Abstract

"Improving the iterative methods in TNO DIANA using physical properties of the underlying model"

Alex Sangers - December 10, 2013 at 16:00 in EEMCS Room 1.2

DIANA is a multi-purpose finite element software package that is dedicated to e.g. structural and geotechnical engineering problems. The solution of one or more systems of linear equations is a computational intensive part of a finite element analysis. For this purpose a number of direct and iterative methods are available in DIANA.

As the demand for larger finite element analysis grows every year, so lead the corresponding models to large linear systems. Iterative methods have proved to be effective to solve these systems.

The purpose of this research is to find out how the iterative methods of DIANA can be improved. One technique considered is deflation, based on the physical properties of the underlying problem. Another technique is scaling the system based on the type of degree of freedom, such as translation, rotation or pressure.

Iterative methods at DIANA

Improving the iterative methods in TNO DIANA using physical properties of the underlying model.

Alex Sangers

Delft Institute of Applied Mathematics
TNO DIANA

December 10, 2013

Content

Finite element analysis at DIANA (1/2)

Some relevant quantities:

u - displacements

ε - strain

σ - stress

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u - displacements

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σ - stress

In case of linear elastic behavior:

$$\varepsilon = B_m u,$$

with B_m the differential matrix,

$$\sigma = D_m \varepsilon,$$

with D_m the elasticity matrix.

Finite element analysis at DIANA (2/2)

K and f are assembled by

$$K^{e_m} = \int_{e_m} B_m^T D_m B_m \; dV, \qquad f^{e_m} = \int_{e_m} \sum_i f_i^{e_m} \; dV$$

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Computationally intensive part!

Solution methods for linear systems

Direct and iterative methods.

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Properties of iterative methods:

- Require less memory
- In general less robust
- Effective for important classes of problems

Iterative solution methods available at DIANA

Symmetric case:
Conjugate Gradient (CG)

- Lanczos-based algorithm
- Strategy: orthogonalizes the residuals
- Optimality
- Short-recurrence

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Nonsymmetric case:
(Restarted) Generalized Minimal Residual (GMRES(s))

- Arnoldi-based algorithm
- Strategy: minimizes the residuals
- Optimality (if no restart)
- Long-recurrence

Preconditioning

The convergence of iterative methods depend on eigenvalues and eigenvectors.

For SPD matrix K holds for the m-th CG iteration:

$$||u - u_m||_K \leq 2 \left[\frac{\sqrt{\lambda_{\max}/\lambda_{\min}} - 1}{\sqrt{\lambda_{\max}/\lambda_{\min}} + 1} \right]^m ||u - u_0||_K$$

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Preconditioning $Ku = f \;\Rightarrow\; P^{-1}Ku = P^{-1}f$,

where:

- $P \approx K \Rightarrow P^{-1}K \approx I$.
- $Px = y$ is easy to solve.

Substructuring and Domain decomposition

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Substructuring

$$K \sim \begin{pmatrix} A_1 & & & B_1 \\ & \ddots & & \vdots \\ & & A_{n_s} & B_{n_s} \\ B_1^T & \dots & B_{n_s}^T & C \end{pmatrix}$$

- No parallel implementation
- Partitioning the elements
- No overlap in partitions
- One preconditioner

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Domain decomposition

$$K = \sum_{i=1}^{n_d} L_i^T K_i R_i$$

- Parallel implementation
- Partitioning the nodes
- Overlap allowed in partitions
- Dual preconditioner

Enhancements

Required improvements:

- Jumps in material properties
- Multiple types of degrees of freedom
- Interface elements: ‘coupling’ elements
- Mixture elements: pressure-translation elements
- Nonlinear loops

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Techniques:

- Deflation
- Scaling
- IDR(s)

Deflation

Define

$$\Pi^{\in} = I - Z(Y^T K Z)^{-1} Y^T K,$$
$$\Pi^{\perp} = I - KZ(Y^T K Z)^{-1} Y^T,$$

so that $\Pi^{\perp} K = K \Pi^{\in}$.

