Literature study: High Order Material Point Method

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Outline

- Introduction master project
- Description MPM
- Numerical challenges with MPM
- Benchmark problems
- Prelimenary numerical results
- Outlook

Introduction master project

Objective:

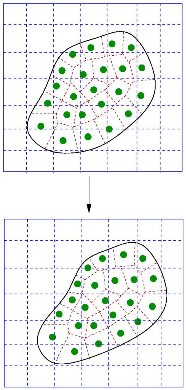
Development of a class of high-order material point methods that:

- solve/reduce some numerical challenges within classical MPM
- reduce computing times
- enable a more accurate prediction of physical quantities

Material point method

Combined particle-mesh approach:

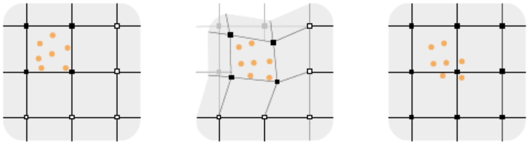
- Material points represent a continuum
- Equation of motion is solved on fixed background grid
- Material points move through domain over time



Description MPM

A time step of MPM consists of three steps:

1. Project particle properties onto nodes
2. Solve equation of motion on background grid
3. Update particle properties from solution at the nodes



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Projection from particles to nodes

Construction of mass matrix $\mathbf{M}^t$ and force vector $\mathbf{F}^{\mathrm{int,t}}$

$$\mathbf{M}^t_{(i,j)} = \int_\Omega \phi_i(x)\phi_j(x)\rho(x)\mathrm{d}x \approx \sum_{p=1}^{n_p} \phi_i(x_p^t)\phi_j(x_p^t)m_p$$

$$\mathbf{F}^{\mathrm{int,t}}_{(i)} = \int_\Omega \sigma(x)\nabla\phi_i(x)\mathrm{d}x \approx \sum_{p=1}^{n_p} \sigma_p^t\nabla\phi_i(x_p^t)V_p^t$$

Solving equation on background grid

- At each time step we obtain

$$\mathbf{M}^t \mathbf{a}^t = \mathbf{F}^t.$$

- Determine lumped mass matrix

$$\mathbf{M}^t \rightarrow \mathbf{M_L}^t.$$

- Solve equation of motion for $i \in \{1, \ldots, n_n\}$

$$\mathbf{a}_i^t = \frac{\mathbf{F}_i^t}{\mathbf{M_L}_{(i,i)}^t}.$$

Update particle properties

Update particle properties (velocity, stress, etc.) from nodal quantities by evaluating basis functions at particle positions.

Example:

$$
\begin{aligned}
v_p^{t+\Delta t} &= v_p^t + \Delta t \sum_{i=1}^{n_n} \phi_i(x_p^t) a_i^t \\
\Delta \epsilon_p^{t+\Delta t} &= \sum_{i=1}^{n_n} \nabla \phi_i(x_p^t) \Delta u_i^{t+\Delta t}
\end{aligned}
$$

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Numerical challenges with MPM

The quality of the MPM solution is affected by:

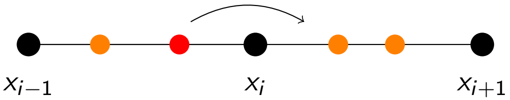
- Grid crossing errors
- Quadrature rule used in MPM

Grid crossing error

Grid crossing:
The movement of a particle from one element to another element

Effect of grid crossing:

- Non-physical increase/decrease of internal force at node
- Influences quality of MPM solution

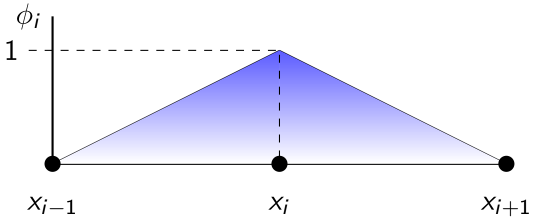


Grid crossing error

Internal force at node x_i is determined by:

$$\mathbf{F}_{(i)}^{\mathrm{int,t}} = \sum_{p=1}^{n_p} \sigma_p^t \nabla \phi_i(x_p^t) V_p^t.$$

In standard MPM, $\nabla \phi_i$ is discontinuous at element boundary:



Quadrature rule MPM

Within MPM particles serve as integration points:

$$\int_\Omega \phi_i(x)\phi_j(x)\rho(x)\mathrm{d}x \approx \sum_{p=1}^{n_p} \phi_i(x_p^t)\phi_j(x_p^t)m_p$$

This numerical integration rule is in general not exact.

