathcal{O}(\epsilon)$ from the still water level $z' = 0$, therefore Φ' can be expanded in a Taylor series around $z' = 0$,

$$\Phi'(x', y', \epsilon\zeta', t') = \Phi'|_{z'=0} + \epsilon\zeta'\frac{\partial\Phi'}{\partial z'}\bigg|_{z'=0} + \frac{(\epsilon\zeta')^2}{2!}\frac{\partial^2\Phi'}{\partial z'^2}\bigg|_{z'=0} + \mathcal{O}\left(\epsilon^3\right).$$

Substitution of this expansion into equation (1.4.6) gives

$$\frac{\partial}{\partial z'}\left(\Phi'|_{z'=0} + \epsilon\zeta'\frac{\partial\Phi'}{\partial z'}\bigg|_{z'=0} + \ldots\right) = \frac{\partial\zeta'}{\partial t'}$$

$$+\epsilon\left[\frac{\partial}{\partial x'}\left(\Phi'|_{z'=0} + \epsilon\zeta'\frac{\partial\Phi'}{\partial z'}\bigg|_{z'=0} + \ldots\right)\frac{\zeta'}{\partial x'} + \frac{\partial}{\partial y'}\left(\Phi'|_{z'=0} + \epsilon\zeta'\frac{\partial\Phi'}{\partial z'}\bigg|_{z'=0} + \ldots\right)\frac{\partial\zeta'}{\partial y'}\right].$$

For equation (1.4.7) this results in

$$\frac{\partial}{\partial t'}\left(\Phi'|_{z'=0} + \epsilon\zeta'\frac{\partial\Phi'}{\partial z'}\bigg|_{z'=0} + \ldots\right) + \frac{1}{2}\epsilon\left(\nabla'\left(\Phi'|_{z'=0} + \epsilon\zeta'\frac{\partial\Phi'}{\partial z'}\bigg|_{z'=0} + \ldots\right)\right)^2 + \frac{2\pi g}{\omega^2 L}\zeta' = -P_a'$$

Hence the zeroth order, $\mathcal{O}(1)$, terms of the free surface boundary conditions are given by

$$\frac{\partial\zeta'}{\partial t'} = \frac{\partial\Phi'}{\partial z'} \qquad ; \quad z' = 0, \qquad (1.4.9)$$

$$\frac{\partial\Phi'}{\partial t'} + \frac{2\pi g}{\omega^2 L}\zeta' = -P_a' \qquad ; \quad z' = 0 \qquad (1.4.10)$$

These conditions can be combined by taking de derivative with respect to time of equation (1.4.10), this results in

$$\frac{\partial^2\Phi'}{\partial t'^2} + \frac{2\pi g}{\omega^2 L}\frac{\partial\Phi'}{\partial z'} = 0, \qquad (1.4.11)$$

keeping in mind the assumption of a constant environmental pressure P_a.

After returning to the physical variables, the following linearised equations for small-amplitude waves are obtained

$$\Delta \Phi = 0 \qquad\qquad ; \quad -h(\boldsymbol{x}) \leq z \leq 0, \qquad (1.4.12)$$

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \Phi}{\partial z} \qquad\qquad ; \quad z = 0, \qquad (1.4.13)$$

$$\frac{\partial \Phi}{\partial t} + g\zeta = -\frac{P_a}{\rho} \qquad\qquad ; \quad z = 0, \qquad (1.4.14)$$

$$\frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial h}{\partial y} = 0 \qquad\qquad ; \quad z = -h(\boldsymbol{x}). \qquad (1.4.15)$$

The linearisation of the Bernoulli equation (1.2.3) is given by

$$P = -\rho g z - \rho \frac{\partial \Phi}{\partial t}. \qquad (1.4.16)$$

The first term on the right hand side represent the hydrostatic pressure and the second term the dynamic pressure.

1.5 2D progressive water waves on a constant depth

In this section the solution of the two-dimensional version of equations (1.4.12) - (1.4.15) is determined. To do this the assumption of a constant depth is made, hence the derivatives of h will be zero, and the environmental pressure P_a is set equal to zero. The solution of the two-dimensional set of equations corresponds to the solution of the three-dimensional system, as is described at the end of this section. Because in two dimensions the solution is more easily obtained the choice to solve this system is made. The solution that is obtained in this section shows the way in which variables like the wave frequency ω and the wave number k_0 influence the shape of the solution. The form of the solution obtained for the velocity potential Φ will also be used in the derivation of the equations for the combined phenomenon of diffraction and reflection, see section 2.1, and the Mild-Slope equation, see chapter 3. The solution is derived with the method of separation of variables and the calculations made by Dingemans (1997) are followed.

The two-dimensional form of equations (1.4.12) - (1.4.15) with a constant depth and $P_a = 0$ are given by

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad\qquad ; \quad -h(x) \leq z \leq 0, \qquad (1.5.1)$$

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \Phi}{\partial z} \qquad\qquad ; \quad z = 0, \qquad (1.5.2)$$

$$\frac{\partial \Phi}{\partial t} + g\zeta = 0 \qquad\qquad ; \quad z = 0, \qquad (1.5.3)$$

$$\frac{\partial \Phi}{\partial z} = 0 \qquad\qquad ; \quad z = -h(x). \qquad (1.5.4)$$

With the method of separation of variables the unknown velocity potential Φ is written as

$$\Phi(x, z, t) = X(x)Z(z)T(t).$$

Substituting this expression for Φ into equation (1.5.1), while assuming $T(t) \neq 0$, gives

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -\frac{1}{Z}\frac{d^2 Z}{dz^2}.$$

The method of separation of variables states that both sides are equal to a constant, $-k_0^2$ is chosen as this constant. This gives

$$\frac{1}{X}\frac{d^2 X}{dx^2} = -k_0^2 \quad \text{and} \quad \frac{1}{Z}\frac{d^2 Z}{dz^2} = k_0^2.$$

The general solutions for $X(x)$ and $Z(z)$ are given by

$$X(x) = A\cos(k_0 x) + B\sin(k_0 x) \quad \text{and} \quad Z(z) = Ce^{k_0 z} + De^{-k_0 z}. \tag{1.5.5}$$

Application of the boundary condition, equation (1.5.4), at $z = -h$ to the solution of $Z(z)$ gives

$$\frac{dZ}{dz} = 0, \quad \text{hence} \quad Ce^{-k_0 h} - De^{k_0 h} = 0. \tag{1.5.6}$$

We set $\Gamma/2 = Ce^{-k_0 h} = De^{k_0 h}$ and substitute this into the solution of $Z(z)$, this gives

$$Z(z) = \frac{\Gamma}{2}\left(e^{k_0(z+h)} + e^{-k_0(z+h)}\right) = \Gamma\cosh(k_0(z+h)). \tag{1.5.7}$$

The solutions that are obtained for $X(x)$ and $Z(z)$ are substituted in the expression for Φ, hence Φ can now be written as

$$\Phi(x, z, t) = \Gamma\cosh(k_0(z+h))\left(A\cos(k_0 x) + B\sin(k_0 x)\right)T(t). \tag{1.5.8}$$

Due to the form of the solution in the x-coordinate it is seen that the separation constant k_0 denotes the wave number. To determine the structure of $T(t)$ boundary conditions (1.5.2) and (1.5.3) at the free surface $z = 0$ are applied. This gives

$$\frac{\partial \zeta}{\partial t} = k\Gamma\sinh(k_0 h)\left(A\cos(k_0 x) + b\sin(k_0 x)\right)T(t), \tag{1.5.9}$$

$$-g\zeta = \Gamma\cosh(k_0 h)\left(A\cos(k_0 x) + B\sin(k_0 x)\right)\frac{dT}{dt}. \tag{1.5.10}$$

Equations (1.5.9) and (1.5.10) can be combined by taking the derivative of (1.5.10) with respect to t. This results in

$$\frac{d^2 T}{dt^2} + gk_0\tanh(k_0 h)T = 0, \tag{1.5.11}$$

where it is assumed that $A\cos(k_0 x) + B\sin(k_0 x) \neq 0$. Let

$$\omega^2 = gk_0\tanh(k_0 h) > 0, \quad h > 0 \text{ and all } k_0, \tag{1.5.12}$$

then the general solution for $T(t)$ is given by

$$T(t) = E\cos(\omega t) + F\sin(\omega t).$$

The solution for Φ obtained with the method of separation of variables for a single value of the wave number k is given by

$$\Phi(x, z, t) = \Gamma\cosh(k_0(z+h))\left(A\cos(k_0x) + b\sin(k_0x)\right)\left(E\cos(\omega t) + F\sin(\omega t)\right). \quad (1.5.13)$$

This solution shows that ω represents the wave frequency, due to the form of the solution of $T(t)$. Equation (1.5.12) is called the dispersion relation, with this relation for each given depth h and wave frequency ω the corresponding wave number k_0 can be determined. A progressive wave solution can be obtained of the form

$$\Phi(x, z, t) = A\cosh(k_0(z+h))\sin(k_0x - \omega t).$$

Substitution of this solution into the kinematic boundary condition for $z = 0$, equation (1.5.3), gives

$$-\omega A\cosh(k_0h)\cos(k_0x - \omega t) + g\zeta = 0,$$

hence the solution of the free-surface elevation $\zeta(x, t)$ results in

$$\zeta(x, t) = \frac{A\omega}{g}\cosh(k_0h)\cos(k_0x - \omega t). \quad (1.5.14)$$

The amplitude $a(x, t)$ of the free surface elevation $\zeta(x, t)$ is given by

$$a(x, t) = \frac{A\omega}{g}\cosh(k_0h).$$

The solutions, derived with the method of separation of variables, of $\zeta(x, t)$ and $\Phi(x, z, t)$ is given by

$$\zeta(x, t) \quad = \quad a\cos(k_0x - \omega t), \quad (1.5.15)$$

$$\Phi(x, z, t) \quad = \quad \frac{ag}{\omega}\frac{\cosh(k_0(z+h))}{\cosh(k_0h)}\sin(k_0x - \omega t). \quad (1.5.16)$$

By the definition of the velocity potential Φ the velocity u in the horizontal direction and w in the vertical direction respectively can be determined. This results in

$$u(x, z, t) = \frac{\partial\Phi}{\partial x} = \frac{agk_0}{\omega}\frac{\cosh(k_0(z+h))}{\cosh(k_0h)}\cos(kx - \omega t), \quad (1.5.17)$$

$$w(x, z, t) = \frac{\partial\Phi}{\partial z} = \frac{agk_0}{\omega}\frac{\sinh(k_0(h+z))}{\cosh(k_0h)}\sin(k_0x - \omega t). \quad (1.5.18)$$

Solution for three-dimensional wave motion

As stated earlier the solution of the three-dimensional wave motion is similar to the solution of the two-dimensional system, equations (1.5.15) and (1.5.16). Wave motion is a behaviour

in three dimensions, therefore the solution of the three-dimensional system, equations (1.4.12) - (1.4.15), is given here. The only change in the three-dimensional solution occurs in the term $k_0 x$ since now both k_0 and x are a vector, namely

$$\boldsymbol{x} = \left(\begin{array}{c} x \\ y \end{array} \right) \quad \text{and} \quad \boldsymbol{k_0} = \left(\begin{array}{c} k_1 \\ k_2 \end{array} \right).$$

Define $\tilde{k} = |\boldsymbol{k_0}| = \sqrt{k_1^2 + k_2^2}$ and the three-dimensional form of equations (1.5.15) - (1.5.18) becomes

$$\zeta(\boldsymbol{x}, t) = a \cos(\boldsymbol{k_0} \cdot \boldsymbol{x} - \omega t), \tag{1.5.19}$$

$$\Phi(\boldsymbol{x}, z, t) = \frac{ag}{\omega} \frac{\cosh(\tilde{k}(z+h))}{\cosh(\tilde{k}h)} \sin(\boldsymbol{k_0} \cdot \boldsymbol{x} - \omega t), \tag{1.5.20}$$

$$u(\boldsymbol{x}, z, t) = \frac{ag\boldsymbol{k_0}}{\omega} \frac{\cosh(\tilde{k}(z+h))}{\cosh(\tilde{k}h)} \cos(\boldsymbol{k_0} \cdot \boldsymbol{x} - \omega t), \tag{1.5.21}$$

$$w(\boldsymbol{x}, z, t) = \frac{ag\boldsymbol{k_0}}{\omega} \frac{\sinh(\tilde{k}(z+h))}{\cosh(\tilde{k}h)} \sin(\boldsymbol{k_0} \cdot \boldsymbol{x} - \omega t), \tag{1.5.22}$$

with the dispersion relation $\omega^2 = g\tilde{k} \tanh(\tilde{k}h)$.

1.6 Wave characteristics

In the previous section the solutions for the velocity potential Φ, the free-surface elevation ζ and the velocity components u and w are obtained with the method of separation of variables. Using these solutions, expressions for the wave velocity c, wave length L, wave energy E and the wave power P can be determined. In this section these expressions are derived and some simplifications are made for the expression for the wave velocity c and wave length L.

The wave characteristics (wave speed c, wave number k_0 and wave frequency ω) are given by the following relations

$$c = \frac{L}{T}, \quad k_0 = \frac{2\pi}{L} \quad \text{and} \quad \omega = \frac{2\pi}{T}.$$

Combining these three relation and using the expression of the dispersion relation, equation (1.5.12), gives

$$c = \frac{\omega}{k_0} = \sqrt{\frac{gL}{2\pi} \tanh\left(\frac{2\pi h}{L}\right)}.$$

1.6.1 Wave classification

Waves can be classified into three categories; i.e. deep-water waves, intermediate-water waves and shallow-water waves. This classification is made on the ratio $\frac{h}{L}$ known as the relative depth, where h is the water height and L the wave length (figure 1). The values of $\frac{h}{L}$ and the corresponding approximations are based on the description made by Sorensen (1997).

- For $\frac{h}{L} > 0.5$ we have deep-water waves. In this scenario the waves do not interact with the sea bottom, hence the wave characteristics will be independent of the water height. The shape of a particle track line in deep-water wave flow is a circle.

- For $0.05 < \frac{h}{L} < 0.5$ the waves are called intermediate-water waves and some interaction with the bottom occurs. Therefore the water height h will be present in the wave characteristics. The shape of the particle track is an ellipse.

- For $\frac{h}{L} < 0.05$ shallow-water waves occur and the particles have a strong interaction with the bottom. The depth of a particle will determine the shape of the motion of a particle. At the free surface the movement will resemble a flattened off ellipse, deeper in the water column the movement is shaped as a line segment, the particles move only in the horizontal direction. In the wave characteristic the the water height h plays a crucial role.

For deep-water waves the approximation $\tanh(2\pi h/L) \approx 1$ can be made, for shallow-water waves $\tanh(2\pi h/L) \approx \frac{2\pi h}{L}$. Using these approximations we get the following overview of the wave characteristics.

Wave type	Wave speed	Wave length
Deep-water wave	$c_0 = \sqrt{\frac{gL_0}{2\pi}} = \frac{gT}{2\pi}$	$L_0 = \frac{gT^2}{2\pi}$
Intermediate-water waves	$c = c_0 \tanh\left(\frac{2\pi h}{L}\right)$	$L = L_0 \tanh\left(\frac{2\pi h}{L}\right)$
Shallow-water waves	$c = \sqrt{gh}$	$L = \sqrt{gh}T$

Table 1.1: Approximations of the wave speed and wave length for the three different types of waves that are considered.

The subscript 0 denotes the deep-water value of the characteristic. The characteristics for the intermediate-water waves are valid for all values of h and L, the characteristics for deep-water waves and shallow-water waves are special limit cases only valid for the restrictions on the relative depth. Since the relative depth is not always known a priori only the expressions for the intermediate-water waves are used.

1.6.2 Wave energy and wave power

An important characteristic of gravity waves is their mechanical wave energy. This energy is transmitted forward as the wave propagates.

Wave energy

Sorensen (1997) describes that the mechanical wave energy is determined by the sum of the kinetic and potential energy. The kinetic energy for a unit width of wave crest and one wave length can be determined by the integral over one wave length and the water depth of one-half

times the mass of a differential element times the velocity of that element squared. This leads to

$$E_k = \int_0^L \int_{-h}^0 \frac{1}{2}\rho \ dzdx \ (u^2 + w^2), \tag{1.6.1}$$

where expressions (1.5.17) for the horizontal velocity u and (1.5.18) for the vertical velocity w is used. After some manipulations (see Appendix, A.3) the kinematic energy is given by

$$E_k = \frac{\rho g H^2 L}{16}.$$

The potential energy solely due to the wave form can be determined by subtracting the potential energy of a mass of still water, $\rho g L h \frac{h}{2}$, from the potential energy of the wave form. This gives

$$E_p = \int_0^L \rho g(h + \zeta) \left(\frac{h + \zeta}{2}\right) \ dx \ - \ \rho g L h \left(\frac{h}{2}\right), \tag{1.6.2}$$

where equation (1.5.15) for the elevation of the free surface ζ is used. Noting that the wave amplitude a equals $\frac{1}{2}H$, half of the wave height, the potential energy is given by (see Appendix, A.3)

$$E_p = \frac{\rho g H^2 L}{16}.$$

The mechanical wave energy E is determined by the sum of the kinematic and potential energy, hence it is given by

$$E = E_k + E_p = \frac{\rho g H^2 L}{16} + \frac{\rho g H^2 L}{16} = \frac{\rho g H^2 L}{8}. \tag{1.6.3}$$

Usually we are interested in the energy per unit length, the total mechanical energy E divided by the wave length L, hence

$$\bar{E} = \frac{E}{L} = \frac{\rho g H2}{8}. \tag{1.6.4}$$

Wave power

The wave power P is the energy per unit time transmitted in the direction of the wave propagation. This can be calculated by the wave period average of the integral over one wave period and the water depth of the dynamic pressure and the velocity u in the horizontal direction. This results in as is given in Sorenson (1997)

$$P = \frac{1}{T} \int_0^T \int_{-h}^0 \left(-\rho \frac{\partial \Phi}{\partial t}\right) u \ dzdt, \tag{1.6.5}$$

where $\partial \Phi / \partial t$ is determined by the expression given in equation (1.5.16). After some straight-

forward calculations (see Appendix, A.4) the wave power is given by

$$P = \frac{\rho g H^2 L}{16T}\left(1 + \frac{2k_0 h}{\sinh(2k_0 h)}\right),$$

$$= \frac{\bar{E}L}{2T}\left(1 + \frac{2k_0 h}{\sinh(2k_0 h)}\right),$$

$$= \frac{n_0 \bar{E} L}{T} = \bar{E} n_0 c = \bar{E} c_g \quad \text{with} \quad n_0 = \frac{1}{2}\left(1 + \frac{2k_0 h}{\sinh(2k_0 h)}\right). \tag{1.6.6}$$

The term c_g denotes the group velocity and is treated in section 1.6.3. From the expression for P it can be seen that the group velocity is also the velocity of the transport of wave energy. The value of n_0 increases as the water depth decreases, namely for deep-water waves n_0 equals 0.5 while for shallow-water waves we have that $n_0 = 1.0$.

1.6.3 Group velocity

Until now only one wave of the form $\cos(k_0 x - \omega t)$ is considered. Usually wave motion is composed of many wave frequencies, expressed as the summation of all the individual wave solutions, which moves with a group velocity c_g. The derivation of the group velocity is described by Dingemans (1997).

Consider two waves with slightly different wave frequencies ω_1 and ω_2 and wave numbers k_1 and k_2, for simplicity the same phase shift α and amplitude a is used. The individual waves are given by

$$\zeta_1 = a\cos(k_1 x - \omega_1 t + \alpha) = a\cos(\gamma_1 + \alpha) \quad \text{and} \quad \zeta_2 = a\cos(k_2 x - \omega_2 t + \alpha) = a\cos(\gamma_2 + \alpha).$$

To determine the free surface elevation of the combined wave the waves ζ_1 and ζ_2 are summed, this results in

$$\zeta_1 + \zeta_2 = 2a\cos\left(\frac{(\gamma_1 + \alpha) - (\gamma_2 + \alpha)}{2}\right)\cos\left(\frac{(\gamma_1 + \alpha) + (\gamma_2 + \alpha)}{2}\right),$$

$$= 2a\cos\left(\frac{k_1 - k_2}{2}x - \frac{\omega_1 - \omega_2}{2}t\right)\cos\left(\frac{k_1 + k_2}{2}x - \frac{\omega_1 + \omega_2}{2}t + \alpha\right). \tag{1.6.7}$$

Define

$$k = \frac{k_1 + k_2}{2} \quad \text{and} \quad \omega = \frac{\omega_1 + \omega_2}{2}.$$

The wave frequencies ω_1 and ω_2 and wave numbers k_1 and k_2 were just slightly different chosen, therefore it follows that

$$k_1 - k_2 = \Delta k \ll k \quad \text{and} \quad \omega_1 - \omega_2 = \Delta\omega \ll \omega.$$

Substitution of these expressions into equation (1.6.7) gives the expression for the combined free surface elevation

$$\zeta(x,t) = 2a \cos \left(\frac{\Delta k}{2} x - \frac{\Delta \omega}{2} t \right) \cos(kx - \omega t + \alpha).$$

The free surface elevation consist of the carrier part (the term $\cos(kx - \omega t + \alpha)$) and the wave envelope (the other part). The whole group moves with speed $c_g = x/t = \Delta \omega / \Delta k$. For this group velocity the phases γ_1 and γ_2 remain equal so that the two components are in phase. If the differences between the two waves numbers and wave frequencies become infinitesimal, the group velocity is given by

$$c_g = \frac{d\omega}{dk_0}. \tag{1.6.8}$$

Calculations show that (see Appendix, A.5) the relation between the group velocity and the wave velocity is given by

$$c_g = n_0 c \qquad \text{with} \quad n_0 = \frac{1}{2} \left(1 + \frac{2k_0 h}{\sinh(2k_0 h)} \right).$$

For deep-water waves it holds that $n_0 = 0.5$ hence the group moves at half the speed of the individual waves. For shallow-water waves we have that $n_0 = 1.0$, thus there is no difference between the velocity of the group and the velocity of the individual waves.

