Development of the Helmholtz Solver based on Schwarz Domain Decomposition Preconditioning

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Presentation Overview

Introduction

The inhomogeneous wave equation is

$$\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right) u(\mathbf{x}, t) = f(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^d \tag{1}$$

Separation of variables leads to

$$\begin{gathered} \left(-\nabla^2 - k^2\right) \phi(\mathbf{x}) = f(\mathbf{x}), \\ k = \frac{2\pi}{\lambda}. \end{gathered} \tag{2}$$

$$f(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_0).$$

Helmholtz Boundary Value Problems

(BVP-1):

$$\begin{cases} \left(-\nabla^2 - k^2\right) u(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_0), & \text{in } \Omega \in \mathbb{R}^d. \\ u(\mathbf{x}) = 0 & \text{for } \mathbf{x} \in \partial\Omega, \\ k \in \mathbb{N}\backslash\{0\} \text{ and } d \in \{1,2,3\}. \end{cases} \tag{3}$$

(BVP-2):

$$\begin{cases} \left(-\nabla^2 - k^2\right) u(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_0), & \text{in } \Omega \in \mathbb{R}^d. \\ \left(\frac{\partial u(\mathbf{x})}{\partial \mathbf{n}} - iku(\mathbf{x})\right) = 0, & \text{for } \mathbf{x} \in \partial\Omega, \\ k \in \mathbb{N}\backslash\{0\} \text{ and } d \in \{1,2,3\}. \end{cases} \tag{4}$$

Finite Difference Discretization

Assume we have BVP-1 on unit domain Ω and $h = \frac{1}{n}$.

$$A\mathbf{u} = \mathbf{f}, \text{ on } \Omega. \tag{5}$$

- A becomes indefinite for large wave numbers.
- Iterative solution methods, GMRES

Numerical Solver Problems

- Pollution Error.
  - Accuracy issue of the solution.
  - Refinement requirements [1] [2].
    $kh < 1$, or even better $k^3 h^2 \leq 1$ with
- Problem size increasing with the wave number k.

[1] A. Sheikh (2014). "Development Of The Helmholtz Solver Based On A Shifted Laplace Preconditioner And A Multigrid Deflation Technique". Thesis

[2] A. Deraemaeker, I. Babuska, and P. Bouillard (1999). "Dispersion and pollution of the FEM solution for the Helmholtz equation in one, two and three dimensions". In: *INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING* 46.4, pp. 471–499

Helmholtz solvers

Attempts at wave-number-independent solvers

- Adapted preconditioned DEF Scheme (APD) (Deflation Preconditioner)[3]
- Two-Level Domain Decomposition Preconditioner with grid coarse space and DtN coarse space [4]

[3] V. Dwarka and C. Vuik (2020). "Scalable convergence using two-level deflation preconditioning for the helmholtz equation". In: *SIAM Journal on Scientific Computing* 42.2, A901–A928. ISSN: 1064-8275

[4] M. Bonazzoli, V. Dolean, I. G. Graham, E. A. Spence, and P. H. Tournier (2018). "Two-level preconditioners for the helmholtz equation". In: *Lecture Notes in Computational Science and Engineering*. Vol. 125. Springer Verlag, pp. 139–147. ISBN: 14397358 (ISSN). DOI: 10.1007/978-3-319-93873-8_11

One-Level Schwarz Preconditioner

2D Unit Square Domain

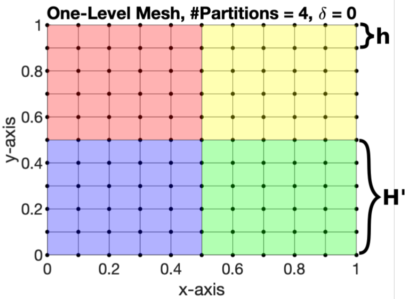

Figure: Non-overlapping subdomains

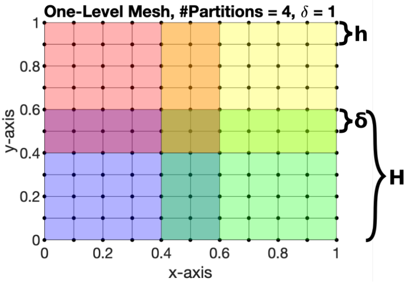

Figure: Overlapping subdomains

One-Level Schwarz Preconditioner

We introduce discrete projection-like operators as

$$P_i = R_i^T A_i^{-1} R_i, \qquad \text{for } i = 1, \dots, N \tag{6}$$

Multiplicative Schwarz operator:

$$M_{\mathrm{MS}}^{-1} = I - (I - P_N)(I - P_{N-1}) \dots (I - P_1). \tag{7}$$

Additive Schwarz operator:

$$M_{\mathrm{AS}}^{-1} = \sum_{i=1}^{N} P_i = \sum_{i=1}^{N} R_i^T A_i^{-1} R_i. \tag{8}$$