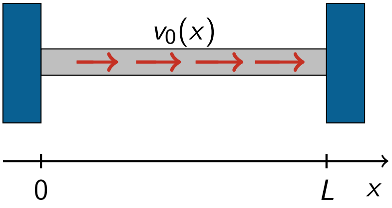
Benchmark problems

Three benchmark problems are considered:

- Vibrating string with initial velocity
- Soil column under self weight
- Vibrating bar with dynamic traction [1]

[1] M. Steffen, R. M. Kirby, M. Berzins. *Analysis and reduction of quadrature errors in the material point method (MPM). Int. J. Numer. Methods. Engrg, 76 (2008), pp. 922-948.*

Vibrating string



$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2}$$

Boundary conditions:

$$u(0,t) = 0$$

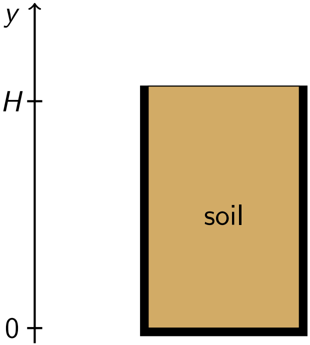
$$u(L,t) = 0$$

Initial conditions:

$$u(x,0) = 0$$

$$\frac{\partial u}{\partial t}(x,0) = v_0 \sin\left(\frac{\pi x}{L}\right)$$

Soil column under self weight



$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial y^2} - g$$

Boundary conditions:

$$u(0,t) = 0$$

$$\frac{\partial u}{\partial y}(H,t) = 0$$

Initial conditions:

$$u(y,0) = 0$$

$$\frac{\partial u}{\partial t}(y,0) = 0$$

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Implementation MPM

- Modified Lagrangian algorithm [2]
- Euler-Cromer time integration scheme
- Piecewise linear basis functions
- 1D implementation in Matlab

[2] (AL)-Kafaji, I, *Formulation of a dynamic material point method (MPM) for geomechanical problems*, Institut für Geotechnik der Universität Stuttgart, (2013)

Results

In this presentation focus on:

- Investigation of grid crossing error
- Spatial convergence

Numerical Approximation:

$$u_{ex} = u_{num} + \mathcal{O}(\Delta x^n) + \mathcal{O}(\Delta t)$$

RMS Error:

$$e^{RMS} = \sqrt{\frac{1}{n_p}\sum_{p=1}^{n_p}\left(u_{num}(x_p,t) - u_{ex}(x_p,t)\right)^2}$$

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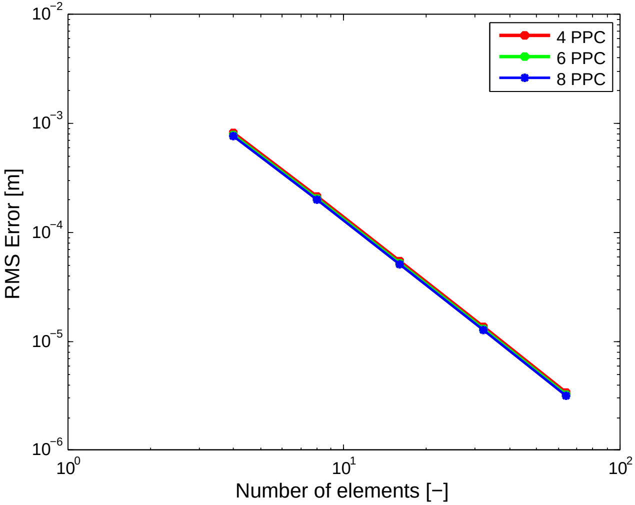
Vibrating string