Chapter 2

Wave transformations for surface gravity waves

When modelling the wave motion in a harbour or near the shore it is not sufficient to use the equations derived in the section 1.4. Near the shore the water depth decreases and objects such as wave breakers or tidal entrances influence the behaviour of the waves. The four major phenomena observed for a single wave motion are refraction, diffraction, shoaling and reflection. The combined effects of diffraction and reflection is described in section 2.1 and refraction and shoaling are treated in section 2.2. In this section only the basic equations are considered, extensions such as interaction of the waves with a current are not included.

2.1 Diffraction and Reflection

> *Definition*
>
> **Diffraction** - The bending and spreading of a wave around an edge of
> an object. The wave energy is transferred laterally along the wave crests.
> **Reflection** - The return of all or part of the wave energy due to the
> encounter of a boundary.

Calculations based on diffraction and reflection are important since wave height is in some degree determined by the characteristics of the present structures, to reduce problems as silting and harbour resonance and naturally occurring changes in hydography are affected by diffraction and reflection.

Figure 2.1 (Shore Protection Manual (1984)) shows two situations, the first one is the hypothetical case that no diffraction and reflection occurs. The incoming waves are not disturbed by the presence of the breakwater. The second situation on the right shows the wave behaviour when diffraction and reflection do occur. In region III there is a superposition of the incident wave and the reflected wave. This superposition leads to the appearance of short-crested waves. In region I the bending of the diffracted waves occurs, the energy flow is from region II into region I. The waves present in region II are not affected by the effects of diffraction and reflection.

The following assumptions, as described in the Shore Protection Manual (1984), are made when modelling the effects of diffraction and reflection,
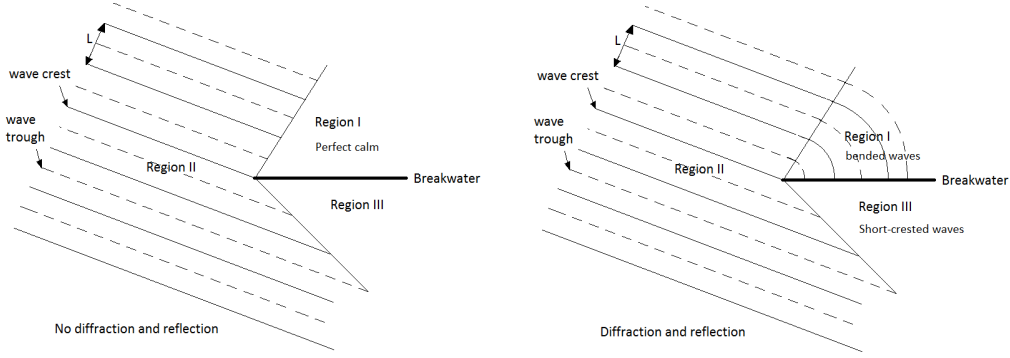
Figure 2.1: The left figure shows the situation when no diffraction and reflection takes place. The figure on the right shows the wave behaviour when diffraction and reflection do occur.

1. Water is an ideal fluid, i.e. inviscid, irrotational and incompressible.

2. Small-amplitude waves can be described by linear wave theory.

3. The variation of the bottom profile can be neglected.

4. The obstacles are cylindrical extending from bottom to surface.

The second assumption indicates that the linearised time harmonic equations (2.2.4) - (2.2.6) can be used as a starting point for the derivation of the diffraction-reflection model. Due to the third assumption no variation of the bottom profile is present, this means that the derivatives of h are zero. A constant water height also results in a constant wave number k. The linearised time harmonic equations without water height variation are given by

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad ; \quad -h(x,y) \leq z \leq 0, \qquad (2.1.1)$$

$$g\frac{\partial \Phi}{\partial z} - \omega^2 \Phi = 0 \qquad ; \quad z = 0, \qquad (2.1.2)$$

$$\frac{\partial \Phi}{\partial z} = 0 \qquad ; \quad z = -h(x,y). \qquad (2.1.3)$$

To determine the equations for the diffraction-reflection phenomenon the derivation of Berkhoff (1976) is followed. The solution Φ of the diffraction-reflection model can be written as a superposition of a known incident wave potential $\tilde{\Phi}$ and a scattered wave potential Φ_d, hence $\Phi = \tilde{\Phi} + \Phi_d$. The scattered wave potential is given by

$$\Phi_d(x,y,z) = \sum_{j=0} Z_j(z)\varphi_j(x,y) \quad \text{with} \quad Z_j(z) = C_j \cosh(k_j(z+h)), \qquad (2.1.4)$$

and the incident wave potential

$$\tilde{\Phi}(x,y,z) = -\frac{1}{2}\frac{Hg}{\omega}i\frac{\cosh(k_0(z+h))}{\cosh(k_0 h)}\tilde{\varphi}(x,y) \quad \text{with} \quad \tilde{\varphi}(x,y) = e^{ik_0(x\cos(\beta)+y\sin(\beta))} \qquad (2.1.5)$$

with β the angle between the direction of wave propagation and and the positive x-axis.

The full reflection condition at the contours of the obstacles is defined as

$$\frac{\partial \Phi_d}{\partial n} = -\frac{\partial \tilde{\Phi}}{\partial n}. \tag{2.1.6}$$

This condition can only be fulfilled if

$$\frac{\partial \varphi_j}{\partial n} = 0, \quad j = 1, 2, \dots \qquad \text{and} \qquad \frac{\partial \varphi_0}{\partial n} = -\frac{\partial \tilde{\varphi}}{\partial n}$$

for all values of z on the contours of the obstacles. It can be shown that with this assumption it must hold that $\varphi_j \equiv 0$ for $j = 1, 2, \dots$ (see Berkhoff (1976)). Substitution of Φ_d into the Laplace equation results in the Helmholtz equation

$$\frac{\partial^2 \varphi_0}{\partial x_2} + \frac{\partial^2 \varphi_0}{\partial y^2} + k_0^2 \varphi_0 = 0.$$

We introduce the radiation condition

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial \varphi_d}{\partial r} - i k_0 \varphi_d \right) = 0 \quad \text{with} \quad r = \sqrt{x^2 + y^2}.$$

This radiation condition represents two conditions. The amplitude of the outgoing wave should go to zero for the radial distance going to infinity, i.e. $\varphi_d \to 0$ as $r \to \infty$. This condition is not enough to uniquely define the scattered wave potential, φ_d should also decrease fast enough for an increasing r. Combining these two conditions results in the radiation condition as stated above.

In real-life problems not all the objects are fully reflecting, as is stated in equation (2.1.6), but partially reflecting. Therefore we introduce the partially reflecting boundary condition

$$\frac{\partial \varphi}{\partial n} + a k_0 \varphi = 0 \quad \text{with} \quad \varphi = \tilde{\varphi} + \varphi_d \text{ and } a = a_1 + i a_2.$$

In which a is the partial reflection coefficient.

2.2 Refraction and Shoaling

Definition

Refraction - The change in direction of wave propagation and in wave length due only to a slow variation of the bottom.

Shoaling - Change of wave height due to bottom variation.

To determine a refraction-shoaling model is important for several reasons, e.g. to calculate the wave height in the region of interest such that it is possible to determine whether the design of this region is sensitive for large oscillations.

As described in the Shore Protection Manual (1984) the following assumptions are made to model the combined effects of refraction and shoaling.

1. The mechanical wave energy between wave rays or orthogonals remains the same, no energy dissipation is present.

2. Direction of wave advance is perpendicular to the wave crest.

3. The small-amplitude wave theory applies.

4. The changes in the bottom topography are gradual.

5. The effects of currents, winds and reflections from beaches and underwater geometry are considered negligible.

Figure 2.2 shows the change in direction of an approaching wave due to a slow variation in the bottom. Diverging wave orthogonals indicate a reduce in energy, converging wave orthogonals represent increased wave energy and hence an increasing wave height.
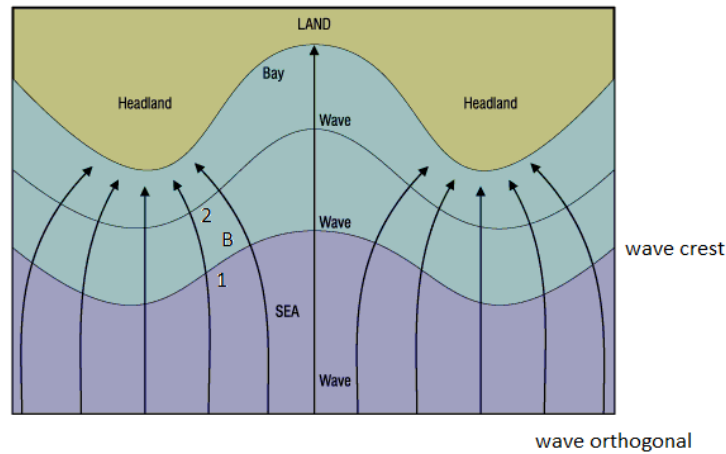
Figure 2.2: Wave refraction pattern

The first assumption states that no energy is dissipated when the wave approaches the shore. This makes it possible to derive a ratio between the wave heights at two points on the wave orthogonal. Suppose that there are two points on a wave orthogonal, where the space between two wave orthogonals is denoted by B_1 respectively B_2. Conservation of energy between wave orthogonals states that, as described by Sorensen (1997), the following must hold

$$B_1 E_1 = B_2 E_2$$

with $E = \frac{\rho g H^2 L}{8}$, the mechanical wave energy. This results in

$$\left(\frac{B \rho g H^2 L}{8} \right)_1 = \left(\frac{B \rho g H^2 L}{8} \right)_2.$$

Rewriting this expression leads to the following relation between the two wave heights H_1 and H_2

$$\frac{H_1}{H_2} = \sqrt{\frac{n_{0,2} L_2}{n_{0,1} L_1}} \sqrt{\frac{B_2}{B_1}}.$$

The first term represents the effect due to shoaling, also known as the shoaling parameter K_s, and the second term the wave set-up due to refraction, the refraction parameter K_r. In most practical applications the values of H, n, L and B are not known. Therefore another model is needed to determine the wave height in the region of interest. The derivation of the refraction-shoaling equations is based on the description made by Berkhoff (1976).

The third assumption implies that the linearised equations (1.4.12) - (1.4.15) can be used as a starting point for the derivation. For convenience the linearised equations are repeated here, where the boundary conditions (1.4.13) and (1.4.13) are combined;

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad\qquad ; \quad -h(x,y) \le z \le 0, \qquad (2.2.1)$$

$$\frac{\partial^2 \Phi}{\partial t^2} + g\frac{\partial \Phi}{\partial z} = 0 \qquad\qquad ; \quad z = 0, \qquad (2.2.2)$$

$$\frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial h}{\partial y} = 0 \qquad\qquad ; \quad z = -h(x,y). \qquad (2.2.3)$$

The small-amplitude wave theory assumes that the solution Φ is simple harmonic in time, hence

$$\Phi(x,y,z,t) = \mathbb{R}\left(\Phi e^{i\omega t}\right).$$

Substituting this into equations (2.2.1) - (2.2.3) gives the time independent form of the linearised small-amplitude wave motion

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad\qquad ; \quad -h(x,y) \le z \le 0, \qquad (2.2.4)$$

$$g\frac{\partial \Phi}{\partial z} - \omega^2 \Phi = 0 \qquad\qquad ; \quad z = 0, \qquad (2.2.5)$$

$$\frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial h}{\partial y} = 0 \qquad\qquad ; \quad z = -h(x,y). \qquad (2.2.6)$$

It is assumed that the changes in bottom topography are gradual, for that reason the new horizontal coordinates are introduced;

$$(\bar{x}, \bar{y}) = \frac{\sigma}{\bar{h}}(x,y)$$

where σ is the mean slope over a distance D and $\bar{h}$ the average water depth. The variation of the water depth results in

$$\nabla h = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right)^T = \frac{\sigma}{\bar{h}}\left(\frac{\partial h}{\partial \bar{x}}, \frac{\partial h}{\partial \bar{y}}\right)^T = \frac{\sigma}{\bar{h}}\overline{\nabla}h \quad \text{with} \quad \overline{\nabla} = \left(\frac{\partial}{\partial \bar{x}}, \frac{\partial}{\partial \bar{y}}\right)^T.$$

The other variables are made dimensionless with the free surface characteristic length $l_0 = g/\omega^2$, hence

$$(x', y', z', h') = \frac{1}{l_0}(x, y, z, h).$$

Substitution the dimensionless variables into equations (2.2.4) - (2.2.6) results in the dimensionless equations, for readability we omit the primes,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad\qquad ; \quad -h \le z \le 0, \qquad (2.2.7)$$

$$\frac{\partial \Phi}{\partial z} - \Phi = 0 \qquad\qquad ; \quad z = 0, \qquad (2.2.8)$$

$$\frac{\partial \Phi}{\partial z} + \epsilon \left(\frac{\partial \Phi}{\partial x} \frac{\partial h}{\partial \bar{x}} + \frac{\partial \Phi}{\partial y} \frac{\partial h}{\partial \bar{y}} \right) = 0 \qquad\qquad ; \quad z = -h, \qquad (2.2.9)$$

where $\epsilon = \sigma/\mu$ with $\mu = \bar{h}/l_0$. The fourth assumption states that he water depth varies slowly, hence the variation of the potential Φ in the horizontal plane will be much less that the variation in the vertical direction. Therefore x and y in the potential function are replaced for the scaled variables $(x', y') = \epsilon(x, y)$. Equations (2.2.7) - (2.2.9) transform into the scaled equations

$$\epsilon^2 \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad\qquad ; \quad -h \le z \le 0, \qquad (2.2.10)$$

$$\frac{\partial \Phi}{\partial z} - \Phi = 0 \qquad\qquad ; \quad z = 0, \qquad (2.2.11)$$

$$\frac{\partial \Phi}{\partial z} + \epsilon^2 \left(\frac{\partial \Phi}{\partial x} \frac{\partial h}{\partial \bar{x}} + \frac{\partial \Phi}{\partial y} \frac{\partial h}{\partial \bar{y}} \right) = 0 \qquad\qquad ; \quad z = -h. \qquad (2.2.12)$$

The velocity potential Φ is written as

$$\Phi(x, y, z) = A(x, y, z) e^{iS(x,y,z)},$$

where $A(x, y, z)$ denotes the amplitude function and $S(x, y, z)$ the phase function. Substitution of this representation into equations (2.2.10) - (2.2.12) gives (see Appendix, A.7)

$$\epsilon^2 \left(\frac{1}{A} \nabla^2 A + (\nabla S)^2 \right) + \frac{1}{A} \frac{\partial^2 A}{\partial z^2} - \left(\frac{\partial S}{\partial z} \right)^2 = 0 \qquad ; \quad -h \le z \le 0, \qquad (2.2.13)$$

$$\frac{\partial A}{\partial z} - A = 0 \qquad\qquad ; \quad z = 0, \qquad (2.2.14)$$

$$\frac{\partial A}{\partial z} + \epsilon^2 \left(\nabla A \cdot \overline{\nabla} h \right) = 0 \qquad\qquad ; \quad z = -h, \qquad (2.2.15)$$

$$\epsilon^2 \nabla \cdot \left[A^2 \nabla S \right] + \frac{\partial}{\partial z} \left[A^2 \frac{\partial S}{\partial z} \right] = 0 \qquad\qquad ; \quad -h \le z \le 0, \qquad (2.2.16)$$

$$\frac{\partial S}{\partial z} = 0 \qquad\qquad ; \quad z = 0, \qquad (2.2.17)$$

$$\frac{\partial S}{\partial z} + \epsilon^2 \left(\nabla S \cdot \overline{\nabla} h \right) = 0 \qquad\qquad ; \quad z = -h, \qquad (2.2.18)$$

Integration of equation (2.2.16), while making use of the boundary conditions (2.2.17) and

(2.2.18), gives

$$\epsilon^2 \int_{-h}^{0} \nabla \cdot A^2 \nabla S \ dz + A^2 \frac{\partial S}{\partial z}\Big|_{-h}^{0} = 0,$$

$$\int_{-h}^{0} \nabla \cdot A^2 \nabla S \ dz + A^2 \left(\nabla S \cdot \overline{\nabla} h\right)\big|_{z=-h} = 0.$$

Applying Leibniz integral rule (see Appendix, A.6) to the integral on the left hand side results in

$$\nabla \cdot \int_{-h}^{0} A^2 \nabla S \ dz = 0. \tag{2.2.19}$$

A solution of equation (2.2.19) is sought, therefore A and S are expanded with respect to ϵ^2. The choice of ϵ^2 is made since in equations (2.2.13) - (2.2.18) only the term ϵ^2 appears. The expansions are given by

$$A(x, y, z) = A_0(x, y, z) + \epsilon^2 A_1(x, y, z) + \dots,$$
$$S(x, y, z) = \epsilon^m \left[S_0(x, y, z) + \epsilon^2 S_1(x, y, z) + \dots\right],$$

where m is a constant that needs to be determined. Substitution of these expansions into equation (2.2.16) gives for the zeroth order, $\mathcal{O}(1)$, approximation

$$\frac{\partial}{\partial z}\left[A_0^2 \frac{\partial S_0}{\partial z}\right] = 0,$$

this implies that

$$A_0^2 \frac{\partial S_0}{\partial z} = C \text{ constant and independent of } z.$$

From the boundary condition $\frac{\partial S_0}{\partial z} = 0$ at $z = 0$ it follows that the constant C must be zero. Therefore it must hold that

$$\frac{\partial S_0}{\partial z} = 0 \quad \text{for} \quad -h \leq z \leq 0,$$

that is S_0 is independent of z. The zeroth order of S is now given by

$$S = \epsilon^m S_0(x, y).$$

The lowest possible order of equation (2.2.13) is given by

$$\frac{1}{A_0} \frac{\partial^2 A_0}{\partial z^2} + \epsilon^2 \epsilon^{2m} \left(\nabla S_0\right)^2 = 0. \tag{2.2.20}$$

There are three options for m, namely;

1. If $m > -1$ it follows that $\epsilon^2 \epsilon^{2m} \to 0$ as $\epsilon \to 0$. With this choice equation (2.2.20) reduces to

$$\frac{1}{A_0} \frac{\partial^2 A_0}{\partial z^2} = 0.$$

   Boundary condition $\frac{\partial A_0}{\partial z} = 0$, $\mathcal{O}(1)$ order of equation (2.2.15), states that A_0 is independent of z. Boundary condition (2.2.14) at $z = 0$ results in the trivial solution $A_0 \equiv 0$. Which indicates that the wave motion does not have an amplitude, this is not a desired solution of the refraction-shoaling equations.

2. If $m < -1$ it follows that $\epsilon^2 \epsilon^{2m} \to \infty$ as $\epsilon \to 0$. In this case equation (2.2.20) reduces to

$$(\nabla S_0)^2 = 0$$

hence S_0 would also be independent of x and y. This means that there is no periodicity of the solution in space, which is not the desired solution of the refraction-shoaling model.

3. The only remaining option for m is $m = -1$ which makes sure that neither the trivial solution for A_0 nor a space independent phase function S_0 is obtained.

With the choice of $m = -1$ the resulting equation is given by

$$\frac{1}{A_0} \frac{\partial^2 A_0}{\partial z^2} = (\nabla S_0)^2 \,.$$

S_0 is independent of z, hence it must hold that the term on the left hand side is also independent of z. Let there be a function $\kappa(x, y)$ such that the eikonal equation is obtained

$$(\nabla S_0)^2 = \kappa(x, y),$$

then it must hold that

$$\frac{1}{A_0} \frac{\partial^2 A_0}{\partial z^2} = \kappa^2(x, y) \quad \text{or} \quad \frac{\partial^2 A_0}{\partial z^2} - \kappa^2 A_0 = 0.$$

The $\mathcal{O}(1)$ boundary conditions of equation (2.2.14) and (2.2.15) are given by

$$\frac{\partial A_0}{\partial z} = 0 \quad \text{for} \quad z = -h \quad \text{and} \quad \frac{\partial A_0}{\partial z} - A_0 = 0 \quad \text{for} \quad z = 0.$$

The solution of $A_0(x, y, z)$ can be obtained with separation of variables as is done in section 1.5. Then it follows that

$$A_0(x, y, z) = f(z, h)a(x, y) = \frac{\cosh(\kappa(z + h))}{\cosh(\kappa h)} a(x, y), \tag{2.2.21}$$

where the dimensionless dispersion relation is given by

$$1 = \kappa \tanh(\kappa h).$$

A more useful representation of equation (2.2.19) can be determined with the use of the solution of A_0, equation (2.2.21). Since S_0 is independent of z it can be taken out of the integral, this results in

$$\nabla \cdot \int_{-h}^{0} A_0^2 \, dz \, \nabla S_0 = 0.$$

Substitution of the solution of $A_0(x, y, z)$ into the equation above gives

$$\nabla \cdot \int_{-h}^{0} \left(\frac{\cosh(\kappa(z + h))}{\cosh(\kappa h)} \right)^2 \, dz \, a^2 \nabla S_0 = 0.$$

After some calculations (see Appendix, A.8) the following equation is obtained

$$\nabla \cdot \left(cc_g a^2 \nabla S_0 \right) = 0.$$

After returning to the physical variables, the following equations describe the combined phenomenon of refraction and shoaling.

$$(\nabla S_0)^2 = k_0^2, \tag{2.2.22}$$

$$\nabla \cdot \left(c c_g a^2 \nabla S_0 \right) = 0, \tag{2.2.23}$$

where k_0 can be determined with the dispersion relation

$$\omega^2 = g k_0 \tanh(k_0 h).$$

Equations (2.2.22) and (2.2.23) are both first order differential equations, therefore it is not possible to impose boundary conditions along the whole boundary of the solution domain.