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- Rigid body mode deflation:
  Z is approximate null space of element matrices corresponding to 'near-rigid bodies'.

Rigid body mode deflation: Example

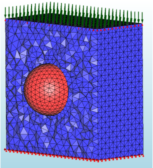

Figure: Stiff sphere in block.

Young's modulus $E(\underline{x}) = \begin{cases} 1 & \text{if } \underline{x} \in \text{block}, \\ 10^6 & \text{if } \underline{x} \in \text{sphere}. \end{cases}$

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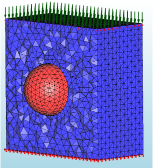

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'Near-rigid body'!

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Suppose only one element in sphere with nodes
$\underline{x}_1 = (x_1, y_1, z_1)$ and $\underline{x}_2 = (x_2, y_2, z_2)$.

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$$\begin{array}{c} \\ \\ x_1 \\ x_2 \\ y_1 \\ y_2 \\ z_1 \\ z_2 \\ \\ \\ \end{array} \qquad Z = \begin{pmatrix} & & & \emptyset & & \\ & & & & & \\ 1 & 0 & 0 & 0 & -z_1 & y_1 \\ 1 & 0 & 0 & 0 & -z_2 & y_2 \\ 0 & 1 & 0 & z_1 & 0 & -x_1 \\ 0 & 1 & 0 & z_2 & 0 & -x_2 \\ 0 & 0 & 1 & -y_1 & x_1 & 0 \\ 0 & 0 & 1 & -y_2 & x_2 & 0 \\ & & & & & \\ & & & \emptyset & & \end{pmatrix}$$

Scaling

Example:

Translation degrees of freedom:	$\mathcal{O}(t)$
Rotation degrees of freedom:	$\mathcal{O}(r)$
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Right preconditioning:

$$KP^{-1}x = f, \quad u = P^{-1}x,$$

with $P = \text{diag}(\{t, r, p\})$.

IDR(s)

Nonsymmetric systems: No 'best' algorithm.

IDR(s) forces residuals r_n in subspace $\mathcal{G}_j$ of decreasing dimension. Some freedom in algorithm remains.

GMRES(s)	IDR(s)
Long recurrences	Short recurrences
Optimal if no restart	Not optimal

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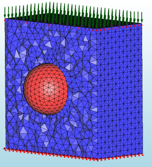
- Large systems of equations
- Ill-conditioned problems

Rigid body mode deflation so far...



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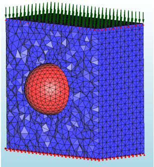

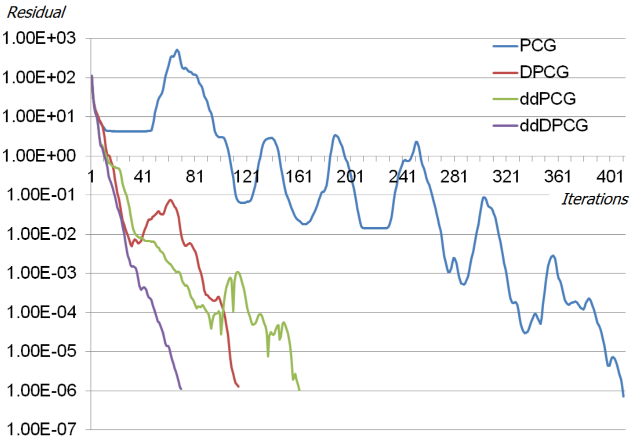

Figure: Convergence for *Block5*.

Analogue behavior for three stiff blocks in a block.

Challenges

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  - Young's modulus of materials
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  - Spring elements
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- Scaling
  - Identifying orders of magnitude (units, expected solution)
  - Combination with other preconditioners
- Nonlinear iteration loop
  - When to reuse information?
  - How to reuse information (rigid bodies, Ritz vectors, etc.)?

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