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Vibrating string

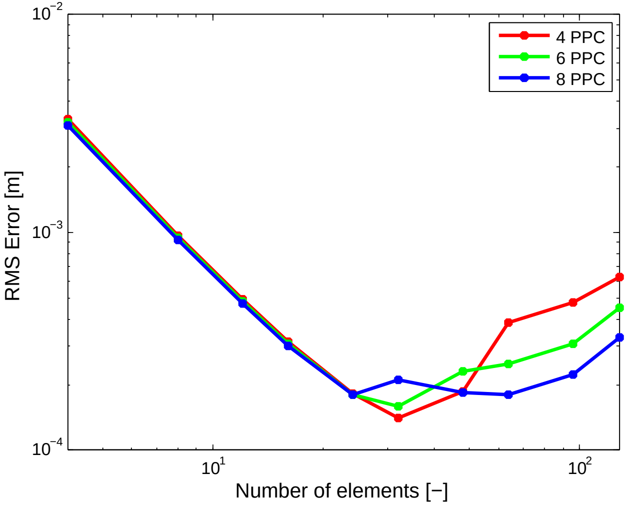
For the vibrating string problem, when particles (almost) stay equally distributed:

- MPM shows 2$^{\mathrm{nd}}$ order convergence in space.
- Increasing number of PPC, decreases the RMS error.

Order of convergence consistent with literature. [3]

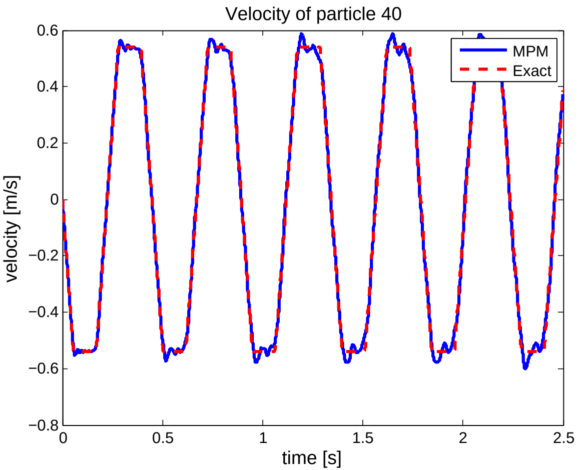
[3] M. Gong. *Improving the Material Point Method. The University of New Mexico (2015)*

Soil column under self weight



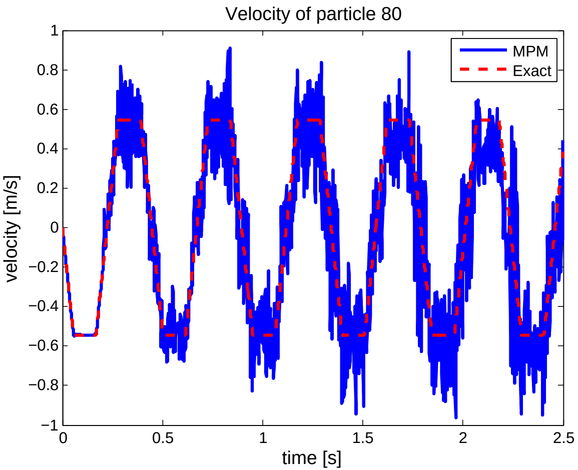
Soil column under self weight

40 elements, no grid crossing



Soil column under self weight

80 elements, grid crossing



Soil column under self weight

- Convergence only for a low number of elements.
- Grid crossings seriously effect quality of the solution.