Chapter 3

The Mild-Slope equation

In the previous chapter the equations that describe the individual phenomena of refraction-shoaling and diffraction-reflection are treated. However when a wave enters a harbour or approaches the shore these four effects occur simultaneously. Therefore it is desired that there exists an equations which combines these phenomena. In 1976 J.C.W. Berkhoff derived the so called Mild-Slope equation in his dissertation at the Technical University of Delft. The description of the Mild-Slope equation as presented here is based on the derivation that Berkhoff made in his dissertation. In section 3.1 the derivation of the Mild-Slope equation is presented and in section 3.2 the corresponding boundary conditions are treated.

Waves in a harbour or at the shore are also influenced by the effects of bottom friction and wave breaking. The Mild-Slope equation can be extended to include these (non-linear) effects, this is described in section 3.3.

3.1 Derivation of the Mild-Slope equation

The derivation starts with the assumption that it is valid to use the linearised small-amplitude wave equations. As done for the refraction-shoaling equation, it is also assumed that the changes in bottom topography are small or as written in Mei (1989)

$$\frac{\nabla h}{k_0 h} \ll 1.$$

With these assumptions it is reasonable to start the derivation with the time-indendent dimensionless equations (2.2.7) - (2.2.9)

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0 \qquad\qquad ; \quad -h \leq z \leq 0, \qquad (3.1.1)$$

$$\frac{\partial \Phi}{\partial z} - \Phi = 0 \qquad\qquad ; \quad z = 0, \qquad (3.1.2)$$

$$\frac{\partial \Phi}{\partial z} + \epsilon \left(\frac{\partial \Phi}{\partial x} \frac{\partial h}{\partial \bar{x}} + \frac{\partial \Phi}{\partial y} \frac{\partial h}{\partial \bar{y}} \right) = 0 \qquad\qquad ; \quad z = -h. \qquad (3.1.3)$$

The three-dimensional velocity potential, as is done for the refraction-shoaling and diffraction-reflection equations, can be written as

$$\Phi(x, y, z, t) = Z(h, z)\varphi(x, y, z) = \frac{\cosh(\kappa(h + z))}{\cosh(\kappa h)}\varphi(x, y, z, t), \qquad (3.1.4)$$

where κ is determined by the dimensionless dispersion relation $1 = \kappa \tanh(\kappa h)$. Subtitution of equation (3.1.4) into equations (3.1.1) - (3.1.3) gives

$$Z\nabla^2\varphi + 2\nabla Z \cdot \nabla\varphi + \varphi\nabla^2 Z + \varphi\frac{\partial^2 Z}{\partial z^2} + 2\frac{\partial Z}{\partial z}\frac{\partial\varphi}{\partial z} + Z\frac{\partial^2\varphi}{\partial z^2} = 0 \quad ; \quad -h \le z \le 0, \quad (3.1.5)$$

$$\frac{\partial\varphi}{\partial z} = 0 \qquad\qquad\qquad\qquad\qquad\qquad ; \quad z = 0, \qquad\qquad (3.1.6)$$

$$Z\frac{\partial\varphi}{\partial z} + \epsilon\left[(\nabla Z \cdot \nabla\bar{h})\varphi + (\nabla\varphi \cdot \nabla\bar{h})Z\right] = 0 \qquad ; \quad z = -h, \qquad (3.1.7)$$

with $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)^T$. Since Z is not directly dependent on x and y but indirectly due to the pressence of $h(x,y)$, the derivatives with respect to x and y are given by

$$\nabla Z = \left(\frac{\partial Z}{\partial h}\frac{\partial h}{\partial x}, \frac{\partial Z}{\partial h}\frac{\partial h}{\partial y}\right) = \epsilon\left(\frac{\partial Z}{\partial h}\frac{\partial h}{\partial\bar{x}}, \frac{\partial Z}{\partial h}\frac{\partial h}{\partial\bar{y}}\right) = \epsilon\frac{\partial Z}{\partial h}\overline{\nabla}h.$$

The second derivatives of Z with respect to x, y and z result in

$$\frac{\partial^2 Z}{\partial z^2} = \kappa^2 Z \quad \text{and} \quad \nabla^2 Z = \epsilon^2\left(\frac{\partial^2 Z}{\partial h^2}\overline{\nabla}h \cdot \overline{\nabla}h + \frac{\partial Z}{\partial h}\overline{\nabla}^2 h\right).$$

Substituting this into equations (3.1.5) - (3.1.7) gives

$$\epsilon^2\left[\varphi\left(\frac{\partial^2 Z}{\partial h^2}\overline{\nabla}h \cdot \overline{\nabla}h + \frac{\partial Z}{\partial h}\overline{\nabla}^2 h\right)\right] + \epsilon\left[2\frac{\partial Z}{\partial h}\overline{\nabla}h \cdot \nabla\varphi\right]$$

$$+ Z\nabla^2\varphi + \kappa^2\varphi Z + 2\frac{\partial Z}{\partial z}\frac{\partial\varphi}{\partial z} + Z\frac{\partial^2\varphi}{\partial z^2} = 0 \qquad ; \quad -h \le z \le 0, \qquad (3.1.8)$$

$$\frac{\partial\varphi}{\partial z} = 0 \qquad\qquad\qquad\qquad\qquad\qquad ; \quad z = 0, \qquad\qquad (3.1.9)$$

$$\epsilon^2\left[\varphi\frac{\partial Z}{\partial h}\overline{\nabla}h \cdot \overline{\nabla}h\right] + \epsilon\left[\nabla\varphi \cdot \overline{\nabla}h Z\right] + Z\frac{\partial\varphi}{\partial z} = 0 \qquad ; \quad z = -h. \qquad (3.1.10)$$

Multiplying equation (3.1.8) with Z and integrating it over the water depth, while using the boundary conditions (3.1.9) and (3.1.10), gives (see Appendix, A.9)

$$\epsilon^2\left[\int_{-h}^{0} z\left(\frac{\partial^2 Z}{\partial h^2}\overline{\nabla}h \cdot \overline{\nabla}h + \frac{\partial Z}{\partial h}\overline{\nabla}^2 h\right)\varphi \, dz\right] + \epsilon\left[\int_{-h}^{0}\frac{\partial Z^2}{\partial h}\overline{\nabla}h \cdot \nabla\varphi \, dz\right]$$

$$\int_{-h}^{0} Z^2\left(\nabla^2\varphi + \kappa^2\varphi\right) \, dz + \epsilon\left[Z^2\nabla\varphi \cdot \overline{\nabla}h\right]_{z=-h} + \epsilon^2\left[Z\varphi\frac{\partial Z}{\partial h}\overline{\nabla}h \cdot \overline{\nabla}h\right]_{z=-h} = 0.$$
$$(3.1.11)$$

It is assumed that $\varphi(x,y,z)$ is only a weak function of the coordinate z, since the dependency of the velocity potential Φ with respect to the vertical coordinate z has been largely been taken into account by $Z(h,z)$. This implies that $\varphi(x,y,z)$ can be expanded in a Taylor

serie with respect to the coordinate σz. Due to boundary condition (3.1.9), the symmetry boundary condition, the expansion only consists of even power of σz, this gives

$$\varphi(x, y, z) = \varphi_0(x, y) + \sigma^2 z^2 \varphi_1(x, y) + \sigma^4 z^4 \varphi_2(x, y) + \dots.$$

Substituting this Taylor serie into equation (3.1.11) and noting that φ_0 is independent of z, gives

$$\epsilon^2 \left[\varphi_0 \overline{\nabla} h \cdot \overline{\nabla} h \int_{-h}^0 Z \frac{\partial^2 Z}{\partial h^2}\, dz + \varphi_0 \overline{\nabla}^2 h \int_{-h}^0 \frac{1}{2} \frac{\partial Z^2}{\partial h}\, dz \right] + \epsilon \overline{\nabla} h \cdot \nabla \varphi_0 \int_{-h}^0 \frac{\partial Z^2}{\partial h}\, dz$$

$$+ \left(\nabla^2 \varphi_0 + \kappa^2 \varphi_0 \right) \int_{-h}^0 Z^2\, dz + \epsilon \overline{\nabla} h \cdot \nabla \varphi_0\, Z^2 \big|_{z=-h}$$

$$+ \epsilon^2 \varphi_0 \overline{\nabla} h \cdot \overline{\nabla} h \frac{1}{2} \left. \frac{\partial Z^2}{\partial h} \right|_{z=-h} + \mathcal{O}\left(\epsilon^n \sigma^m \right) = 0,$$

with $n \geq 0$ and $m \geq 2$. Applying Leibniz integral rule to $\int_{-h}^0 \frac{\partial Z^2}{\partial h}\, dz$ gives

$$\int_{-h}^0 \frac{\partial}{\partial h} Z^2\, dz = \frac{\partial}{\partial h} \int_{-h}^0 Z^2\, dz - Z^2 \big|_{z=-h}.$$

Neclecting the $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\epsilon^n \sigma^m)$, $m \geq 2, n \geq 0$ terms gives

$$\epsilon \overline{\nabla} h \cdot \nabla \varphi_0 \left[\frac{\partial}{\partial h} \int_{-h}^0 Z^2\, dz - Z^2 \big|_{z=-h} \right] + \left(\nabla^2 \varphi_0 + \kappa^2 \varphi_0 \right) \int_{-h}^0 Z^2\, dz + \epsilon \overline{\nabla} h \cdot \nabla \varphi_0\, Z^2 \big|_{z=-h} = 0.$$

This results in

$$\epsilon \overline{\nabla} h \cdot \nabla \varphi_0 \left[\frac{\partial}{\partial h} \int_{-h}^0 Z^2\, dz \right] + \left(\nabla^2 \varphi_0 + \kappa^2 \varphi_0 \right) \int_{-h}^0 Z^2\, dz = 0. \qquad (3.1.12)$$

After some manipulations (see Appendix, A.9) equation (3.1.12) can be written as

$$\nabla \cdot \int_{-h}^0 Z^2\, dz \nabla \varphi_0 + \kappa^2 \varphi_0 \int_{-h}^0 Z^2\, dz = 0. \qquad (3.1.13)$$

The obtained equation includes the term

$$\int_{-h}^0 Z^2\, dz \quad \text{with} \quad Z(z, h) = \frac{\cosh(\kappa(z + h))}{\cosh(\kappa h)}.$$

In section 2.2 this term is also present and it is shown that it equals $\frac{n_0 \omega^2}{g}$. Substituting this into equation (3.1.13) gives

$$\boxed{\begin{aligned}
\nabla \cdot \left(\frac{n_0 \omega^2}{g} \nabla \varphi_0 \right) + \kappa^2 \frac{n_0 \omega^2}{g} \varphi_0 &= 0 \iff \\
\nabla \cdot \left(\frac{n_0 \omega^2}{g \kappa^2} \nabla \varphi_0 \right) + \frac{n_0 \omega^2}{g} \varphi_0 &= 0 \iff \\
\nabla \cdot (c c_g \nabla \varphi_0) + \omega^2 \frac{c_g}{c} \varphi_0 &= 0,
\end{aligned}}$$

$$\text{(3.1.14)}$$

where it is used that $c = \omega/\kappa$ and $c_g = n_0 c$. Equation (3.1.14) is known as the Mild-Slope equation which combines the effects of refraction, shoaling, diffraction and reflection. Substituting $\varphi = A e^{iS}$ into equation (3.1.14) will result in the equation for refraction and shoaling. Assuming a constant water depth will result in the diffraction-reflection equation.

3.2   Boundary conditions

The Mild-Slope equation cannot be solved without the appropriate boundary conditions for the considered domain. There are two distinct boundaries, i.e.

- The closed boundary of the harbour, where complete reflection, partial reflection or no reflection of the wave energy occurs.

- At the open boundary there is an incoming wave, which is described by the wave-maker condition. Waves which approach the open boundary completely pass it, thus the no reflection boundary condition is needed.

Closed boundary

The situation of the closed boundary is sketched in figure 3.1.
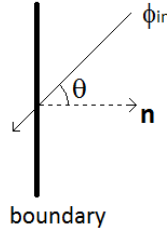
Figure 3.1: An element of the closed boundary where there is an incoming wave φ_{in} at an angle θ with respect to the normal vector **n** perpendicular to the boundary.

At the boundary (partial) reflection occurs, therefore the reflection coefficient R is introduced with $0 \leq R \leq 1$. R denotes the amount of energy that is reflected due to the interaction with the boundary. For $R = 1$ there is full reflection, for $0 < R < 1$ partial reflection and for $R = 0$ no reflection of the wave energy. At the boundary the velocity potential φ is determined by summation of the incoming wave and the reflected part of the incoming wave, hence

$$\varphi = \varphi_{in} + R\varphi_{in} = (1 + R)\varphi_{in}. \tag{3.2.1}$$

The amount of the wave energy that goes through the boundary depends on the angle θ of the incoming wave and the reflection coefficient R. The part of the wave energy perpendicular to the boundary minus the amount that is reflected goes through, hence this is given by

$$\frac{\partial \varphi}{\partial n} = \frac{\partial \varphi_{in}}{\partial n} - R\frac{\partial \varphi_{in}}{\partial n} = -ik_0(1 - R)\varphi_{in}\cos(\theta). \tag{3.2.2}$$

Combining the expressions (3.2.1) and (3.2.2) gives the condition for the closed boundary;

$$\frac{\partial \varphi}{\partial n} = -ik_0\left(\frac{1 - R}{1 + R}\right)\varphi\cos(\theta) \quad \text{with} \quad 0 \leq R \leq 1. \tag{3.2.3}$$

Open boundary

At the open boundary there is the sketched situation of figure 3.2
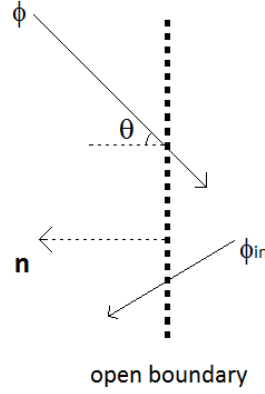
Figure 3.2: An element of the open boundary where there is a prescribed incoming wave φ_{in} and an outgoing wave φ at an angle θ with respect to the normal vector $\mathbf{n}$ perpendicular to the open boundary.

The condition at the open boundary is determined by a superposition of the influences due to the incoming wave φ_{in} and the outgoing wave φ. Since there is no reflection at the open boundary the amount of the incoming wave that crosses it is given by

$$\frac{\partial \varphi_{in}}{\partial n} = -ik_0\varphi_{in}(\boldsymbol{e}_{in} \cdot \boldsymbol{n}).$$

The outgoing wave φ is influenced by the contribution of the incoming wave at the angle θ, namely

$$\frac{\partial(\varphi - \varphi_{in})}{\partial n} = -ik_0(\varphi - \varphi_{in})\cos(\theta).$$

Superposing these two conditions leads to the condition of the open boundary

$$\frac{\partial \varphi}{\partial n} = -ik_0\varphi_{in}(\boldsymbol{e}_{in} \cdot \boldsymbol{n}) - ik_0(\varphi - \varphi_{in})\cos(\theta). \tag{3.2.4}$$

The boundary conditions (3.2.3) and (3.2.4) depend on the wave number k_0. The wave number changes when wave breaking and bottom friction is taken into account. In the case of energy

dissipation the term ik_0 should be replaced by p. The expression for p is derived in section 3.3.

The boundary conditions, equations (3.2.3) and (3.2.4), depend on the angle θ. Unfortunately θ is not known a priori, hence an approximation for the numerical implementation is needed. The same approximation can be used for both boundary conditions, therefore only the approximation of the condition for the closed boundary is described. The condition for the open boundary will follow automatically. The condition for the open boundary including energy dissipation is given by

$$\frac{\partial \varphi}{\partial n} = -p \left(\frac{1 - R}{1 + R} \right) \varphi \cos(\theta).$$

The reflection coefficient R is also an unknown parameter depending on many (non-linear) processes. Its value is determined experimentally by matching the outcome of the model to the measurements on the domain. Since the measurements contain error, the value of R will not be very accurate. The term $\cos(\theta)$ is approximated by the first order Taylor series, i.e.

$$\cos(\theta) = \sqrt{1 - \sin^2(\theta)} \approx 1 - \frac{1}{2} \sin^2(\theta).$$

Since the value of R is not very accurate it is assumed that this approximation is sufficient. The solution of φ is of the form $\varphi = \varphi_0 e^{-p(x \cos(\theta) + y \sin(\theta))}$, the second derivative with respect to s (parallel to the boundary) is given by

$$\frac{\partial^2 \varphi}{\partial s^2} = p^2 \sin^2(\theta) \varphi.$$

Substuting this into the condition for the closed boundary gives

$$\frac{\partial \varphi}{\partial n} = - \left(\frac{1 - R}{1 + R} \right) \left\{ p\varphi - \frac{1}{2p} \frac{\partial^2 \varphi}{\partial s^2} \right\}.$$

For the boundary condition of the open boundary this yields

$$\frac{\partial \varphi}{\partial n} = -p\varphi_{in}(\boldsymbol{e}_{in} \cdot \boldsymbol{n}) - p(\varphi - \varphi_{in}) + \frac{1}{2p} \left(\frac{\partial^2 \varphi}{\partial s^2} - \frac{\partial^2 \varphi_{in}}{\partial s^2} \right).$$

The condition of the open boundary does not contain the reflection coefficient R, therefore it might be necessary to use a higher order Taylor series as an approximation of $\cos(\theta)$. However this is not taken into account in HARES.

3.3   Dissipation of wave energy

In the derivation of the Mild-Slope equation we assumed that there was no loss of energy. However for shallow-water waves the waves interact with the bottom, hence energy loss due to bottom friction will be present. Due to the assumption of a decreasing water depth shoaling occurs. When no energy dissipation is present the waves would become infinitely high. In real life this is not possible and waves will break after reaching a maximal height for a certain water depth. Wave breaking will lead to energy dissipation.

For a good approximation of the wave behaviour in a harbour it seems reasonable to include energy dissipation in the Mild-Slope equation. The equation for energy conservation including energy dissipation, as stated by Dingemans (1997), is given by

$$\nabla \cdot (c_g E) + D = 0 \quad \text{with} \quad D = WE,$$

where D represent the dissipation of wave energy, E the mechanical wave energy and W energy source term. The energy source term consists of the summation of the energy dissipation due to bottom friction W_f and the dissipation due to wave breaking W_b. The Mild-Slope equation with dissipation of energy is given by

$$\nabla \cdot (cc_g \nabla \varphi_0) + \omega^2 \frac{c_g}{c} \varphi_0 - W \frac{\partial \varphi_0}{\partial t} = 0. \tag{3.3.1}$$

Similarly, as done by Dingemans (1997), the Mild-Slope equation for the elevation of the free surface ζ is obtained by

$$\nabla \cdot (cc_g \nabla \zeta) + \omega^2 \frac{c_g}{c} \zeta - W \frac{\partial \zeta}{\partial t} = 0. \tag{3.3.2}$$

It is assumed that the wave motion is harmonic in time, therefore we substitute $\zeta = \mathbb{R}\left(\tilde{\zeta} e^{i\omega t}\right)$ into equation (3.3.2). This results in the time independent form of the Mild-Slope equation;

$$\nabla \cdot \left(cc_g \nabla \tilde{\zeta}\right) + \omega^2 \frac{c_g}{c} \tilde{\zeta} - i\omega W \tilde{\zeta} = 0. \tag{3.3.3}$$

Expression for p

The expression for p, the modified wave number due to the presence of energy dissipation, is easily derived using the one dimensional version of equation (3.3.3). The one dimensional form, with constant coefficients, is given by

$$\frac{n_0}{k_0^2} \frac{\partial^2 \tilde{\zeta}}{\partial x^2} + \left(n_0 - \frac{iW}{\omega}\right) \tilde{\zeta} = 0 \quad \Longleftrightarrow \quad \frac{\partial^2 \tilde{\zeta}}{\partial x^2} + k_0^2 \left(1 - \frac{iW}{\omega n_0}\right) \tilde{\zeta} = 0.$$

Substituting the general solution $\tilde{\zeta} = \tilde{\zeta}_0 e^{px}$ into this equation gives the expression for p, that is

$$p = \pm i k_0 \sqrt{1 - \frac{iW}{\omega n_0}}.$$

In HARES the modified wave number p is not yet implemented, only the term ik_0 is taken into account. Although p is not yet implemented from now on it is used in the derivations.