Upper bound for condition number preconditioner system:

$$\kappa(M_{\mathrm{AS}}^{-1} A) \leq C \left(\frac{1}{\delta^2 H^2} \right), \tag{9}$$

Two-Level Schwarz Preconditioner

Introduce a coarse space, which gives the two-level additive Schwarz operator is of the form

$$M_{\mathrm{AS2}}^{-1} = \sum_{i=0}^{N} P_i = R_0^T A_0^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i, \tag{10}$$

Leads to new upper bound for condition number

$$\kappa(M_{\mathrm{AS2}}^{-1} A) \leq C \left(1 + \frac{H}{\delta}\right) \tag{11}$$

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This upper bound for the condition number is **parallel scalable**!

GDSW Preconditioner

Advantages of using GDSW:

- More freedom in constructing the coarse grid,
- GDSW coarse spaces are flexible in adding additional coarse functions,
- The GDSW preconditioner only requires the trace of the interface.

GDSW Preconditioner

R_0 from Equation (10) is replaced by Φ giving the operator

$$M_{\mathrm{GDSW}}^{-1} = \sum_{i=0}^{N} P_i = \Phi^T A_0^{-1} \Phi + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i, \tag{12}$$

with

$$\kappa\left(M_{\mathrm{GDSW}}^{-1} A\right) \leq C\left(1 + \frac{H}{\delta}\right)\left(1 + \log\left(\frac{H}{h}\right)\right)^2, \tag{13}$$

and for certain GDSW coarse space we even find

$$\kappa(M_{\mathrm{GDSW}}^{-1} A) \leq C\left(1 + \frac{H}{\delta}\right). \tag{14}$$

Deflation

Higher-order coarse correction operator

Deflation:

$$P_D = I - P = I - AZ\left(E\right)^{-1} Z^T, \quad Z \in \mathbb{R}^{(n+1)^d \times r^d}. \tag{15}$$

For APD preconditioner: coarse correction operator $Z = I_{2h}^h$ with weight ε

$$I_{2h}^h \left[u_{2h}\right]_i = \left\{ \begin{array}{ll} \left(\frac{1}{8}\left[u_{2h}\right]_{(i-2)/2} + \left(\frac{3}{4} - \varepsilon\right)\left[u_{2h}\right]_{(i)/2} + \frac{1}{8}\left[u_{2h}\right]_{(i+2)/2}\right) & \text{if } i \text{ is even,} \\ \frac{1}{2}\left(\left[u_{2h}\right]_{(i-1)/2} + \left[u_{2h}\right]_{(i+1)/2}\right) & \text{if } i \text{ is odd} \end{array} \right) \tag{16}$$

Quadratic approximation using the rational Bézier curve.

Parallel Computing

- More memory storage,
- better computational performance.

Distributed memory message-passing models using the MPI library.

Trilinos: software suite with robust, scalable, parallel solver algorithms.
FROSch: Schwarz preconditioner with GDSW-type coarse space.

Test Problem Domain Decomposition Preconditioner

One-Level Additive Schwarz Preconditioner

Note: 2D Poisson problem unit square domain.

H' ($\delta = 1$)	$n^2 = 6400$	$n^2 = 25600$
0.5	17	22
0.25	24	33
0.125	30	43
0.0625	45	50+

Table: Number of GMRES iterations for the test problem using a one-level AS preconditioner.

Research Questions

Research Questions 1

Does a Helmholtz solver that uses a two-level additive Schwarz preconditioner combined with **first-order** grid coarse space show **numerical scalability** and **efficiency**?

- What causes the solver to be inscalable?

Research Questions

Research Questions 2

Does a Helmholtz solver that uses a two-level additive Schwarz preconditioner and a **higher-order** coarse space from the **deflation setting** show **numerical scalability** and **efficiency**?

- What causes the solver to be inscalable if this is the case?
- How much better does the Helmholtz solver perform when the **higher-order coarse space from the deflation setting** is used as the **multiplicative** coupling of the coarse level?

Research Questions

Research Questions 3

When using higher-order coarse spaces shows promising performance, does using a **GDSW-type coarse space**, improve the Helmholtz solver further?

- Can higher-order coarse spaces be used in GDSW-type coarse spaces?
- Does this solver show improved performance?

Research Questions

Research Questions 4

If time permits, does the numerically scalable and efficient Helmholtz solver show **parallel scalability** when it is transformed into a **parallel algorithm**? To do this we could use **FROSch** of **Trilinos**.

- Do memory or communication problems arise?
- How much does a restricted additive Schwarz preconditioner improve the parallel algorithm further?