Similar observations by Bardenhagen et al.[4] and Steffen et al.[5]

[4] S. G. Bardenhagen, E. M. Kober. *The generalized interpolation material point method. Comput Model Engrg Sci, 5 (2004), pp. 477-495.*

[5] M. Steffen, R. M. Kirby, M. Berzins. *Analysis and reduction of quadrature errors in the material point method (MPM). Int. J. Numer. Methods. Engrg, 76 (2008), pp. 922-948.*

High order MPM

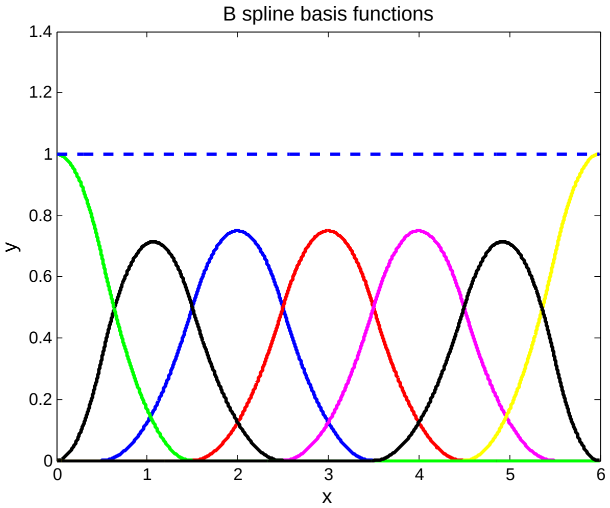
Quadratic B-spline basis functions:

- are positive functions
- have continuous derivatives
- lead to lower quadrature errors [6]
- possess partition of unity property

B-spline basis functions (seem) well suited for MPM!

[6] M. Steffen, R. M. Kirby, M. Berzins. *Analysis and reduction of quadrature errors in the material point method (MPM). Int. J. Numer. Methods. Engrg, 76 (2008), pp. 922-948.*

B-splines

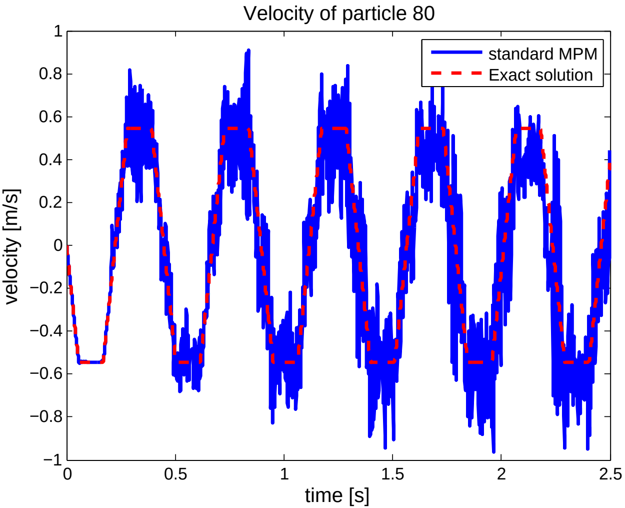


Outlook

B-spline MPM already implemented, next steps:

- Comparison with standard MPM (Benchmark problems)
- Apply an MLS approach to decrease quadrature error
- Decrease quadrature error with B-spline approximation
- Investigate 3^{rd} order B-spline basis functions
- Extension of B-spline MPM to 2D

Linear vs. B-spline MPM



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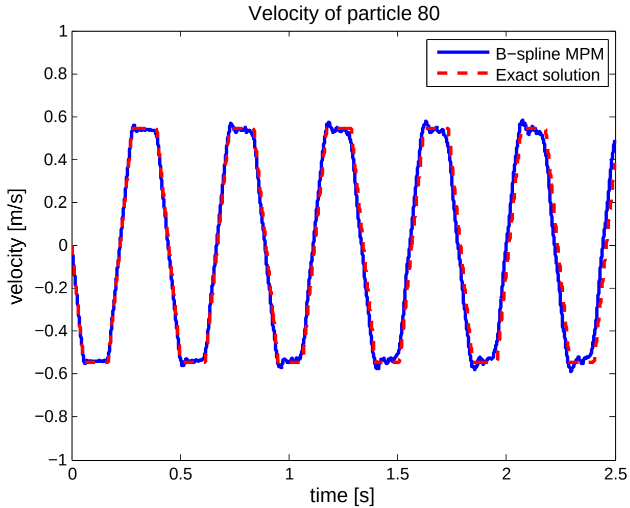
Linear vs. B-spline MPM



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