3.3.1 Bottom friction coefficient W_f

For the derivation of the bottom friction coefficient W_f the approach as described in Visser (1984) and Dingemans (1997) is followed. The dissipation of wave energy in the bottom boundary layer is given by

$$\langle D_f \rangle = \left\langle \tau^b \cdot \boldsymbol{u}^b \right\rangle \quad \text{with} \quad \tau^b = c_f \rho \left| \boldsymbol{u}^b \right| \boldsymbol{u}^b, \tag{3.3.4}$$

where $\langle \cdot \rangle$ denotes the mean over one wave period. Visser (1984) states that linear wave theory for the horizontal velocity near the bottom can be used which yields that $\boldsymbol{u}^b$ is expressed as

$$\boldsymbol{u}^b = \frac{a\omega}{\sinh(kh)} \cos(\omega t),$$

with $a(x, y, t) = |\zeta(x, y, t)|$. Substituting this into equation (3.3.4) gives

$$\langle D_f \rangle = \left\langle c_f \rho \left| \frac{a\omega \cos(\omega t)}{\sinh(kh)} \right| \left(\frac{a\omega \cos(\omega t)}{\sinh(kh)} \right)^2 \right\rangle = c_f \rho \left(\frac{a\omega}{\sinh(kh)} \right)^3 \left\langle |\cos(\omega t)|^3 \right\rangle.$$

With $\left\langle |\cos(\omega t)|^3 \right\rangle = \frac{4}{3\pi}$ we get

$$\langle D_f \rangle = \frac{4}{3\pi} c_f \rho \frac{a^3 \omega^3}{\sinh^3(kh)}.$$

The bottom friction coefficient W_f is obtained by dividing the equation above by the energy per unit length $\bar{E}$, equation (1.6.4), i.e.

$$W_f = \frac{\langle D_f \rangle}{\bar{E}} = \frac{8}{3\pi} c_f \frac{a\omega^3}{\sinh^3(kh)}.$$

3.3.2   Wave breaking coefficient W_b

The wave breaking coefficient W_b is determined by the derivation as obtained by Battjes and Janssen (1978). The characteristic breaker height, as described by Southgate (1993), is defined as

$$\gamma_s = \left(\frac{H}{h} \right)_{\text{b,shallow}} \qquad \text{and} \qquad \gamma_d = \left(\frac{H}{L} \right)_{\text{b,deep}}.$$

Shallow-water waves break due to limitations on the water depth while for deep-water waves the steepness of the wave is crucial. The maximal value for γ_d is by default 0.14. The well-known Miche's criterion, Battjes and Janssen (1978), for the maximal possible wave height H_m is given by

$$H_m = \frac{2\pi\gamma_d}{k} \tanh\left(\frac{\gamma_s}{2\pi\gamma_d} kh \right) = \frac{0.88}{k} \tanh\left(\frac{\gamma_s}{0.88} kh \right).$$

The probability that at a given point a height is associated with a breaking or broken wave Q_b can be determined by

$$\frac{1 - Q_b}{\ln Q_b} = -\left(\frac{H_{rms}}{H_m} \right)^2,$$

where H_{rms} denotes the root mean square wave height. In HARES $H_{rms} = 2a$ is used. The power dissipation in the bore per unit span is given by

$$D_b' = \frac{1}{4} \rho g (h_2 - h_1)^3 \sqrt{\frac{g(h_1 + h_2)}{2h_1 h_2}},$$

with h_1 and h_2 the water heights at both sides of the bore. The following estimates are made

$$h_2 - h_1 \approx H \quad \text{and} \quad \sqrt{\frac{g(h_1 + h_2)}{2h_1 h_2}} \approx \sqrt{\frac{g}{h}}.$$

This results in

$$D_b' \sim \frac{1}{4}\rho g H^3 \sqrt{\frac{g}{h}}.$$

If the waves are periodic with frequency $f = \frac{1}{T}$, the average power dissipation per unit length for shallow-water waves in the breaking process is then given by

$$D_b = \frac{D_b'}{L} = \frac{f D_b'}{c} \sim \frac{f D_b'}{\sqrt{gh}} = \frac{1}{4}f\rho g \frac{H^3}{h}. \tag{3.3.5}$$

For shallow water the ration H/h is approximately 1, hence $D_b \sim \frac{1}{4}f\rho g H^2$. For random waves the expected value of the dissipated power per unit area is of interest. Hence equation (3.3.5) needs to be multiplied by Q_b and H set to H_m. This results in

$$D_b = \frac{\alpha}{4}f\rho g Q_b H_m^2.$$

The obtained wave breaking coefficient W_b is given by

$$W_b = \frac{D_b}{\bar{E}} = \frac{2\alpha}{T}Q_b\frac{H_m^2}{H_{rms}^2} = \frac{2\alpha}{T}Q_b\frac{H_m^2}{4a^2}.$$

3.4   Summary of the Mild-Slope equation

In this chapter we derived the Mild-Slope equation where the dissipation of wave energy is included and given its corresponding boundary conditions. The Mild-Slope with dissipation is given by equation (3.3.3), with $c = \omega/k_0$ and $c_g = n_0 c$ this can be written as

$$\nabla \cdot \left(\frac{n_0}{k_0^2} \nabla \tilde{\zeta} \right) + \left(n_0 - \frac{iW}{\omega} \right) \tilde{\zeta} = 0, \tag{3.4.1}$$

with k_0 the wave number dependent on the water height h, W the energy source for the dissipation of energy and ω the constant wave frequency. The energy source W is given by

$$W = W_f + W_b = \frac{8}{3\pi} c_f \frac{a\omega^3}{\sinh^3(kh)} + \frac{2\alpha}{T} Q_b \frac{H_m^2}{4a^2}. \tag{3.4.2}$$

Including the energy dissipation in the Mild-Slope equation leads to a non-linear contribution. In both W_f and W_b $a(x,y) = |\tilde{\zeta}|$ is present , therefore we have that the last term $\frac{iW}{\omega}\tilde{\zeta}$ results in a non-linear contribution. In chapter 4 it is described how this non-linearity is treated in the numerical implementation of the Mild-Slope equation.

The condition for the open boundary with an incoming wave is given by

$$\frac{\partial \tilde{\zeta}}{\partial n} = -p\tilde{\zeta}_{in} \left(\boldsymbol{e}_{in} \cdot \boldsymbol{n} \right) - p \left(\tilde{\zeta} - \tilde{\zeta}_{in} \right) + \frac{1}{2p} \left(\frac{\partial^2 \tilde{\zeta}}{\partial s^2} - \frac{\partial^2 \tilde{\zeta}_{in}}{\partial s^2} \right), \tag{3.4.3}$$

and for the closed boundary we have

$$\frac{\partial \tilde{\zeta}}{\partial n} = - \left(\frac{1-R}{1+R} \right) \left\{ p\tilde{\zeta} - \frac{1}{2p} \frac{\partial^2 \tilde{\zeta}}{\partial s^2} \right\}. \tag{3.4.4}$$

The modified wave number p is given by

$$p = ik_0 \sqrt{1 - \frac{iW}{\omega n_0}}.$$

Chapter 4

Numerical implementation of the Mild-Slope equation

In chapter 3 we derived the dissipation included Mild-Slope equation and its corresponding boundary conditions. These equations are used in HARES to determine the wave motion in harbours. In this chapter the numerical implementation of the non-linear Mild-Slope equation is described. In section 4.1 we apply the Ritz-Galerkin finite element method to obtain a system of equations. In section 4.2 the implementation of the non-linearity in HARES is described.

4.1 Ritz-Galerkin Finite Element Method

To determine the solution of equation (3.4.1) with the boundary conditions (3.4.3) and (3.4.4) the method of finite elements is used. Berkhoff (1976) states that this is a good method to solve the Mild-Slope equation in a harbour for the following two reasons;

- It has an easy way of representing boundaries of an arbitrary shape.

- It is possible to use small elements in areas where a strong variation of the solution can be expected and large elements in areas where not.

The derivation of the finite element integrals is based on Ritz-Galerkin method as described by Zienkewicz (1971). Integrating equation (3.4.1), after multiplication with the test function $\eta(x, y)$, over the domain Ω gives the weak formulation of the Mild-Slope equation;

$$\int_{\Omega} \left\{ \nabla \cdot \left(\frac{n_0}{k^2} \nabla \tilde{\zeta} \right) + \left(n_0 - \frac{iW}{\omega} \right) \tilde{\zeta} \right\} \eta \, d\Omega = 0 \quad \forall \eta.$$

As observed in chapter 3 the term W, the dissipation of energy due to bottom friction and wave breaking, leads to a non-linearity in the Mild-Slope equation. For the derivation of the integrals it is assumed that W is piecewise constant on each element.

Application of Gauss divergence theorem results in

$$-\int_{\Omega} \frac{n_0}{k^2} \nabla \tilde{\zeta} \cdot \nabla \eta \, d\Omega + \int_{\Omega} \left(n_0 - \frac{iW}{\omega} \right) \tilde{\zeta} \eta \, d\Omega + \int_{\Gamma} \frac{n_0}{k^2} \eta \frac{\partial \tilde{\zeta}}{\partial n} \, d\Gamma = 0 \quad \forall \eta, \qquad (4.1.1)$$

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where the boundary of Ω is denoted by Γ. The boundary Γ of the domain consist of an open boundary and a closed boundary, therefore it is divided into Γ_1 for the open boundary and Γ_2 for the closed boundary. Substituting the boundary conditions (3.4.3) and (3.4.4) in equation (4.1.1) gives

$$\int_\Omega \left\{ \left(n_0 - \frac{iW}{\omega} \right) \tilde{\zeta}\eta - \frac{n_0}{k^2} \nabla\tilde{\zeta} \cdot \nabla\eta \right\} d\Omega - \int_{\Gamma_2} \frac{n_0}{k^2} \left(\frac{1-R}{1+R} \right) \left\{ p\tilde{\zeta} - \frac{1}{2p} \frac{\partial^2 \tilde{\zeta}}{\partial s^2} \right\} \eta \; d\Gamma$$

$$+ \int_{\Gamma_1} \left\{ \frac{n_0}{k^2} \left(-p\tilde{\zeta}_{in}(\boldsymbol{e}_{in} \cdot \boldsymbol{n}) - p(\tilde{\zeta} - \tilde{\zeta}_{in}) + \frac{1}{2p} \left(\frac{\partial^2 \tilde{\zeta}}{\partial s^2} - \frac{\partial^2 \tilde{\zeta}_{in}}{\partial s^2} \right) \right) \right\} \eta \; d\Gamma = 0 \quad \forall \eta. \tag{4.1.2}$$

The boundary conditions contain both the term $\frac{1}{2p} \frac{\partial^2 \tilde{\zeta}}{\partial s^2} \eta$, applying partial integration of the expression gives

$$\frac{1}{2p} \int_{\Gamma_i} \frac{\partial^2 \tilde{\zeta}}{\partial s^2} \eta \; d\Gamma = \frac{1}{2p} \left\{ \eta \frac{\partial\tilde{\zeta}}{\partial s} \Big|_{\Gamma_i} - \int_{\Gamma_i} \frac{\partial\eta}{\partial s} \frac{\partial\tilde{\zeta}}{\partial s} \; d\Gamma \right\}$$

It is assumed that

$$\frac{1}{2p} \left\{ \eta \frac{\partial\tilde{\zeta}}{\partial s} \Big|_{\Gamma_1} + \eta \frac{\partial\tilde{\zeta}}{\partial s} \Big|_{\Gamma_2} \right\} = 0$$

since $\Gamma = \Gamma_1 + \Gamma_2$ is a closed boundary. The term $\frac{1}{2p} \frac{\partial^2 \tilde{\zeta}_{in}}{\partial s^2}$ is only present on the open boundary, partial integration will lead to the term $\frac{1}{2p} \eta \frac{\partial\tilde{\zeta}_{in}}{\partial s} \Big|_{\Gamma_1}$. In the implementation in HARES is it assumed that this term also equals zero, since it only gives a contribution at the two ends of the open boundary. After the application of the partial integration the governing equations are given by

$$\boxed{\begin{aligned} &\int_\Omega \left\{ \left(n_0 - \frac{iW}{\omega} \right) \tilde{\zeta}\eta - \frac{n_0}{k^2} \nabla\tilde{\zeta} \cdot \nabla\eta \right\} d\Omega - \int_{\Gamma_2} \frac{n_0}{k^2} \left(\frac{1-R}{1+R} \right) \left\{ p\tilde{\zeta}\eta + \frac{1}{2p} \frac{\partial\tilde{\zeta}}{\partial s} \frac{\partial\eta}{\partial s} \right\} d\Gamma \\ &+ \int_{\Gamma_1} \frac{n_0}{k^2} \left\{ -p\tilde{\zeta}_{in} (\boldsymbol{e}_{in} \cdot \boldsymbol{n}) \eta - p(\tilde{\zeta} - \tilde{\zeta}_{in})\eta + \frac{1}{2p} \left(\frac{\partial\tilde{\zeta}_{in}}{\partial s} \frac{\partial\eta}{\partial s} - \frac{\partial\tilde{\zeta}}{\partial s} \frac{\partial\eta}{\partial s} \right) \right\} d\Gamma = 0. \end{aligned}} \tag{4.1.3}$$

The unknown solution of $\tilde{\zeta}$ is approximated by a finite linear combination of basis functions $\zeta_j(x,y)$;

$$\tilde{\zeta}(x,y) \approx \tilde{\zeta}^n(x,y) = \sum_{j=1}^{n} a_j \zeta_j(x,y).$$

Substituting this approximation into equation (4.1.2) and setting $\eta \to \zeta_i(x,y)$ gives the following system of equations

$$\sum_{j=1}^{n_{el}} a_j \left\{ \int_{\Omega} \left[\left(n_0 - \frac{iW}{\omega} \right) \zeta_i \zeta_j - \frac{n_0}{k^2} \nabla \zeta_i \cdot \nabla \zeta_j \right] d\Omega \right\}$$

$$+ \sum_{j=1}^{n_{bel1}} a_j \left\{ -\int_{\Gamma_1} \frac{n_0}{k^2} \left(p \zeta_i \zeta_j + \frac{1}{2p} \frac{\partial \zeta_j}{\partial s} \frac{\partial \zeta_i}{\partial s} \right) d\Gamma \right\}$$

$$+ \sum_{j=1}^{n_{bel2}} a_j \left\{ -\int_{\Gamma_2} \frac{n_0}{k^2} \left(\frac{1-R}{1+R} \right) \left(p \zeta_j \zeta_i + \frac{1}{2p} \frac{\partial \zeta_j}{\partial s} \frac{\partial \zeta_i}{\partial s} \right) d\Gamma \right\}$$

$$= \int_{\Gamma_1} \frac{n_0}{k^2} \left(p\tilde{\zeta}_{in} (\boldsymbol{e}_{in} \cdot \boldsymbol{n}) \zeta_i - p\tilde{\zeta}_{in} \zeta_i - \frac{1}{2p} \frac{\partial \tilde{\zeta}_{in}}{\partial s} \frac{\partial \zeta_i}{\partial s} \right) d\Gamma. \tag{4.1.4}$$

This can also be written in the matrix-vector notation $\boldsymbol{Sa} = \boldsymbol{f}$. Our domain Ω and boundary Γ are divided into n_{el} internal elements and n_{bel} boundary elements, with $n = n_{el} + n_{bel}$. The internal elements are shaped as a triangle, where ζ_i is piecewise linear, and the boundary elements are linear line segments. There are two boundaries, Γ_1 and Γ_2, hence $n_{bel} = n_{bel1} + n_{bel2}$ where n_{beli} $(i = 1, 2)$ denotes the amount of boundary element on boundary Γ_i. Matrix $\boldsymbol{S}$ and vector $\boldsymbol{f}$ are determined by the summation over all the contributions from the internal and boundary elements, this results in

$$S_{ij} = \sum_{l=1}^{n_{el}} S_{ij}^{el} + \sum_{l=1}^{n_{bel1}} S_{ij}^{bel1} + \sum_{l=1}^{n_{bel2}} S_{ij}^{bel2},$$

$$f_i = \sum_{l=1}^{n_{el}} f_i^{el} + \sum_{l=1}^{n_{bel1}} f_i^{bel1} + \sum_{l=1}^{n_{bel2}} f_i^{bel2}.$$

The basis functions in each element satisfy the following property

$$\zeta_i(x, y) = \alpha_i + \beta_i x + \gamma_i y, \tag{4.1.5}$$
$$\zeta_i(x_j, y_j) = \delta_{ij},$$

where δ_{ij} denotes the Kronecker delta function and the coefficients α_i, β_i and γ_i can be uniquely determined (see van Kan et al. (2008, p. 110)).

4.1.1 Internal elements

For the internal elements, they have no connection to the boundary, the following expression for S_{ij} and f_i are found

$$S_{ij} = \int_{\Omega} \left[\left(n_0 - \frac{iW}{\omega} \right) \zeta_i \zeta_j - \frac{n_0}{k^2} \nabla \zeta_i \cdot \nabla \zeta_j \right] d\Omega \quad \text{and} \quad f_i = 0. \tag{4.1.6}$$

In HARES they make the assumption that n_0, k, p, W, ω and h are constant for each element and substituting the basis function, equation (4.1.5), into equation (4.1.6) gives

$$S_{ij}^{el} = \left(n_0 - \frac{iW}{\omega}\right)\int_{el}\zeta_i\zeta_j\ d\Omega - \frac{n_0}{k^2}(\beta_i\beta_j + \gamma_i\gamma_j)\int_{el}1\ d\Omega \quad \text{and} \quad f_i^{el} = 0. \qquad (4.1.7)$$

4.1.2   Boundary elements on the open boundary

The elements on the open boundary, Γ_1, are described by the following equations

$$S_{ij} = -\int_{\Gamma 1}\frac{n_0}{k^2}\left(p\zeta_i\zeta_j + \frac{1}{2p}\frac{\partial\zeta_j}{\partial s}\frac{\partial\zeta_i}{\partial s}\right)\ d\Gamma,$$

$$f_i = \int_{\Gamma_1}\frac{n_0}{k^2}\left(p\tilde{\zeta}_{in}(\boldsymbol{e}_{in}\cdot\boldsymbol{n})\zeta_i - p\tilde{\zeta}_{in}\zeta_i - \frac{1}{2p}\frac{\partial\tilde{\zeta}_{in}}{\partial s}\frac{\partial\zeta_i}{\partial s}\right)\ d\Gamma.$$

The assumption that the variables are constant on each element also holds for R, thus inserting the basis function into equations (4.1.8) and (4.1.9) results in

$$S_{ij}^{bel1} = -\frac{n_0}{k^2}\left\{p\int_{\Gamma_1}\zeta_i\zeta_j\ d\Gamma - \frac{1}{2p}\int_{\Gamma_1}\frac{\partial\zeta_j}{\partial s}\frac{\partial\zeta_i}{\partial s}\ d\Gamma,\right\} \qquad (4.1.8)$$

$$f_i^{bel1} = \frac{n_0}{k^2}\left\{p\int_{\Gamma_1}\left\{\tilde{\zeta}_{in}(\boldsymbol{e}_{in}\cdot\boldsymbol{n}) - p\tilde{\zeta}_{in}\right\}\zeta_i\ d\Gamma - \frac{1}{2p}\int_{\Gamma_1}\frac{\partial\tilde{\zeta}_{in}}{\partial s}\frac{\partial\zeta_i}{\partial s}\ d\Gamma\right\}. \qquad (4.1.9)$$

4.1.3   Boundary elements on the closed boundary

For the elements on the closed boundary Γ_2 we found the following expressions for S_{ij} and f_i;

$$S_{ij} = -\int_{\Gamma_2}\frac{n_0}{k^2}\left(\frac{1-R}{1+R}\right)\left(p\zeta_i\zeta_j + \frac{1}{2p}\frac{\partial\zeta_j}{\partial s}\frac{\partial\zeta_i}{\partial s}\right)\ d\Gamma \qquad \text{and} \qquad f_i = 0.$$

With the assumption of constant variables in each element and the substitution the basis function gives for each element on the closed boundary

$$S_{ij}^{bel2} = -\frac{n_0}{k^2}\left(\frac{1-R}{1+R}\right)\left\{p\int_{\Gamma_2}\zeta_i\zeta_j\ d\Gamma + \frac{1}{2p}\int_{\Gamma_2}\frac{\partial\zeta_j}{\partial s}\frac{\partial\zeta_i}{\partial s}\ d\Gamma\right\} \qquad \text{and} \qquad f_i^{bel2} = 0. \quad (4.1.10)$$

4.1.4   Summary of the integrals

In this section we derived the integrals, obtained with the Ritz-Galerkin finite element method, that are solved in HARES. The internal elements are shape like a triangle and have piecewise linear basis functions, the boundary elements for the open and closed boundary are linear line segments. Ror each element the assumption that the variables n_0, k, W, ω, p, h and R are constant is made. The equations that are obtained in this section are given below

Internal element

The integral for an internal element is given by

$$S_{ij}^{el} = \left(n_0 - \frac{iW}{\omega}\right)\int_{el}\zeta_i\zeta_j\ d\Omega - \frac{n_0}{k^2}(\beta_i\beta_j + \gamma_i\gamma_j)\int_{el}1\ d\Omega \quad \text{and} \quad f_i^{el} = 0. \qquad (4.1.11)$$

Boundary element on the open boundary

For an element on the open boundary the following expression yields

$$S_{ij}^{be_{l1}} = -\frac{n_0}{k^2} \left\{ p \int_{\Gamma_1} \zeta_i \zeta_j \; d\Gamma - \frac{1}{2p} \int_{\Gamma_1} \frac{\partial \zeta_j}{\partial s} \frac{\partial \zeta_i}{\partial s} \; d\Gamma \right\}, \tag{4.1.12}$$

$$f_i^{be_{l1}} = \frac{n_0}{k^2} \left\{ p \int_{\Gamma_1} \left\{ \tilde{\zeta}_{in}(\boldsymbol{e}_{in} \cdot \boldsymbol{n}) - p\tilde{\zeta}_{in} \right\} \zeta_i \; d\Gamma - \frac{1}{2p} \int_{\Gamma_1} \frac{\partial \tilde{\zeta}_{in}}{\partial s} \frac{\partial \zeta_i}{\partial s} \; d\Gamma \right\}. \tag{4.1.13}$$

Boundary element on the closed boundary

The integral for a boundary element on the closed boundary results in

$$S_{ij}^{be_{l2}} = -\frac{n_0}{k^2} \left(\frac{1-R}{1+R} \right) \left\{ p \int_{\Gamma_2} \zeta_i \zeta_j \; d\Gamma + \frac{1}{2p} \int_{\Gamma_2} \frac{\partial \zeta_j}{\partial s} \frac{\partial \zeta_i}{\partial s} \; d\Gamma \right\} \qquad \text{and} \qquad f_i^{be_{l2}} = 0. \tag{4.1.14}$$

In HARES these integrals are exactly determined by Gaussian integration. As can be seen from the expressions for f_i^{el}, $f_i^{be_{l1}}$ and $f_i^{be_{l2}}$ the only contribution on the right-hand side of the matrix-vector notation $\boldsymbol{Sa} = \boldsymbol{f}$ is given by the prescribed incoming wave $\tilde{\zeta}_{in}$.

4.2 Non-linearity in the Mild-Slope equation

As stated earlier the dissipation included Mild-Slope equation is non-linear in the unknown free surface elevation parameter $\tilde{\zeta}(x,y)$. Hence the system $\boldsymbol{Sa} = \boldsymbol{f}$ should actually be written as $\boldsymbol{S}(W(\boldsymbol{a}))\boldsymbol{a} = \boldsymbol{f}$. Therefore the solution is obtained in an iterative manner. In the first iteration is is assumed that $W = 0$ and the system $\boldsymbol{S}(0)\boldsymbol{a}^1 = \boldsymbol{f}$ is solved. A solution for $\boldsymbol{a}^1$ is obtained and $W(\boldsymbol{a}^1)$ is determined with equation (3.4.2) for each element. The next iteration step is solving the new system $\boldsymbol{S}(W(\boldsymbol{a}^1))\boldsymbol{a}^2 = \boldsymbol{f}$ and after that $\boldsymbol{a}^2$ is determined. This is repeated until a good approximation of the elevation of the free surface $\zeta(x,y,t)$ is obtained. This procedure can also be put into an algorithm, this gives

$$
\begin{array}{l}
\qquad W^0 = 0 \; ; \text{ initial value for the dissipation term} \\[1em]
\text{for} \quad i = 1, 2, \ldots \\
\qquad \text{Solve } \boldsymbol{a}^i \text{ from } \boldsymbol{S}(W^{i-1})\boldsymbol{a}^i = \boldsymbol{f} \\
\qquad W^i = W(\boldsymbol{a}^i) \\
\text{end}
\end{array}
$$

The term $\boldsymbol{S}(W^{i-1})$ can be determined with equations (4.1.11), (4.1.12) and (4.1.14) and $W^i = W(\boldsymbol{a}^i)$ by equation (3.4.2).

Chapter 5

ILU Bi-CGSTAB

In the previous chapter the system $\boldsymbol{Sa} = \boldsymbol{f}$ is derived where vector $\boldsymbol{a}$ contains the unknowns. This system could be solved with the calculation of $\boldsymbol{a} = \boldsymbol{S}^{-1}\boldsymbol{f}$, but in general computing the inverse takes a lot of time. Therefore numerical methods are derived which deal with solving a general system $\boldsymbol{Ax} = \boldsymbol{b}$ in a smart way such that the solution is obtained faster. The solution of the system $\boldsymbol{Sa} = \boldsymbol{f}$ is in HARES numerically approximated with the Bi-CGSTAB method preconditioned with the Incomplete LU (ILU) factorization. In section 5.1 a description of the unpreconditioned Bi-CGSTAB method and its algorithm is given, section 5.2 treats the Incomplete LU preconditioner and section 5.3 gives the algorithm for preconditioned Bi-CGSTAB method.

5.1 Bi-CGSTAB

The Bi-CGSTAB method is in 1992 derived by van der Vorst as an improvement of the Conjugate Gradient Squared (CG-S) method by Sonneveld (1989), which on its turn was an improvement of the Bi-Conjugate Gradient (Bi-CG) method. These methods involve solving the linear system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ with $\boldsymbol{A} \in \mathbb{R}^{N \times N}$ and $\boldsymbol{x}, \boldsymbol{b} \in \mathbb{R}^N$. In general matrix $\boldsymbol{A}$ is a sparse matrix and $\boldsymbol{A}$ does not have to satisfy nice properties, such as a symmetric positive definite matrix, for the methods to work.

The methods Bi-CG, CG-S and Bi-CGSTAB are Krylov subspace methods, where the Krylov subspace of dimension m is defined as

$$\mathcal{K}_m(\boldsymbol{A}; \boldsymbol{r}_0) = span\left\{\boldsymbol{r}_0, \boldsymbol{A}\boldsymbol{r}_0, \boldsymbol{A}^2\boldsymbol{r}_0, \ldots, \boldsymbol{A}^{m-1}\boldsymbol{r}_0\right\},$$

with $\boldsymbol{r}_0 = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_0$.

Bi-CG

Bi-CG uses a basis $\boldsymbol{r}_0, \ldots \boldsymbol{r}_{i-1}$ which is constructed for $\mathcal{K}_i(\boldsymbol{A}; \boldsymbol{r}_0)$ such that $\boldsymbol{r}_j \perp span\{\boldsymbol{r}_0^*, \ldots, \boldsymbol{r}_{j-1}^*\}$ with $(j \leq i)$ and $\boldsymbol{r}_0^*, \ldots, \boldsymbol{r}_{i-1}^*$ forms a basis for $\mathcal{K}_i(\boldsymbol{A}^T; \boldsymbol{r}_0^*)$ such that $\boldsymbol{r}_j^* \perp span\{\boldsymbol{r}_0, \ldots, \boldsymbol{r}_{j-1}\}$ with $(j \leq i)$.
The Bi-CG method is derived by Fletcher (1976) where he assumes that the residuals $\boldsymbol{r}_i$ and $\boldsymbol{r}_i^*$ can be written as

$$\boldsymbol{r}_i^{Bi-CG} = P_i(\boldsymbol{A})\boldsymbol{r}_0 \quad \text{and} \quad \boldsymbol{r}_i^{*Bi-CG} = P_i(\boldsymbol{A}^T)\boldsymbol{r}_0^*,$$

where $P_i(\boldsymbol{A})$ is a polynomial of degree at most i. The bi-orthogonality of the residuals $\boldsymbol{r}_i$ and $\boldsymbol{r}_j^*$ can also be written as

$$\left(P_i(\boldsymbol{A})\boldsymbol{r}_0, P_j(\boldsymbol{A}^T)\boldsymbol{r}_0^*\right) = 0 \quad \text{for } j < i. \tag{5.1.1}$$

With this expression $\boldsymbol{r}_i$ as well as $\boldsymbol{r}_j^*$ need to be constructed, therefore matrix-vector products with both $\boldsymbol{A}$ and $\boldsymbol{A}^T$ need to be determined.

CG-S

The CG-S method is Sonneveld (1989), he used that equation (5.1.1) can be written as

$$\left(P_i(\boldsymbol{A})\boldsymbol{r}_0, P_j(\boldsymbol{A}^T)\boldsymbol{r}_0^*\right) = (P_j(\boldsymbol{A})P_i(\boldsymbol{A})\boldsymbol{r}_0, \boldsymbol{r}_0^*) = 0 \quad \text{for } j < i.$$

CG-S constructs residuals that can be written as

$$\boldsymbol{r}_i^{CG-S} = P_i(\boldsymbol{A})\boldsymbol{r}_i^{Bi-CG} = P_i^2(\boldsymbol{A})\boldsymbol{r}_0.$$

The benefit of the CG-S algorithm is that $\boldsymbol{r}_j^*$ do not need to be formed and hence matrix-vector products with $\boldsymbol{A}^T$ are not computed any more and the convergence of CG-S can be twice as fast as the convergence of Bi-CG. The downside of this method is that its convergence behaviour is not very smooth and the squaring of the residual polynomial may lead to a build-up of rounding errors.

Bi-CGSTAB

To improve the convergence behaviour of the CG-S method van der Vorst (1992) derived the more smoothly converging variant of CG-S called Bi-CGSTAB. The residual $\boldsymbol{r}_i$ is written as

$$\boldsymbol{r}_i^{Bi-CGSTAB} = Q_j(\boldsymbol{A})\boldsymbol{r}_i^{Bi-CG} = Q_j(\boldsymbol{A})P_j(\boldsymbol{A})\boldsymbol{r}_0,$$

with

$$Q_j(\boldsymbol{A}) = (I - \omega_1\boldsymbol{A})(I - \omega_2\boldsymbol{A})\ldots(I - \omega_j\boldsymbol{A}), \tag{5.1.2}$$

where the coefficients ω_i are chosen such that the residual is minimized. The Bi-CGSTAB algorithm, see van der Vorst (1992), is given in algorithm 5.1.

Bi-CGSTAB is a finite method, this means that in finite precision arithmetic after at most N iterations the exact solution $\boldsymbol{x}$ is obtained. The algorithm presented in algorithm 5.1 ω is chosen to minimize the norm of the residual. Another choice of ω is possible and its form should be depending on the specific problem that is solved. There are three possibilities when breakdown in the Bi-CGSTAB algorithm occurs. It could happen when $\rho_i = (\boldsymbol{r}_0*, \boldsymbol{r}_{i-1}) = 0$ with $\boldsymbol{r}_{i-1} \neq 0$, when $(\boldsymbol{r}_0^*, \boldsymbol{v}_i) = 0$ or when $(\boldsymbol{t}, \boldsymbol{s}) = 0$. One iteration of Bi-CGSTAB computes 4 inner products, 2 matvec products and $12N$ flops. The vectors $\boldsymbol{x}$, $\boldsymbol{b}$, $\boldsymbol{r}_i$, $\boldsymbol{r}_0^*$, $\boldsymbol{p}$, $\boldsymbol{v}$ and $\boldsymbol{t}$ and matrix $\boldsymbol{A}$ need to be stored.

5.2   ILU preconditioner

In preconditioning the trick is to find a matrix $\boldsymbol{K}$ such that $\boldsymbol{K}^{-1}\boldsymbol{A}$ has better properties than matrix $\boldsymbol{A}$ for the numerical algorithm that is used. Due to the better properties of $\boldsymbol{K}^{-1}\boldsymbol{A}$ less iterations are needed for a good approximation of the solution $\boldsymbol{x}$. As described by van der Vorst (1992) a good preconditioner $\boldsymbol{K}$ satisfies the following properties

Bi-CGSTAB

1. $\boldsymbol{x}_0$ is an initial guess; $\boldsymbol{r}_0 = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_0$;
2. $\boldsymbol{r}_0^*$ is an arbitrary vector, such that $(\boldsymbol{r}_0, \boldsymbol{r}_0^*) \neq 0$, e.g. $\boldsymbol{r}_0^* = \boldsymbol{r}_0$;
3. $\rho_0 = \alpha = \omega_0 = 1$;
4. $\boldsymbol{v}_0 = \boldsymbol{p}_0 = 0$;

5.   For $i = 1, 2, \ldots$
6. $\rho_i = (\boldsymbol{r}_0^*, \boldsymbol{r}_{i-1})$; $\beta = (\rho_i/\rho_{i-1})(\alpha/\omega_{i-1})$;
7. $\boldsymbol{p}_i = \boldsymbol{r}_{i-1} + \beta(\boldsymbol{p}_{i-1} - \omega_{i-1}\boldsymbol{v}_{i-1})$;
8. $\boldsymbol{v}_i = \boldsymbol{A}\boldsymbol{p}_i$;
9. $\alpha = \rho_i/(\boldsymbol{r}_0^*, \boldsymbol{v}_i)$;
10. $\boldsymbol{s} = \boldsymbol{r}_{i-1} - \alpha\boldsymbol{v}_{i-1}$;
11. $\boldsymbol{t} = \boldsymbol{A}\boldsymbol{s}$;
12. $\omega_i = (\boldsymbol{t}, \boldsymbol{s})/(\boldsymbol{t}, \boldsymbol{t})$;
13. $\boldsymbol{x}_i = \boldsymbol{x}_{i-1} + \alpha\boldsymbol{p}_i + \omega_i\boldsymbol{s}$;
14.   if $\boldsymbol{x}_i$ is accurate enough then quit
15. $\boldsymbol{r}_i = \boldsymbol{s} - \omega_i\boldsymbol{t}$;
16.   end

Alg. 5.1: The unpreconditioned Bi-CGSTAB algorithm.

ILU

1.   For $i = 1, 2, \ldots, n$
2.   For $k = 1, \ldots, i - 1$ and for $(i, k) \in NZ(\boldsymbol{A})$
3. $a_{ik} = a_{ik}/a_{kk}$;
4.   For $j = k + 1, \ldots, n$ and for $(i, j) \in NZ(\boldsymbol{A})$
5. $a_{ij} = a_{ij} - a_{ik}a_{kj}$;
7.   end
8.   end
9.   end

Alg. 5.2: Incomplete LU factorization algorithm (Meijerink and van der Vorst (1977)).

1. $\boldsymbol{K}$ is a good approximation to $\boldsymbol{A}$ in some sense.

2. The cost of the construction of K is not prohibitive.

3. The system $\boldsymbol{Ky} = \boldsymbol{z}$ is much easier to solve than the original system.

In HARES the incomplete LU factorization (ILU) is used as a preconditioner of matrix $\boldsymbol{S}$, ILU is a variant of Gaussian elimination where some elements in the LU factorization are discarded. Gaussian elimination of a matrix results in a $\boldsymbol{LU}$ factorization, where $\boldsymbol{L}$ is a lower triangular matrix and $\boldsymbol{U}$ an upper triangular matrix. The standard ILU method uses that matrices $\boldsymbol{L}$ and $\boldsymbol{U}$ have the same zero pattern as $\boldsymbol{A}$, i.e. if $a_{i,j} = 0$ $(1 \leq i, j \leq N)$ for a certain combination (i, j) then $u_{i,j} = l_{i,j} = 0$ and if $a_{i,j} \neq 0$ then $u_{i,j} \neq 0$ and $l_{i,j} \neq 0$. The diagonal of $\boldsymbol{L}$ is set equal to one, i.e. $l_{i,i} = 1$, and the diagonal of $\boldsymbol{U}$ is determined in the ILU algorithm. In general it is impossible to match $\boldsymbol{A}$ with $\boldsymbol{LU}$ when $\boldsymbol{L}$ and $\boldsymbol{U}$ have the same zero-pattern as $\boldsymbol{A}$, the extra elements of $\boldsymbol{LU}$ are called the fill-in elements. Matrix $\boldsymbol{A}$ can be written as

$$\boldsymbol{A} = \boldsymbol{LU} - \boldsymbol{R},$$

where matrix $\boldsymbol{R}$ is the residual of the factorization, containing the fill-in elements, and $\boldsymbol{K} = \boldsymbol{LU}$ is the preconditioner.

There are three ways of applying the preconditioner $\boldsymbol{K}$, at the left or at the right side of matrix $\boldsymbol{A}$ or when the preconditioner $\boldsymbol{K}$ is available in factored form on both sides of matrix $\boldsymbol{A}$. The ILU preconditioner $\boldsymbol{K} = \boldsymbol{LU}$ is available in factored form, with $\boldsymbol{K}_1 = \boldsymbol{L}$ and $\boldsymbol{K}_2 = \boldsymbol{U}$. The preconditioned system $\boldsymbol{Ax} = \boldsymbol{b}$ is given by

$$\boldsymbol{L}^{-1}\boldsymbol{AU}^{-1}\boldsymbol{y} = \boldsymbol{L}^{-1}\boldsymbol{b} \quad \text{with} \quad \boldsymbol{y} = \boldsymbol{Ux}.$$

Denote the set of non-zero elements of matrix $\boldsymbol{A}$ as $NZ(\boldsymbol{A})$, i.e. the set of pairs (i, j), $1 \leq i, j \leq N$ such that $a_{i,j} \neq 0$. The incomplete factorization of matrix $\boldsymbol{A}$ is determined such that the elements of $\boldsymbol{A} - \boldsymbol{LU}$ are zero in the elements of $NZ(\boldsymbol{A})$. The algorithm of the incomplete LU factorization is given in algorithm 5.2.

Preconditioned Bi-CGSTAB

1. $\boldsymbol{x}_0$ is an initial guess; $\boldsymbol{r}_0 = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_0$;
2. $\boldsymbol{r}_0^*$ is an arbitrary vector such that $(\boldsymbol{r}_0^*, r_0) \neq 0$, e.g. $\boldsymbol{r}_0^* = r_0$.
3. $\rho_0 = \alpha = w_0 = 1$;
4. $\boldsymbol{v}_0 = \boldsymbol{p}_0 = 0$;

5.   For $i = 1, 2, \ldots$
6. $\rho_i = (\boldsymbol{r}_0^*, \boldsymbol{r}_{i-1})$; $\beta = (\rho_i / \rho_{i-1})(\alpha / w_{i-1})$;
7. $\boldsymbol{p}_i = \boldsymbol{r}_{i-1} + \beta(\boldsymbol{p}_{i-1} - w_{i-1}\boldsymbol{v}_{i-1})$;
8.   Solve $\boldsymbol{y}$ for $\boldsymbol{K}\boldsymbol{y} = \boldsymbol{p}_i$;
9. $\boldsymbol{v}_i = \boldsymbol{A}\boldsymbol{y}$;
10. $\alpha = \rho_i / (\boldsymbol{r}_0^*, \boldsymbol{v}_i)$;
11. $\boldsymbol{s} = \boldsymbol{r}_{i-1} - \alpha\boldsymbol{v}_i$;
12.   Solve $\boldsymbol{z}$ from $\boldsymbol{K}\boldsymbol{z} = \boldsymbol{s}$;
13. $\boldsymbol{t} = \boldsymbol{A}\boldsymbol{z}$;
14. $w_i = (\boldsymbol{t}, \boldsymbol{s}) / (\boldsymbol{t}, \boldsymbol{t})$;
15. $\boldsymbol{x}_i = \boldsymbol{x}_{i-1} + \alpha\boldsymbol{y} + w_i\boldsymbol{z}$;
16.   if $\boldsymbol{x}_i$ is accurate enough then quit;
17. $\boldsymbol{r}_i = \boldsymbol{s} - w_i\boldsymbol{t}$;
18.   end

Alg. 5.3: The preconditioned Bi-CGSTAB algorithm.

The ILU preconditioner used in HARES is not the ILU(0) preconditioner as presented above but the ILU variant derived by van der Ploeg (1994), which uses a special reordering of the unknowns.

5.3 Preconditioned Bi-CGSTAB

The Bi-CGSTAB algorithm given in table 5.1 can be easily adapted in the case when a preconditioner is applied to the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. The preconditioned Bi-CGSTAB algorithm as described by van der Vorst (1992) is given in algorithm 5.3.

Chapter 6

Numerical methods proposed for improvement

In this section alternative numerical methods are presented which could lead to an improvement of the current numerical implementation of the non-linear Mild-Slope equation. Section 6.1 treats the matrix solver IDR(s) and section 6.2 the shifted Laplace preconditioner.

6.1 IDR(s)

The IDR(s) algorithm (see Sonneveld and van Gijzen (2008)) is a Krylov subspace type method for which the residuals $\boldsymbol{r}_n = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_n$ are in the Krylov subspace $\mathcal{K}^n(\boldsymbol{A}; \boldsymbol{r}_0)$. The residuals of a Krylov method satisfy the following recursion method

$$\boldsymbol{r}_{n+1} = \boldsymbol{r}_n - \alpha \boldsymbol{A}\boldsymbol{v}_n - \sum_{l=1}^{\hat{l}} \gamma_l \Delta \boldsymbol{r}_{n-l},$$

where $\boldsymbol{v}_n$ is any computable vector in $\mathcal{K}^n(\boldsymbol{A}; \boldsymbol{r}_0) \setminus \mathcal{K}^{n-1}(\boldsymbol{A}; \boldsymbol{r}_0)$ and $\Delta \boldsymbol{r}_n = \boldsymbol{r}_{n+1} - \boldsymbol{r}_n$. The IDR($s$) method is based on the following theorem (see Sonneveld and van Gijzen (2008), also for the proof).

Theorem 1. *(IDR theorem) Let $\boldsymbol{A}$ be any matrix in $\mathbb{C}^{N \times N}$, let v_0 be any non-zero vector in $\mathbb{C}^N$, and let $\mathcal{G}_0$ be the full Krylov space $\mathcal{K}^N(\boldsymbol{A}, \boldsymbol{v}_0)$. Let $\mathcal{S}$ denote any (proper) subspace of $\mathbb{C}^N$ such that $\mathcal{S}$ and $\mathcal{G}_0$ do not share a non-trivial invariant subspace of $\boldsymbol{A}$, and define the sequence $\mathcal{G}_j$, $j = 1, 2, \ldots$ as*

$$\mathcal{G}_j = (\boldsymbol{I} - \omega_j \boldsymbol{A})(\mathcal{G}_{j-1} \cap \mathcal{S}),$$

where the ω_j's are non-zero scalars. Then

(i) $\mathcal{G}_j \subset \mathcal{G}_{j-1}$ for all $\mathcal{G}_{j-1} \neq \{\boldsymbol{0}\}$, $j > 0$.

(ii) $\mathcal{G}_j = \{\boldsymbol{0}\}$ for some $j \leq N$.

The IDR(s) algorithm is based on generating residuals $\boldsymbol{r}_n$ which are forced to be in the subspace $\mathcal{G}_j$, where j is non-decreasing with increasing n. According to theorem 1 ultimately $\boldsymbol{r}_n \in \mathcal{G}_M = \{\boldsymbol{0}\}$ with $M \leq N$. The residual $\boldsymbol{r}_{n+1}$ is in the space $\mathcal{G}_{j+1}$ if

$$\boldsymbol{r}_{n+1} = (\boldsymbol{I} - \omega_{j+1}\boldsymbol{A})\boldsymbol{v}_n, \tag{6.1.1}$$

with $\boldsymbol{v}_n \in \mathcal{G}_j \cap \mathcal{S}$. The main problem becomes finding $\boldsymbol{v}_n$. The following expression is chosen for $\boldsymbol{v}_n$;

$$\boldsymbol{v}_n = \boldsymbol{r}_n - \sum_{l=1}^{s} \gamma_l \Delta \boldsymbol{r}_{n-l}.$$

Let there be a $N \times s$ matrix $\boldsymbol{P} = (\boldsymbol{p}_1 \ \boldsymbol{p}_2 \ \ldots \ \boldsymbol{p}_s)$ such that $\mathcal{S}$ is the left null-space of $\boldsymbol{P}$, i.e. $\mathcal{S} = \mathcal{N}\left(\boldsymbol{P}^H\right)$. Since $\boldsymbol{v}_n$ is contained in $\mathcal{S}$ it also holds that

$$\boldsymbol{P}^H \boldsymbol{v}_n = \boldsymbol{P}^H \left(\boldsymbol{r}_n - \boldsymbol{c}\Delta\boldsymbol{R}_n\right) = 0 \quad \Rightarrow \quad \boldsymbol{P}^H \Delta\boldsymbol{R}_n \boldsymbol{c} = \boldsymbol{P}^H \boldsymbol{r}_n,$$

with $\Delta\boldsymbol{R}_n = (\Delta\boldsymbol{r}_{n-1} \ \Delta\boldsymbol{r}_{n-2} \ \ldots \ \Delta\boldsymbol{r}_{n-s})$ and $\boldsymbol{c} \in \mathbb{C}^s$ contains the coefficients γ_l. We obtained an $s \times s$ linear system for the coefficients γ_l which is uniquely solvable. Vector $\boldsymbol{c}$ can be determined and hence we are able to compute $\boldsymbol{v}_n$ and $\boldsymbol{r}_{n+1} \in \mathcal{G}_{j+1}$. By theorem 1 it follows that $\mathcal{G}_{j+1} \subset \mathcal{G}_j$, therefore it also holds that $\boldsymbol{r}_{n+1} \in \mathcal{G}_j$. Using $\boldsymbol{r}_{n+1}$ we can compute $\Delta\boldsymbol{R}_{n+1}$, $\boldsymbol{v}_{n+1} \in (\mathcal{G}_j \cap \mathcal{S})$ and hence $\boldsymbol{r}_{n+2} \in \mathcal{G}_{j+1}$. This needs to repeated $s+1$ times such that all the elements of $\Delta\boldsymbol{R}_n$ are in $\mathcal{G}_{j+1}$. Then we have that $\boldsymbol{v}_{n+(s+1)} \in (\mathcal{G}_{j+1} \cap \mathcal{S})$ and therefore the computed residual $\boldsymbol{r}_{n+(s+1)}$ is contained in the subspace $\mathcal{G}_{j+2}$ of $\mathcal{G}_{j+1}$.

The IDR(s) algorithm is given in algorithm 6.1. From the algorithm we can see that for the first residual in $\mathcal{G}_{j+1}$ the coefficient ω can be chosen freely (in this algorithm minimizing the norm of $\boldsymbol{r}_{n+1}$) but for the computations of the other residuals in $\mathcal{G}_{j+1}$ it remains the same. Also the matrix $\boldsymbol{P}$ can be chosen freely in the beginning of the algorithm.

In the current algorithm the initialization is done using a simple Krylov method. However, as long as the $\Delta\boldsymbol{x}_i$, $i = 0, \ldots, s-1$ are in the complete Krylov subspace, they can be chosen freely. In the algorithm ω should be selected, currently it is based on a strategy proposed by Sleijpen and van der Vorst (1995).

The IDR theorem states that dimension reduction takes place but not by how much. The following theorem describes at which rate the dimension reduction takes place.

Theorem 2. (***Extended IDR theorem***) *Let $\boldsymbol{A}$ be any vector in $\mathbb{C}^{N \times N}$, let $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_s \in \mathbb{C}^N$ be linearly independent, let $\boldsymbol{P} = (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_s)$, let $\mathcal{G}_0 = \mathcal{K}^N(\boldsymbol{A}; \boldsymbol{r}_0)$ be the full Krylov space corresponding to $\boldsymbol{A}$ and the vector $\boldsymbol{r}_0$, and let the sequence of spaces $\{\mathcal{G}_j, j = 1, 2, \ldots\}$ be defined by*

$$\mathcal{G}_j = (\boldsymbol{I} - \omega_j \boldsymbol{A})\left(\mathcal{G}_{j-1} \cap \mathcal{N}\left(\boldsymbol{P}^H\right)\right),$$

where ω_j are non-zero numbers, such that $\boldsymbol{I} - \omega_j \boldsymbol{A}$ is non-singular. Let $\dim(\mathcal{G}_j) = d_j$; then the sequence $\{d_j, j = 0, 1, 2, \ldots\}$ is monotonically non-increasing and satisfies

$$0 \le d_j - d_{j+1} \le d_{j-1} - d_j \le s.$$

According to the extended IDR theorem the dimension reduction per step is between 0 and s. When the dimension reduction is precisely s it is called the generic case, otherwise we have the non-generic case. The extended IDR theorem leads to the following corollary for the generic case.

Corollary 1. *In the generic case IDR(s) requires at most $N + \frac{N}{s}$ matrix-vector multiplications to compute the exact solution in exact arithmetic.*

IDR(s)

1. **Require:** $\boldsymbol{A} \in \mathbb{C}^{N \times N}$; $\boldsymbol{x}_0, \boldsymbol{b} \in \mathbb{C}^N$; $\boldsymbol{P} \in \mathbb{C}^{N \times s}$; $TOL \in (0,1)$; $MAXIT > 0$
2. **Ensure:** $\boldsymbol{x}_n$ such that $\|\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_n\| \leq TOL$
3.   *{Initialization.}*
4.   Calculate $\boldsymbol{r}_0 = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_0$;

5.   *{Apply s minimum norm steps, to build enough vectors in $\mathcal{G}_0$}*
6.   For $n = 0, 1, \ldots, s-1$
7. $\boldsymbol{v} = \boldsymbol{A}\boldsymbol{r}_n$; Select ω;
8. $\Delta \boldsymbol{x}_n = \omega \boldsymbol{r}_n$; $\Delta \boldsymbol{r}_n = -\omega \boldsymbol{v}$;
9. $\boldsymbol{r}_{n+1} = \boldsymbol{r}_n + \Delta \boldsymbol{r}_n$; $\boldsymbol{x}_{n+1} = \boldsymbol{x} + \Delta \boldsymbol{x}_n$;
10.   end
11. $\Delta \boldsymbol{R}_{n+1} = (\Delta \boldsymbol{r}_n \ldots \Delta \boldsymbol{r}_0)$; $\Delta \boldsymbol{X}_{n+1} = (\Delta \boldsymbol{x}_n \ldots \Delta \boldsymbol{x}_0)$;

12.   *{Building $\mathcal{G}_j$ spaces for $j = 1, 2, 3, \ldots$}*
13. $n = s$
14.   *{Loop over $\mathcal{G}_j$ spaces}*
15.   while $\|\boldsymbol{r}_n\| > TOL$ and $n < MAXIT$
16.   *{Loop inside $\mathcal{G}_j$ space}*
17.   for $k = 0, 1, \ldots, s$
18.   Solve $\boldsymbol{c}$ from $\boldsymbol{P}^H \Delta \boldsymbol{R}_n \boldsymbol{c} = \boldsymbol{P}^H \boldsymbol{r}_n$;
19. $\boldsymbol{v} = \boldsymbol{r}_n - \Delta \boldsymbol{R}_n \boldsymbol{c}$;
20.   if $k = 0$
21.   *{Entering $\mathcal{G}_{j+1}$}*
22. $\boldsymbol{t} = \boldsymbol{A}\boldsymbol{v}$;
23.   Select ω;
24. $\Delta \boldsymbol{r}_n = -\Delta \boldsymbol{R}_n \boldsymbol{c} - \omega \boldsymbol{t}$;
25. $\Delta \boldsymbol{x}_n = -\Delta \boldsymbol{X}_n \boldsymbol{c} + \omega \boldsymbol{v}$;
26.   else
27.   *{Subsequent vectors in $\mathcal{G}_{j+1}$}*
28. $\Delta \boldsymbol{x}_n = -\Delta \boldsymbol{X}_n \boldsymbol{c} + \omega \boldsymbol{v}$;
29. $\Delta \boldsymbol{r}_n = -\boldsymbol{A}\Delta \boldsymbol{x}_n$;
30.   end
31. $\boldsymbol{r}_{n+1} = \boldsymbol{r}_n + \Delta \boldsymbol{r}_n$;
32. $\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \Delta \boldsymbol{x}_n$;
33. $n = n + 1$;
34. $\Delta \boldsymbol{R}_n = (\Delta \boldsymbol{r}_{n-1} \ldots \Delta \boldsymbol{r}_{n-s})$;
35. $\Delta \boldsymbol{X}_n = (\Delta \boldsymbol{x}_{n-1} \ldots \Delta \boldsymbol{x}_{n-s})$;
36.   end
37.   end

Alg. 6.1: The IDR(s) algorithm

6.1.1   IDR(s)-*biortho*

It is possible to make some adjustments to the IDR(s) algorithm as presented previously. The residual of equation 6.1.1 can also be written as

$$\boldsymbol{r}_{n+1} = \boldsymbol{r}_n - \omega_{j+1}\boldsymbol{A}\boldsymbol{v}_n - \boldsymbol{G}_n\boldsymbol{c} \quad \text{with} \quad \boldsymbol{G}_n = \Delta\boldsymbol{R}_n.$$

The corresponding recursion for the iterate is obtained by multiplying the equation above by $\boldsymbol{A}^{-1}$, hence

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n + \omega_{j+1}\boldsymbol{v}_n + \boldsymbol{U}_n\boldsymbol{c} \quad \text{with} \quad \boldsymbol{U}_n = A^{-1}\boldsymbol{G}_n = \Delta\boldsymbol{X}_n.$$

For $\boldsymbol{u}_{n+1} = \boldsymbol{x}_{n+2} - \boldsymbol{x}_{n+1}$ and $\boldsymbol{g}_{n+1} = -(\boldsymbol{r}_{n+2} - \boldsymbol{r}_{n+1})$ we find the following iterates

$$\boldsymbol{u}_{n+1} = \omega_{j+1}\boldsymbol{v}_{n+1} + \boldsymbol{U}_{n+1}\boldsymbol{c} \quad \text{and} \quad \boldsymbol{g}_{n+1} = \boldsymbol{A}\boldsymbol{u}_{n+1}.$$

The next residual and iterate can be determined by

$$\boldsymbol{r}_{n+k+1} = \boldsymbol{r}_{n+k} - \boldsymbol{g}_{n+k} \quad \text{and} \quad \boldsymbol{x}_{n+k+1} = \boldsymbol{x}_{n+k} + \boldsymbol{u}_{n+k},$$

with $\boldsymbol{r}_{n+k+1}, \boldsymbol{r}_{n+k}, \boldsymbol{g}_{n+k} \in \mathcal{G}_{j+1}$. To compute a new residual in $\mathcal{G}_{j+1}$ we could also use a more general linear combination of vectors in $\mathcal{G}_{j+1}$;

$$\boldsymbol{r}_{n+k+1} = \boldsymbol{r}_{n+k} - \sum_{i=1}^{k}\beta_i\boldsymbol{g}_{n+i}.$$

And for the vector $\boldsymbol{g}_{n+k}$ we can use

$$\boldsymbol{g}_{n+k} = \bar{\boldsymbol{g}} - \sum_{i=1}^{k-1}\alpha_i\boldsymbol{g}_{n+i} \quad \text{with} \quad \bar{\boldsymbol{g}} = -(\boldsymbol{r}_{n+k+1} - \boldsymbol{r}_{n+k}) = -\Delta\boldsymbol{r}_{n+k}.$$

The values of α_i and β_i are chosen such that intermediate residuals and $\boldsymbol{g}_{n+k}$ have desirable properties. Analogous $\boldsymbol{x}_{n+k+1}$ and $\boldsymbol{u}_{n+k}$ can be determined with a linear combination using the same parameters α_i and β_i. In IDR(s)-*biortho* α_i is chosen such that the vector $\boldsymbol{g}_{n+k}$ is orthogonal to $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_{k-1}$ and β_i such that the intermediate residual $\boldsymbol{r}_{n+k+1}$ is orthogonal to $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k$.

6.2   Shifted Laplace Preconditioner

As described in chapter 4 applying the finite element method to the Mild-Slope equation leads to a discrete Mild-Slope operator, which for the inner iteration results in solving the linear system $\boldsymbol{S}\boldsymbol{a} = \boldsymbol{f}$. The global vector $\boldsymbol{f} \in \mathbb{C}^N$ contains the source term of the given incoming wave and global matrix $\boldsymbol{S} \in \mathbb{C}^{N \times N}$ has the following form

$$\boldsymbol{S} = (-\boldsymbol{L} - \boldsymbol{C} + z_1\boldsymbol{M}) = (\boldsymbol{L}^* + \boldsymbol{C}^* - z_1\boldsymbol{M}^*) \quad \text{with} \quad \boldsymbol{L}^* = -\boldsymbol{L},\ \boldsymbol{C}^* = -\boldsymbol{C} \text{ and } \boldsymbol{M}^* = -\boldsymbol{M}.$$

Matrix $\boldsymbol{L}$ is the discretization of $\nabla \cdot \left(\frac{n_0}{k^2}\nabla\tilde{\zeta}\right)$, $\boldsymbol{C}$ corresponds to the boundary conditions and $\boldsymbol{M}$ the mass matrix for $\tilde{\zeta}$ with $z_1 = \left(n_0 - \frac{iW}{\omega}\right)$. As described by Erlangga et al. (2004) the shifted Laplace preconditioner is of the form

$$\boldsymbol{K} = (\boldsymbol{L}^* + \boldsymbol{C}^* - z_2\boldsymbol{M}^*) = (-\boldsymbol{L} - \boldsymbol{C} + z_2\boldsymbol{M}),$$

that is the same form as the discrete Mild-Slope operator $\boldsymbol{S}$. The precondioned system is given by

$$\boldsymbol{K}^{-1}\boldsymbol{S}\boldsymbol{a} = \boldsymbol{K}^{-1}\boldsymbol{f} \quad \Rightarrow \quad (-\boldsymbol{L} - \boldsymbol{C} + z_2\boldsymbol{M})^{-1}(-\boldsymbol{L} - \boldsymbol{C} + z_1\boldsymbol{M})\boldsymbol{a} = (-\boldsymbol{L} - \boldsymbol{C} + z_2\boldsymbol{M})^{-1}\boldsymbol{f}.$$

The shift parameter z_2 needs to be chosen in such a way that it is easy to obtain the inverse of the preconditioner P. As mentioned by van Gijzen et al. (2007) a suitable value for the shift parameter is $z_2 = -|z_1|i$. Hence the shifted Laplace preconditioner is given by

$$\boldsymbol{K} = (-\boldsymbol{L} - \boldsymbol{C} - i|z_1|\boldsymbol{M}).$$

Matrix $\boldsymbol{K}$ is still a full matrix and therefore computing its inverse takes a long computational time. Hence one should approximate it, e.g. by taking the incomplete LU factorization of it.

The matrix $\boldsymbol{K}^{-1}\boldsymbol{S}$ has, with the exact shifted Laplace preconditioner, has a special spectrum for certain properties of the matrices $\boldsymbol{L}$, $\boldsymbol{C}$ and $\boldsymbol{M}$. The complex numbers z_1 and z_2 can be written as

$$z_1 = \alpha_1 + i\beta_1 \quad \text{and} \quad z_2 = \alpha_2 + i\beta_2.$$

van Gijzen et al. (2007) derived that, when $\boldsymbol{L}$ and $\boldsymbol{C}$ are symmetric positive semidefinite real matrices and $\boldsymbol{M}$ a symmetric positive definite real matrix and $\beta_2 < 0$ all the eigenvalues of the preconditioned system are located inside or on a circle with center $c = \frac{z_1 - \bar{z}_2}{z_2 - \bar{z}_2}$ and radius $R = \left|\frac{z_2 - z_1}{z_2 - \bar{z}_2}\right|$.

Chapter 7

Numerical experiments

The following experiments are performed to gain insight in the behaviour of the several numerical methods.

- Incompete LU factorization (ILU(0)) preconditioner combined with

    CG-S; Bi-CGSTAB; IDR(2); IDR(4);

- The shifted Laplace (SL) preconditioner combined with

    CG-S; Bi-CGSTAB; IDR(2); IDR(4);

- The incomplete LU factorization (ILU(0)) of the shifted Laplace (ILU(0)-SL) preconditioner combined with

    CG-S; Bi-CGSTAB; IDR(2); IDR(4);

The test are done on the data of the harbour of Scheveningen (as is shown on the title page), provided by Svasek Hydraulics. The domain is divided into 126.504 internal triangular elements and 2.213 boundary line segments. This results in 63.253 unknowns, hence we have that $\boldsymbol{S} \in \mathbb{C}^{63.253 \times 63.253}$. We considered 25 outer iterations for the non-linear part, figure 7.1 shows the number of iterations for each outer iteration that were needed for the numerical method to converge.
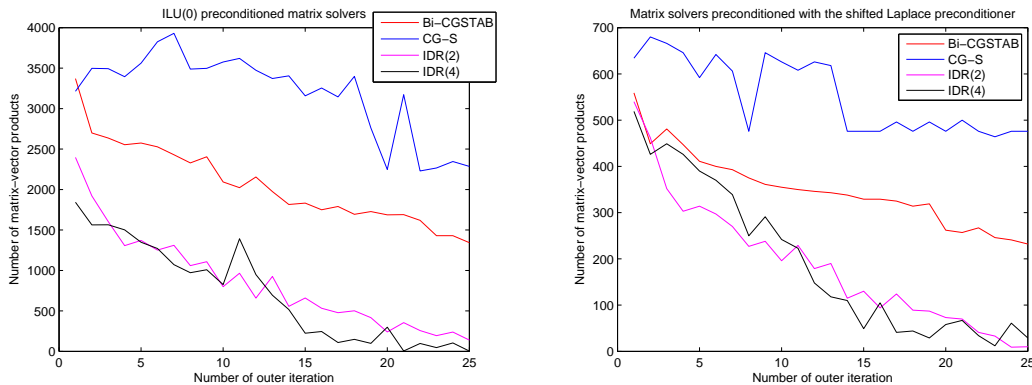
(a) Results for the incomplete LU factorization preconditioner

(b) Results for the shifted Laplace preconditioner
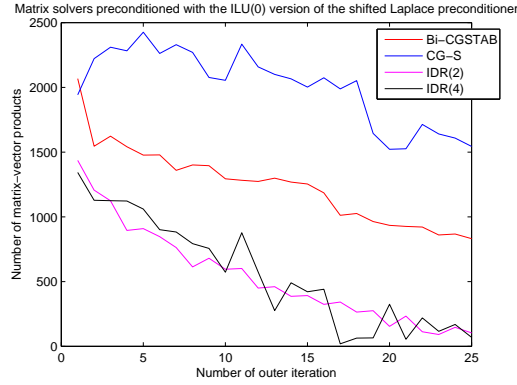
(c) Results for the incomplete LU factorization of
the shifted Laplace preconditioner

Figure 7.1: The number of iterations that are needed for the numerical method to converge
depending on the type of preconditioner.

Figure 7.1 shows the number of matrix-vector products needed for each outer iteration such
that the numerical method converges with the three types of preconditioners. As mentioned
in section 6.1 we can choose our starting vector $\Delta \boldsymbol{X}_s$ as long as its elements are in the
complete subspace. In the implementation of IDR(s) we used, when the number of outer
iterations is larger than or equal to s, that $\Delta \boldsymbol{X}_s$ contains the solutions of the s previous
outer iterations. This is why the amount of iterations needed for the numerical methods
IDR(2) and IDR(4) decrease fast as the number of outer iterations increases, while for the
methods CG-S and Bi-CGSTAB there is not a significant decrease of the amount of iterations
needed. CG-S and Bi-CGSTAB use only the previous solution. Figure 7.1(b) shows that
using the shifted Laplace preconditioner the number of iterations is decreased by a factor
four compared with using the incomplete LU factorization as a preconditioner. Using the full
shifted Laplace preconditioner is very expensive, see table 7.1. That is why the incomplete
LU factorization of the shifted Laplace preconditioner is determined. Figure 7.1(c) shows that
this preconditioner uses less iterations than the ILU(0) preconditioner but more than the full
shifted Laplace preconditioner. The total time needed for the 25 outer iterations is given in
table 7.1.

|		Numerical method			
Preconditioner		Bi-CGSTAB	CG-S	IDR(2)	IDR(4)
	ILU(0)	$3.0192 \cdot 10^3$	$4.1053 \cdot 10^3$	$1.2303 \cdot 10^3$	$1.2863 \cdot 10^3$
Total time (s)	SL	$2.4946 \cdot 10^4$	$3.7362 \cdot 10^4$	$1.2566 \cdot 10^4$	$1.3273 \cdot 10^4$
	ILU(0)-SL	$1.8525 \cdot 10^3$	$2.6634 \cdot 10^3$	$0.7361 \cdot 10^3$	$0.9741 \cdot 10^3$

Table 7.1: CPU time until the whole process is completed.

As can be seen from table 7.1 the CPU time for IDR(2) and IDR(4) are of the same order, while
for Bi-CGSTAB and CG-S the computational time is significantly higher. At the moment
it takes more time when the shifted Laplace preconditioner is used than using the ILU(0)

preconditioner, but less iterations are needed. The algorithm which is currently used applies the preconditioning with the standard direct method of Matlab. Hence for the matrix $\boldsymbol{K}$ this costs more time than for the lower triangular matrix $\boldsymbol{L}$. Applying the incomplete LU factorization of the shifted Laplace preconditioner results is about a factor three faster than the current numerical method ILU(0) preconditioner Bi-CGSTAB.

Figure 7.2 shows the convergence behaviour of the four numerical methods in the first outer iteration. Preconditioning with the incomplete LU factorization leads to a greater difference for the amount of iterations that are needed to convergence between the four numerical methods than preconditioning with the shifted Laplace preconditioner. We can also see that the convergence behaviour of CG-S is not as smooth as the convergence behaviour of Bi-CGSTAB and IDR(s), which was one of the disadvantages of using CG-S. Figure 7.2(a) shows that the convergence of the methods CG-S, Bi-CGSTAB and IDR(2) stagnates when using the ILU(0) preconditioner, when using the shifted Laplace and the ILU(0) shifted Laplace this behaviour is only present for CG-S and Bi-CGSTAB.
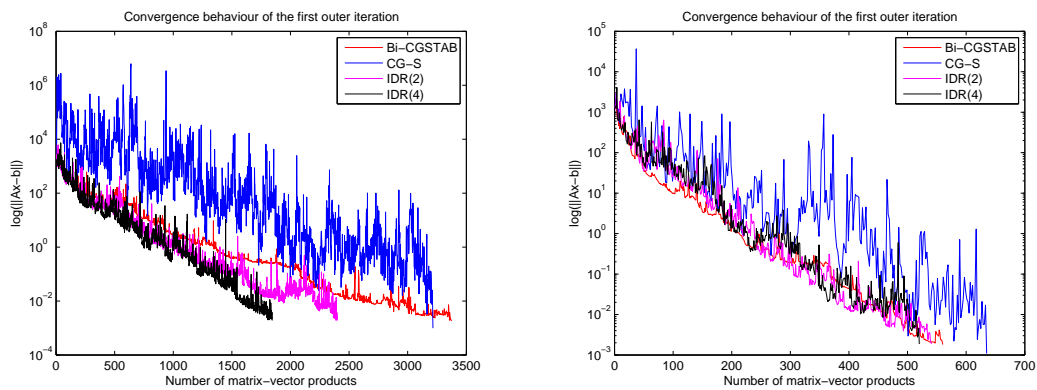

(a) Convergence behaviour for the incomplete LU factorization preconditioner

(b) Convergence behaviour for the shifted Laplace preconditioner
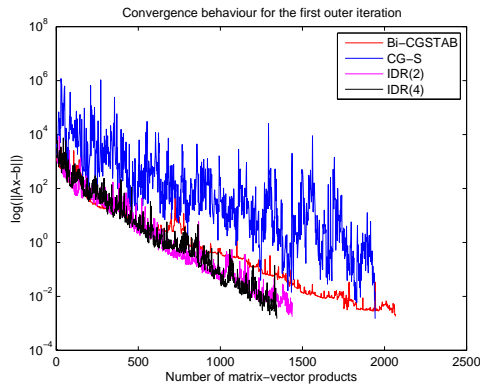
(c) Convergence behaviour for the incomplete LU
factorization of the shifted Laplace preconditioner

Figure 7.2: The convergence behaviour of the first outer iteration of the four numerical
methods with the two preconditioners applied.

Combining the results of figures 7.1 and 7.2 and table 7.1 indicate that using the IDR(s)
method results in less iterations (matrix-vector products) and shorter CPU time than Bi-
CGSTAB or CG-S. Figure 7.3 shows the convergence behaviour of ILU(0)-IDR(s), SL-IDR(s)
and the ILU(0) version of SL-IDR(s) with $s = 2$ and $s = 4$. Using the incomplete LU fac-
torization of the shifted Laplace preconditioner IDR(s) converges faster than just the ILU(0)
preconditioner, but comparing it to the full shifted Laplace preconditioner a lot more itera-
tions are needed.
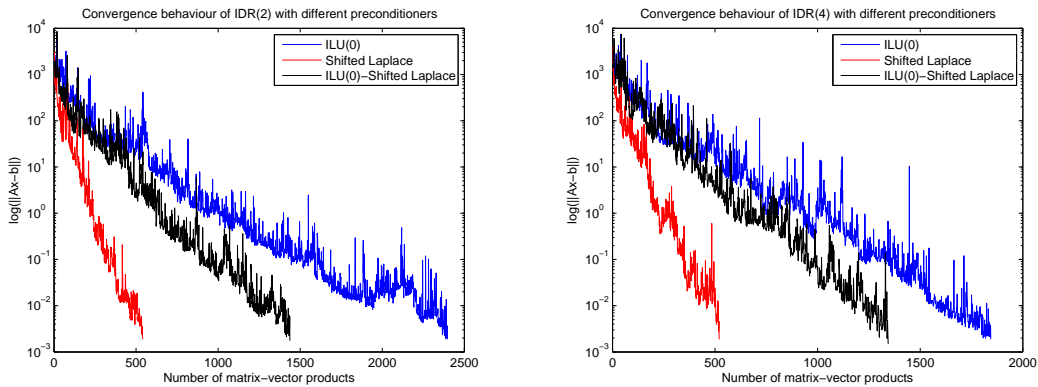
(a) Difference between the ILU(0) and the SL pre-
conditioner for IDR(2)

(b) Difference between the ILU(0) and the SL pre-
conditioner for IDR(4)

Figure 7.3: The convergence behaviour of IDR(2) and IDR(4) combined with the two pre-
conditioners.

Chapter 8

Conclusion and research objectives

This literature study starts with the derivation of the linearised small-amplitude wave equations as described in the work by Dingemans (1997) and Mei (1989). Using these equations the Mild-Slope equation, as described by Berkhoff (1976) in his PhD thesis for the Technical University of Delft, can be derived. The Mild-Slope equations combines the four effects (diffraction, reflection, refraction and shoaling) that influence the wave dynamics in coastal regions. Therefore it is often used to predict the wave behaviour in harbours. The original Mild-Slope equation does not incorporate the loss of wave energy, e.g. caused by wave breaking and bottom friction, but this is present in the actual wave dynamics. Including the energy dissipation leads to a non-linear component in the Mild-Slope equation and hence more advanced numerical methods are needed to solve it.

The Ritz-Galerkin finite element method of the Mild-Slope equation is derived in chapter 4, to get an idea of the shape of the matrices and vectors used in HARES. Hence we might be able to say something about the convergence behaviour of the numerical methods proposed. Currently the matrix solver Bi-CGSTAB, derived by van der Vorst (1992), preconditioned with a special form of the incomplete LU factorization, proposed by van der Ploeg (1994). The iterative method IDR(s), Sonneveld and van Gijzen (2008), and the shifted Laplace preconditioner, van Gijzen et al. (2007), are presented as a possibility to improve HARES. Tests are performed on the data of the harbour of Scheveningen to see whether IDR(s) and the shifted Laplace preconditioner result in a decreased computational time. Comparing figure 7.1(a) with figure 7.1(c) we see that the amount of iterations needed for each outer iteration has decreased. The computational time, CPU time, see table 7.1, has been improved by a factor three when using the incomplete LU factorization of the shifted Laplace preconditioner combined with IDR(s) instead of the ILU(0) preconditioner with Bi-CGSTAB.

Research objectives

In the second part of this research I will investigate the following topics.

- *Which improvements can be made regarding way the non-linearity of the problem is treated?* Currently they implemented Picard iteration and just perform 25 outer iterations. It might be that less outer iterations are also sufficient. Is it possible to implement a method such that only the amount of outer iterations is performed which significantly improve the solution? Can something be said about the error that is made in the obtained result?

- In the performed experiments we obtained a faster convergence when using the shifted Laplace preconditioner. In van Gijzen et al. (2007) an optimal value is derived. However it is not clear whether our damping matrix C satisfies the conditions for this optimal shift. *Can we theoretically say something about the convergence behaviour of the shifted Laplace preconditioner for the Mild-Slope equation?* Knowing this it might be possible to determine an optimal form of the shifted Laplace preconditioner to improve the convergence even further. In figure 7.3 we see that using the incomplete LU factorization of the shifted Laplace preconditioner speeds up the convergence. However using a different approximation could decrease the number of needed matrix-vector multiplications even more. *How should the shifted Laplace preconditioner be approximated?*

- In this literature study some iterative methods based on the Krylov subspace are tested on a test problem. However other methods, such as a direct method, could also give good results. More types of matrix solvers will be tested on a wide range of test problems. At Svašek Hydraulics FORTRAN is used as the computational program. Hence in the end the proposed numerical method should be able to run in FORTRAN.

- The algorithm of IDR (s), as presented in Alg. 6.1, produces in the lines $5 - 11$ the vectors that are contained in $\mathcal{G}_0$. These starting vectors do not have to be formed this way and maybe they can be chosen in such a way that the convergence is speeded up. *Does chosing the initial vectors lead to a faster convergence, and if so how should these vectors be chosen?*

- In section 6.2 it is stated that all the eigenvalues of the preconditioned system are located in a circle for certain conditions of the matrices. In the algorithms of Bi-CGSTAB and IDR(s) the coefficients ω_i present in the polynomial $Q(\boldsymbol{A})$, equation 5.1.2, can be chosen in a desired manner. A good choice would be to choose ω_i such that this polynomial is minimized on a given domain. *When this domain is a circle, how should we choose the coefficients ω_i.*

Appendix A

Calculations not included in the text

A.1 Derivation of the vorticity equation

$$\nabla(a \cdot b) = (a \cdot \nabla)b + (b \cdot \nabla)b + a \times (\nabla \times b) + b \times (\nabla \times a)$$

$$\nabla \times (a \times b) = (b \cdot \nabla)a - (a \cdot \nabla)b + a\nabla \cdot b - b\nabla \cdot a$$

$$\nabla \boldsymbol{v}^2 = 2(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + 2\boldsymbol{v} \times (\nabla \times \boldsymbol{v}) \quad \Rightarrow \quad (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \frac{1}{2}\nabla \boldsymbol{v}^2 - \boldsymbol{v} \times (\nabla \times \boldsymbol{v})$$

$$\nabla \times \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v}\right) = \nabla \times \left[-\nabla\left(\frac{\boldsymbol{p}}{\rho} + gz\right) + \nu\Delta\boldsymbol{v}\right]$$

can be written as

$$\frac{\partial}{\partial t}(\nabla \times \boldsymbol{v}) + \nabla \times \left(\frac{1}{2}\nabla \boldsymbol{v}^2 - \boldsymbol{v} \times (\nabla \times \boldsymbol{v})\right) = \nabla \times \nu\Delta\boldsymbol{v}$$

Introducing $\boldsymbol{\Omega} = \nabla \times \boldsymbol{v}$ gives

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} - \nabla \times (\boldsymbol{v} \times \boldsymbol{\Omega}) = \nu\Delta\boldsymbol{\Omega}$$

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} - (\boldsymbol{\Omega} \cdot \nabla)\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{\Omega} - \boldsymbol{v}\nabla \cdot \boldsymbol{\Omega} + \boldsymbol{\Omega}\nabla \cdot \boldsymbol{v} = \nu\Delta\boldsymbol{\Omega}$$

The terms $\nu\nabla \cdot \boldsymbol{\Omega}$ and $\boldsymbol{\Omega}\nabla \cdot \boldsymbol{v}$ are both equal to zero, hence we get

$$\begin{aligned}
\frac{\partial \boldsymbol{\Omega}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{\Omega} &= (\boldsymbol{\Omega} \cdot \nabla)\boldsymbol{v} + \nu\Delta\boldsymbol{\Omega} \\
\left((\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right)\boldsymbol{\Omega} &= (\boldsymbol{\Omega} \cdot \nabla)\boldsymbol{v} + \nu\Delta\boldsymbol{\Omega} \\
\frac{d\boldsymbol{\Omega}}{dt} &= (\boldsymbol{\Omega} \cdot \nabla)\boldsymbol{v} + \nu\Delta\boldsymbol{\Omega}
\end{aligned}$$

A.2 Derivation of the dimensionless equations

We start with the equations

$$\Delta\Phi = 0 \qquad\qquad ; \quad -h(\boldsymbol{x}) \leq z \leq \zeta(\boldsymbol{x}, t) \qquad\qquad \text{(A.2.1)}$$

$$\frac{\partial\zeta}{\partial t} + \frac{\partial\Phi}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\Phi}{\partial y}\frac{\partial\zeta}{\partial y} = \frac{\partial\Phi}{\partial z} \qquad\qquad ; \quad z = \zeta(\boldsymbol{x}, t) \qquad\qquad \text{(A.2.2)}$$

$$\frac{\partial\Phi}{\partial t} + \frac{1}{2}(\nabla\Phi)^2 + g\zeta = -\frac{P_a}{\rho} \qquad\qquad ; \quad z = \zeta(\boldsymbol{x}, t) \qquad\qquad \text{(A.2.3)}$$

$$\frac{\partial\Phi}{\partial z} + \frac{\partial\Phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial\Phi}{\partial y}\frac{\partial h}{\partial y} = 0 \qquad\qquad ; \quad z = -h(\boldsymbol{x}) \qquad\qquad \text{(A.2.4)}$$

and the scaled variables

$$\Phi = \frac{a\omega L}{2\pi}\Phi', \quad (x, y, z, h) = \frac{L}{2\pi}(x', y', z', h'), \quad t = \frac{1}{\omega}t' \quad \text{and} \quad \zeta = a\zeta'.$$

We will treat each equation separately. Substitution of these dimensionless variables into equation (A.2.1) gives

$$\Delta\Phi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi = 0,$$

$$\frac{2\pi}{L}\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right)\frac{a\omega L}{2\pi}\Phi' = 0,$$

$$a\omega\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}\right)\Phi' = 0,$$

$$\Delta'\Phi' = 0.$$

The same is done for equation (A.2.2)

$$\frac{\partial\Phi}{\partial z} = \frac{\partial\zeta}{\partial t} + \frac{\partial\Phi}{\partial x}\frac{\partial\zeta}{\partial x} + \frac{\partial\Phi}{\partial y}\frac{\partial\zeta}{\partial y},$$

$$\frac{a\omega L}{2\pi}\frac{2\pi}{L}\frac{\partial\Phi'}{\partial z'} = \omega a\frac{\partial\zeta'}{\partial t'} + \frac{a\omega L}{2\pi}\frac{2\pi}{L}a\frac{2\pi}{L}\left(\frac{\partial\Phi'}{\partial x'}\frac{\partial\zeta'}{\partial x'} + \frac{\partial\Phi'}{\partial y'}\frac{\partial\zeta'}{\partial y'}\right),$$

$$\frac{\partial\Phi'}{\partial z'} = \frac{\partial\zeta'}{\partial t'} + \frac{2\pi a}{L}\left(\frac{\partial\Phi'}{\partial x'}\frac{\partial\zeta'}{\partial x'} + \frac{\partial\Phi'}{\partial y'}\frac{\partial\zeta'}{\partial y'}\right),$$

$$\frac{\partial\Phi'}{\partial z'} = \frac{\partial\zeta'}{\partial t'} + \epsilon\left(\frac{\partial\Phi'}{\partial x'}\frac{\partial\zeta'}{\partial x'} + \frac{\partial\Phi'}{\partial y'}\frac{\partial\zeta'}{\partial y'}\right),$$

where $\epsilon = \frac{2\pi a}{L}$. For the dynamical boundary condition, equation (A.2.3), for $z = \zeta$ we get

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla \Phi)^2 + g\zeta = -\frac{P_a}{\rho},$$

$$\frac{a\omega^2 L}{2\pi}\frac{\partial \Phi'}{\partial t'} + \frac{1}{2}a^2\omega^2(\nabla'\Phi')^2 + ga\zeta' = -\frac{P_a}{\rho},$$

$$\frac{\partial \Phi'}{\partial t'} + \frac{1}{2}\frac{2\pi a}{L}(\nabla'\Phi')^2 + \frac{2\pi g}{\omega^2 L}\zeta' = -\frac{2\pi P_a}{\rho a\omega^2 L},$$

$$\frac{\partial \Phi'}{\partial t'} + \frac{1}{2}\epsilon(\nabla'\Phi')^2 + \left(\frac{2\pi g}{\omega^2 L}\right)\zeta' = -P'_a.$$

The last equation for the wetted surface boundary is made dimensionless by

$$\frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{\partial h}{\partial y} = 0.$$

$$a\omega\frac{\partial \Phi'}{\partial z'} + a\omega\left(\frac{\partial \Phi'}{\partial x'}\frac{\partial h'}{\partial x'} + \frac{\partial \Phi'}{\partial y'}\frac{\partial h'}{\partial y'}\right) = 0,$$

$$\frac{\partial \Phi'}{\partial z'} + \frac{\partial \Phi'}{\partial x'}\frac{\partial h'}{\partial x'} + \frac{\partial \Phi'}{\partial y'}\frac{\partial h'}{\partial y'} = 0.$$

For the boundaries $z = \zeta$ and $z = h$ we get the dimensionless expressions

$$z = \zeta \quad \Rightarrow \quad \frac{Lz}{2\pi} = a\zeta' \quad \Rightarrow \quad z' = \frac{2\pi a}{L}\zeta' \quad \text{hence} \quad z' = \epsilon\zeta',$$

and

$$z = -h \quad \Rightarrow \quad \frac{Lz'}{2\pi} = \frac{Lh'}{2\pi} \quad \Rightarrow \quad z' = h'.$$

Now we have obtained the set of equations (1.4.5) - (1.4.8).

A.3 Calculations for the kinetic- and potential wave energy

Kinetic wave energy

The equation for the kinetic wave energy is given by

$$E_k = \int_0^L \int_{-h}^0 \frac{1}{2}\rho \; dzdx \; (u^2 + w^2),$$

with

$$u(x, z, t) = \frac{gk_0 H}{2\omega}\frac{\cosh(k_0(z + h))}{\cosh(k_0 h)}\cos(k_0 x - \omega t),$$

$$w(x, z, t) = \frac{gk_0 H}{2\omega}\frac{\sinh(k_0(z + h))}{\cosh(k_0 h)}\sin(k_0 x - \omega t).$$

Taking the square of both velocity components gives

$$u^2(x,z,t) = \frac{g^2 k_0^2 H^2}{4\omega^2} \frac{\cosh^2(k_0(z+h))}{\cosh^2(k_0 h)} \cos^2(k_0 x - \omega t),$$

$$= \frac{g k_0 H^2}{4 \tanh(k_0 h)} \frac{\cosh^2(k_0(z+h))}{\cosh^2(k_0 h)} \cos^2(k_0 x - \omega t),$$

$$= \frac{g k_0 H^2}{4 \sinh(k_0 h) \cosh(k_0 h)} \cosh^2(k_0(z+h)) \cos^2(k_0 x - \omega t),$$

$$= \frac{g k_0 H^2}{2 \sinh(2 k_0 h)} \cosh^2(k_0(z+h)) \cos^2(k_0 x - \omega t),$$

$$w^2(x,z,t) = \frac{g k_0 H^2}{2 \sinh(2 k_0 h)} \sinh^2(k_0(z+h)) \sin^2(k_0 x - \omega t).$$

Summing the squared velocity components gives

$$u^2 + w^2 = \frac{g k_0 H^2}{2 \sinh(2 k_0 h)} \left(\cosh^2(k_0(z+h)) \cos^2(k_0 x - \omega t) + \sinh^2(k_0(z+h)) \sin^2(k_0 x - \omega t) \right).$$

We would like to find a nice expression for

$$\cosh^2(k_0(z+h)) \cos^2(k_0 x - \omega t) + \sinh^2(k_0(z+h)) \sin^2(k_0 x - \omega t),$$

such that it is possible to integrate it. We will do this by introducing the Euler representation of the cosine, sine, cosine hyperbolic and the sine hyperbolic . We have

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \quad \Rightarrow \quad \cos^2(x) = \frac{e^{2ix} + e^{-2ix} + 2}{4},$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \quad \Rightarrow \quad \sin^2(x) = \frac{e^{2ix} + e^{-2ix} - 2}{-4},$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \quad \Rightarrow \quad \cosh^2(x) = \frac{e^{2x} + e^{-2x} + 2}{4},$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2} \quad \Rightarrow \quad \sinh^2(x) = \frac{e^{2x} + 2e^{-2x} - 2}{4}.$$

Introducing $\alpha = k_0(z+h)$ and $\beta = k_0 x - \omega t$, gives

$$\cosh^2(\alpha) \cos^2(\beta) = \frac{(e^{2\alpha} + e^{-2\alpha} + 2)(e^{2i\beta} + e^{-2i\beta} + 2)}{16},$$

$$= \frac{e^{2\alpha+2i\beta} + e^{2\alpha-2i\beta} + 2e^{2\alpha} + e^{-2\alpha+ei\beta} + e^{-2\alpha-2i\beta} + 2e^{-2\alpha} + 2e^{2i\beta} + e^{-2i\beta} + 4}{16},$$

$$\sinh^2(\alpha) \sin^2(\beta) = \frac{(e^{2\alpha} + e^{-2\alpha} - 2)(e^{2i\beta} + e^{-2i\beta} - 2)}{-16},$$

$$= \frac{-e^{2\alpha+2i\beta} - e^{2a-2i\beta} + 2e^{2\alpha} - e^{-2\alpha+2i\beta} - e^{-2\alpha-2i\beta} + 2e^{-2\alpha} + 2e^{2i\beta} + 2e^{-2i\beta} - 4}{16}.$$

Combining this gives

$$\cosh^2(\alpha)\cos^2(\beta) + \sinh^2(\alpha)\sin^2(\beta) = \frac{4e^{2\alpha} + 4e^{-2\alpha} + 4e^{2i\beta} + 4e^{-2i\beta}}{16},$$

$$= \frac{e^{2\alpha} + e^{-2\alpha} + e^{2i\beta} + e^{-2i\beta}}{4},$$

$$= \frac{1}{2}\cosh(2\alpha) + \frac{1}{2}\cos(2\beta),$$

$$= \frac{1}{2}\cosh(2k_0(z+h)) + \frac{1}{2}\cos(2(k_0 x - \omega t)).$$

Hence for the sum of the squared velocity components we find

$$u^2 + w^2 = \frac{gk_0 H^2}{4\sinh(2k_0 h)}\left[\cosh(2k_0(z+h)) + \cos(2(k_0 x - \omega t))\right].$$

We first consider the integral

$$\int_{-h}^0 \frac{1}{2}\rho(u^2 + w^2)\,dz.$$

This gives

$$\int_{-h}^0 \frac{1}{2}\rho(u^2 + w^2)\,dz = \frac{\rho g k_0 H^2}{8\sinh(2k_0 h)}\int_{-h}^0 \left[\cosh(2k_0(z+h)) + \cos(2(k_0 x - \omega t))\right]\,dz,$$

$$= \frac{\rho g k_0 H^2}{8\sinh(2k_0 h)}\left[\frac{1}{2k_0}\sinh(2k_0(z+h)) + \cos(2(k_0 x - \omega t))z\right]_{-h}^0,$$

$$= \frac{\rho g k_0 H^2}{8\sinh(2k_0 h)}\left[\frac{1}{2k_0}\sinh(2k_0 h) + h\cos(2(k_0 x - \omega t))\right].$$

Now we can determine the outer integral, this gives

$$\frac{\rho g k_0 H^2}{8\sinh(2k_0 h)}\int_0^L \left[\frac{1}{2k_0}\sinh(2k_0 h) + h\cos(2(k_0 x - \omega t))\right]\,dx$$

$$= \frac{\rho g k_0 H^2}{8\sinh(2k_0 h)}\left[\frac{1}{2k_0}\sinh(2k_0 h)x + \frac{h}{2k_0}\sin(2(k_0 x - \omega t))\right]_0^L,$$

$$= \frac{\rho g k_0 H^2}{8\sinh(2k_0 h)}\left[\frac{L}{2k_0}\sinh(2k_0 h) + \frac{h}{2k_0}\sin(2(k_0 L - \omega t)) - \frac{h}{2k_0}\sin(-2\omega t)\right],$$

$$= \frac{\rho g k_0 H^2}{8\sinh(2k_0 h)}\left[\frac{L}{2k_0}\sinh(2k_0 h)\right],$$

$$= \frac{\rho g H^2 L}{16} = E^k.$$

The expression $\sin(2(k_0 L - \omega t)) - \sin(-2\omega t)$ equals zero, this is because $k_0 = 2\pi/L$ hence we get

$$\sin(4\pi - 2\omega t) - \sin(-2\omega t) = 0.$$

Potential wave energy

The equation for the potential wave energy is given by

$$E_p = \int_0^L \rho g(h + \zeta) \left(\frac{h + \zeta}{2} \right) \, dx - \rho g L h \left(\frac{h}{2} \right),$$

with

$$\zeta = \frac{H}{2} \cos(k_0 x - \omega t).$$

We first look at the term $(h + \zeta)^2/2$, this gives

$$\frac{(h + \zeta)^2}{2} = \frac{h^2 + 2h\zeta + \zeta^2}{2},$$

$$= \frac{1}{2} \left[h^2 + hH \cos(k_0 x - \omega t) + \frac{H^2}{4} \cos^2(k_0 x - \omega t) \right],$$

$$= \frac{1}{2} \left[h^2 + hH \cos(k_0 x - \omega t) + \frac{H^2}{8} \left[\cos(2(k_0 x - \omega t)) + 1 \right] \right].$$

For the integral we get

$$\frac{\rho g}{2} \int_0^L \left[h^2 + hH \cos(k_0 x - \omega t) + \frac{H^2}{8} \left[\cos(2(k_0 x - \omega t)) + 1 \right] \right] \, dx$$

$$= \frac{\rho g}{2} \left[h^2 x + \frac{hH}{k_0} \sin(k_0 x - \omega t) + \frac{H^2}{8} \left[\frac{1}{2k_0} \sin(2(k_0 x - \omega t)) + x \right] \right]_0^L,$$

$$= \frac{\rho g}{2} \left[h^2 L + \frac{hH}{k_0} \left[\sin(k_0 L - \omega t) - \sin(-\omega t) \right] + \frac{H^2}{8} \left(\frac{1}{2k_0} \left[\sin(2(k_0 L - \omega t)) - \sin(-2\omega t) \right] + L \right) \right],$$

$$= \frac{\rho g}{2} \left[h^2 L + \frac{H^2}{8} L \right].$$

Hence for the potential wave energy we get

$$E_p = \frac{\rho g h^2 L}{2} + \frac{\rho g H^2 L}{16} - \frac{\rho g h^2 L}{2},$$

$$= \frac{\rho h H^2 L}{16}.$$

A.4   Calculations for the wave power

The equation for the wave power is given by

$$P = \frac{1}{T} \int_0^T \int_{-h}^0 \left(-\rho \frac{\partial \Phi}{\partial t} \right) u \, dzdt,$$

with

$$-\rho \frac{\partial \Phi}{\partial t} = \frac{\rho g H}{2} \frac{\cosh(k_0(z + h))}{\cosh(k_0 h)} \cos(k_0 x - \omega t),$$

$$u = \frac{g H T}{2L} \frac{\cosh(k_0(z + h))}{\sinh(k_0 h)} \cos(k_0 x - \omega t).$$

Thus we get

$$-\rho\frac{\partial\Phi}{\partial t}u = \frac{\rho g^2 H^2 T}{4L}\frac{\cosh^2(k_0(z+h))}{\sinh(k_0 h)\cosh(k_0 h)}\cos^2(k_0 x - \omega t),$$
$$= \frac{\rho g^2 H^2 T}{4L}\frac{[\cosh(2k_0(z+h))+1]}{\sinh(2k_0 h)}\cos^2(k_0 x - \omega t).$$

The integral with respect to z is given by

$$\frac{\rho g^2 H^2}{4L}\int_{-h}^{0}\frac{[\cosh(2k_0(z+h))+1]}{\sinh(2k_0 h)}\cos^2(k_0 x - \omega t)\ dz$$
$$= \frac{\rho g^2 H^2}{4L}\left[\frac{1}{2k_0}\frac{\sinh(2k_0(z+h))}{\sinh(2k_0 h)}\cos^2(k_0 x - \omega t) + z\frac{\cos^2(k_0 x - \omega t)}{\sinh(2k_0 h)}\right]_{-h}^{0},$$
$$= \frac{\rho g^2 H^2}{4L}\left[\frac{\cos^2(k_0 x - \omega t)}{2k_0} + \frac{h\cos^2(k_0 x - \omega t)}{\sinh(2k_0 h)}\right],$$
$$= \frac{\rho g^2 H^2}{4L}\left[\frac{(\cos(2(k_0 x - \omega t))+1)}{4k_0} + h\frac{(\cos(2(k_0 x - \omega t))+1)}{2\sinh(2k_0 h)}\right].$$

For the second integral with respect to t we receive

$$\frac{\rho g^2 H^2}{4L}\int_{0}^{T}\left[\frac{(\cos(2(k_0 x - \omega t))+1)}{4k_0} + h\frac{(\cos(2(k_0 x - \omega t))+1)}{2\sinh(2k_0 h)}\right]\ dt$$
$$= \frac{\rho g^2 H^2}{4L}\left[\frac{\sin(2(k_0 x - \omega t))}{-8\omega k_0} + \frac{t}{4k_0} + \frac{h}{2\sinh(2k_0 h)}\left(\frac{-1}{2\omega}\sin(2(k_0 x - \omega t))+t\right)\right]_{0}^{T},$$
$$= \frac{\rho g^2 H^2}{4L}\left[\frac{\sin(2(k_0 x - \omega T))-\sin(2k_0 x)}{-8\omega k_0} + \frac{T}{4k_0} + \frac{h}{2\sinh(2k_0 h)}\left(\frac{\sin(2k_0 x)-\sin(2(k_0 x - \omega T))}{2\omega}+T\right)\right],$$

$$= \frac{\rho g^2 H^2}{4L}\left[\frac{T}{4k_0} + \frac{hT}{2\sinh(2k_0 h)}\right] = \frac{\rho g^2 H^2 T}{16k_0 L}\left[1 + \frac{2k_0 h}{\sinh(2k_0 h)}\right],$$

$$= \frac{\rho g H^2 L}{16T}\left[1 + \frac{2k_0 h}{\sinh(2k_0 h)}\right] = \frac{E}{2T}\left[1 + \frac{2k_0 h}{\sinh(2k_0 h)}\right] = \frac{n_0 E}{T}.$$

The expression that is obtained for the wave power is

$$P = \frac{n_0 E}{T}.$$

A.5 Derivation of the group velocity

As was derived in section 1.6.3 the expression for the group velocity is given by

$$c_g = \frac{d\omega}{dk_0},$$

with

$$\omega = \sqrt{gk_0 \tanh(k_0 h)}.$$

Performing the calculation gives

$$
\begin{aligned}
c_g &= \frac{d\omega}{dk_0}, \\
&= \frac{g \tanh(k_0 h) + \frac{gk_0 h}{\cosh^2(k_0 h)}}{2\sqrt{gk_0 \tanh(k_0 h)}}, \\
&= \frac{g \tanh(k_0 h)}{2\sqrt{gk_0 \tanh(k_0 h)}} + \frac{gk_0 h}{2\sqrt{gk_0 \tanh(k_0 h)} \cosh^2(k_0 h)}, \\
&= \frac{\omega^2/k_0}{2\omega} + \frac{gk_0 h}{2\omega \cosh^2(k_0 h)}, \\
&= \frac{\omega}{2k_0} + \frac{1}{2} \frac{gk_0 h\omega^2}{\omega gk_0 \sinh(k_0 h)\cosh(k_0 h)}, \\
&= \frac{\omega}{2k_0} + \frac{\omega h}{\sinh(2k_0 h)}, \\
&= \frac{1}{2}\frac{\omega}{k_0}\left(1 + \frac{2k_0 h}{\sinh(2k_0 h)}\right), \\
&= n_0 c.
\end{aligned}
$$

A.6   Leibniz integral rule for variable limits

Suppose that the limits of integration a and b and the integrand $f(x,\alpha)$ all are functions of the parameter α. Then we can apply Leibniz integral rule to obtain

$$\frac{d}{d\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\ dx = \frac{db(\alpha)}{d\alpha}f(b(\alpha),\alpha) - \frac{da(\alpha)}{d\alpha}f(a(\alpha),\alpha) + \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha}f(x,\alpha)\ dx.$$

A.7   Derivation of the refraction equations

We start with the equations

$$
\begin{aligned}
\epsilon^2\left(\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}\right) + \frac{\partial^2\Phi}{\partial z^2} &= 0 && ;\quad -h \le z \le 0, \\
\frac{\partial\Phi}{\partial z} - \Phi &= 0 && ;\quad z = 0, \\
\frac{\partial\Phi}{\partial z} + \epsilon^2\left(\frac{\partial\Phi}{\partial x}\frac{\partial h}{\partial \bar{x}} + \frac{\partial\Phi}{\partial y}\frac{\partial h}{\partial \bar{y}}\right) &= 0 && ;\quad z = -,
\end{aligned}
$$

and we substitute

$$\Phi(x,y,z) = A(x,y,z)e^{iS(x,y,z)}$$

into the equations above. Since both A and S depend on x, y and z the first and second derivatives are the same with respect to x, y and z. Therefore we only determine the derivatives with respect to z, the other ones follow immediately.

$$\frac{\partial \Phi}{\partial z} = \frac{\partial A}{\partial z} e^{iS} + Ai \frac{\partial S}{\partial z} e^{iS},$$

$$\frac{\partial^2 \Phi}{\partial z^2} = \frac{\partial^2 A}{\partial z^2} e^{iS} + \frac{\partial A}{\partial z} i \frac{\partial S}{\partial z} e^{iS} + \frac{\partial A}{\partial z} i \frac{\partial S}{\partial z} e^{iS} + Ai \frac{\partial^2 S}{\partial z^2} e^{iS} - A \left(\frac{\partial S}{\partial z} \right)^2 e^{iS}.$$

Hence for the first equation, the real part is given by

$$\epsilon^2 \left(\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} - A \left[\left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} \right)^2 \right] \right) + \frac{\partial^2 A}{\partial z^2} - A \left(\frac{\partial S}{\partial z} \right)^2 = 0,$$

$$\epsilon^2 \left(\frac{1}{A} \left[\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} \right] + \left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} \right)^2 \right) + \frac{1}{A} \frac{\partial^2 A}{\partial z^2} - \left(\frac{\partial S}{\partial z} \right)^2 = 0.$$

Introducing $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right)$ gives

$$\epsilon^2 \left(\frac{1}{A} \nabla^2 A + (\nabla S)^2 \right) + \frac{1}{A} \frac{\partial^2 A}{\partial z^2} - \left(\frac{\partial S}{\partial z} \right)^2 = 0.$$

For the imaginary part of the first equation is given by

$$\epsilon^2 \left(2 \nabla A \cdot \nabla S + A \nabla^2 S \right) + 2 \frac{\partial A}{\partial z} \frac{\partial S}{\partial z} + A \frac{\partial^2 S}{\partial z^2} = 0.$$

Multiplying the equation above gives

$$\epsilon^2 \left(2A \nabla A \cdot \nabla S + A^2 \nabla^2 S \right) + 2A \frac{\partial A}{\partial z} \frac{\partial S}{\partial z} + A^2 \frac{\partial^2 S}{\partial z^2} = 0,$$

$$\epsilon^2 \nabla \cdot \left[A^2 \nabla S \right] + \frac{\partial}{\partial z} \left[A^2 \frac{\partial S}{\partial z} \right] = 0.$$

For the boundary conditions we get

$$\frac{\partial A}{\partial z} - A = 0 \qquad \text{and} \qquad \frac{\partial S}{\partial z} = 0 \qquad \text{for } z = 0,$$

$$\frac{\partial A}{\partial z} + \epsilon^2 \left(\nabla A \cdot \bar{\nabla} h \right) = 0 \qquad \text{and} \qquad \frac{\partial S}{\partial z} + \epsilon^2 \left(\nabla S \cdot \bar{\nabla} h \right) = 0 \qquad \text{for } z = -h.$$

A.8   Integral for the refraction equation

We start with the integral

$$\nabla \cdot \int_{-h}^{0} \left(\frac{\cosh(\kappa(z+h))}{\cosh(\kappa h)} \right)^2 dz \, a^2 \nabla S_0 = 0.$$

Now we only consider the integral, this gives

$$\int_{-h}^{0} \left(\frac{\cosh(\kappa(z+h))}{\cosh(\kappa h)} \right)^2 dz = \frac{1}{\cosh^2(\kappa h)} \int_{-h}^{0} \frac{1}{2} \left[\cosh(2\kappa(z+h)) + 1 \right] \, dz,$$

$$= \frac{1}{\cosh^2(\kappa h)} \frac{1}{2} \left[\frac{1}{2\kappa} \sinh(2\kappa(z+h)) + z \right]_{-h}^{0},$$

$$= \frac{1}{\cosh^2(\kappa h)} \frac{1}{2} \left[\frac{\sinh(2\kappa h)}{2\kappa} + h \right],$$

$$= \frac{1}{2} \left[\frac{\sinh(2\kappa h)}{2\kappa \cosh^2(\kappa h)} + \frac{h}{\cosh^2(\kappa h)} \right],$$

$$= \frac{1}{2} \left[\frac{\sinh(\kappa h)}{\kappa \cosh(\kappa h)} + \frac{h}{\cosh^2(\kappa h)} \right],$$

$$= \frac{1}{2} \left[\frac{\tanh(\kappa h)}{\kappa} + \frac{h}{\cosh^2(\kappa h)} \right],$$

$$= \frac{1}{2\kappa} \left[\tanh(\kappa h) + \frac{\kappa h \sinh(\kappa h)}{\sinh(\kappa h) \cosh^2(\kappa h)} \right],$$

$$= \frac{1}{2\kappa} \left[\tanh \kappa h) + \frac{2\kappa h \tanh(\kappa h)}{\sinh(2\kappa h)} \right],$$

$$= \frac{\tanh(\kappa h)}{\kappa} \frac{1}{2} \left[1 + \frac{2\kappa h}{\sinh(2\kappa h)} \right],$$

$$= \frac{c^2}{g} n_0 = \frac{cc_g}{g}.$$

Hence the total expression becomes

$$\nabla \cdot \int_{-h}^{0} A_0^2 \, dz \, \nabla S_0 = \nabla \cdot \left(\frac{cc_g}{g} a^2 \nabla S_0 \right) = 0,$$

since g is the gravitational constant we get

$$\nabla \cdot \left(cc_g a^2 \nabla S_0 \right) = 0.$$

A.9   Calculations for the derivation of the Mild-Slope equation

Multiplication of equation (3.1.8) with Z and integrating it over the water depth gives

$$\epsilon^2 \left[\int_{-h}^{0} Z \left(\frac{\partial^2 Z}{\partial h^2} \overline{\nabla} h \cdot \overline{\nabla} h + \frac{\partial Z}{\partial h} \overline{\nabla}^2 h \right) \varphi \, dz \right] + \epsilon \left[\int_{-h}^{0} \frac{\partial Z^2}{\partial h} \overline{\nabla} h \cdot \nabla \varphi \, dz \right]$$

$$+ \int_{-h}^{0} \left[Z^2 (\nabla^2 \varphi + \kappa^2 \varphi) + \frac{\partial Z^2}{\partial z} \frac{\partial \varphi}{\partial z} + Z^2 \frac{\partial^2 \varphi}{\partial z^2} \right] \, dz = 0,$$

where we used that $2Z \frac{\partial Z}{\partial h} = \frac{\partial Z^2}{\partial h}$. Applying partial integration to the last two terms on the second line gives

$$\int_{-h}^{0} \left[\frac{\partial Z^2}{\partial h} \frac{\partial \varphi}{\partial z} + Z^2 \frac{\partial^2 \varphi}{\partial z^2} \right] \, dz = Z^2 \frac{\partial \varphi}{\partial z} \Big|_{-h}^{0} - \int_{-h}^{0} Z^2 \frac{\partial^2 \varphi}{\partial z^2} \, dz + \int_{-h}^{0} Z^2 \frac{\partial^2 \varphi}{\partial z^2} \, dz = Z^2 \frac{\partial \varphi}{\partial z} \Big|_{-h}^{0}.$$

Applying the boundary conditions (3.1.9) and (3.1.10) gives

$$
Z^2 \frac{\partial \varphi}{\partial z}\bigg|_{-h}^{0} = Z^2 \frac{\partial \varphi}{\partial z}\bigg|_{z=0} - Z^2 \frac{\partial \varphi}{\partial z}\bigg|_{z=-h}
$$

$$
= -Z \left(Z \frac{\partial \varphi}{\partial z} \right)_{z=-h} = Z \left[\epsilon^2 \left(\varphi \frac{\partial Z}{\partial h} \overline{\nabla} h \cdot \overline{\nabla} h \right) + \epsilon \left(\nabla \varphi \cdot \overline{\nabla} h Z \right) \right]_{z=-h}.
$$

The equation we obtain is

$$
\epsilon^2 \left[\int_{-h}^{0} z \left(\frac{\partial^2 Z}{\partial h2} \overline{\nabla} h \cdot \overline{\nabla} h + \frac{\partial Z}{\partial h} \overline{\nabla}^2 h \right) \varphi \ dz \right] + \epsilon \left[\int_{-h}^{0} \frac{\partial Z^2}{\partial h} \overline{\nabla} h \cdot \nabla \varphi \ dz \right]
$$

$$
\int_{-h}^{0} Z^2 \left(\nabla^2 \varphi + \kappa^2 \varphi \right) \ dz + \epsilon \left[Z^2 \nabla \varphi \cdot \overline{\nabla} h \right]_{z=-h} + \epsilon^2 \left[Z \varphi \frac{\partial Z}{\partial h} \overline{\nabla} h \cdot \overline{\nabla} h \right]_{z=-h} = 0.
$$

Here we give the manipulations that are done to show that equation (3.1.12) and equation (3.1.13) are equivalent. We will start with equation (3.1.13) and end up with equation (3.1.12)

$$
\nabla \cdot \int_{-h}^{0} Z^2 \ dz \nabla \varphi_0 + \kappa^2 \varphi_0 \int_{-h}^{0} Z^2 \ dz
$$

$$
= \int_{-h}^{0} Z^2 \ dz \nabla^2 \varphi_0 + \nabla \varphi_0 \nabla \cdot \int_{-h}^{0} Z^2 \ dz + \kappa^2 \varphi_0 \int_{-h}^{0} Z^2 \ dz,
$$

$$
= \int_{-h}^{0} Z^2 \ dz \nabla^2 \varphi_0 + \nabla \varphi_0 \left[\nabla h \ Z^2 \big|_{z=-h} + \int_{-h}^{0} \frac{\partial Z^2}{\partial h} \nabla h \ dz \right] + \kappa^2 \varphi_0 \int_{-h}^{0} Z^2 \ dz,
$$

$$
= \int_{-h}^{0} Z^2 \ dz \nabla^2 \varphi_0 + \nabla \varphi_0 \left[\nabla h \ Z^2 \big|_{z=-h} + \frac{\partial}{\partial h} \int_{-h}^{0} Z^2 \ dz \nabla h - \nabla h \ Z^2 \big|_{z=-h} \right] + \kappa^2 \varphi_0 \int_{-h}^{0} Z^2 \ dz,
$$

$$
= \int_{-h}^{0} Z^2 \ dz \nabla^2 \varphi_0 + \frac{\partial}{\partial h} \int_{-h}^{0} Z^2 \ dz \nabla h \cdot \nabla \varphi_0 + \kappa^2 \varphi_0 \int_{-h}^{0)} Z^2 \ dz,
$$

$$
= \int_{-h}^{0} Z^2 \ dz \nabla^2 \varphi_0 + \epsilon \frac{\partial}{\partial h} \int_{-h}^{0} Z^2 \ dz \overline{\nabla} h \cdot \nabla \varphi_0 + \kappa^2 \varphi_0 \int_{-h}^{0} Z^2 \ dz,
$$

$$
= \left(\nabla^2 \varphi_0 + \kappa^2 \varphi_0 \right) \int_{-h}^{0} Z^2 \ dz + \epsilon \overline{\nabla} h \cdot \nabla \varphi_0 \frac{\partial}{\partial h} \int_{-h}^{0} Z^2 \ dz.
$$

The equation we obtained equals equation (3.1.12).

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