Limiting discontinuous Galerkin solutions using multiwavelets

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Literature review

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Outline

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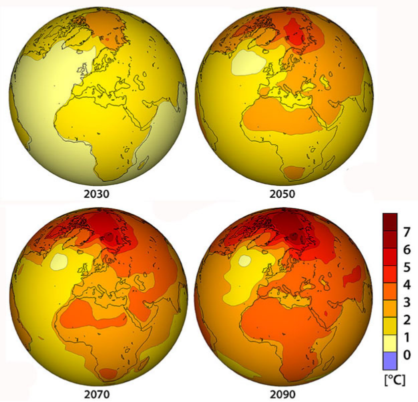
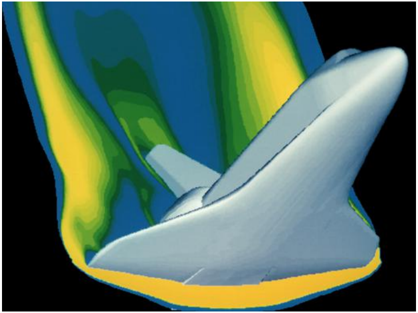
Motivation



Climate modelling: simulation of the mean temperature change

Motivation



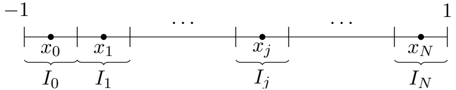
A computer simulation of high velocity air flow around the Space Shuttle during re-entry.

Discontinuous Galerkin: discretization in space

Linear advection equation on $[-1,1]$:

$$u_t + u_x = 0, \;\; x \in [-1,1], t \geq 0,$$
$$u(x,0) = u^0(x), \;\; x \in [-1,1],$$

$u = u(x,t)$, periodic boundary conditions.
Exact solution: $u(x,t) = u^0(x-t)$.



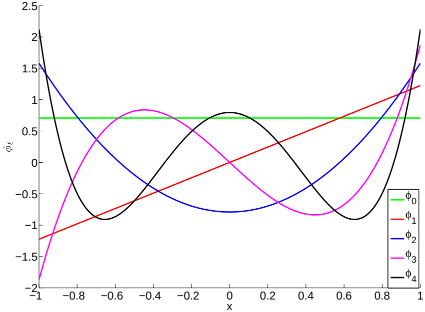
Discretize in space: $x_j = -1 + (j + \frac{1}{2})\Delta x, \quad j = 0, \dots, N.$

(B. Cockburn, Springer, 1998)

Discontinuous Galerkin: approximations

Approximate $u(x,t)$ by a piecewise polynomial of degree k (polynomial on each I_j), using

- Taylor expansion: linear combination of $1, x, x^2, \ldots, x^k$;
- Now: scaled Legendre polynomials $\phi_0, \phi_1, \phi_2, \ldots, \phi_k$.



Example of scaled Legendre polynomials $\phi_0, \ldots, \phi_4$

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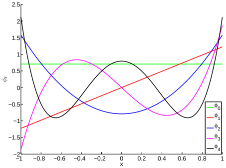
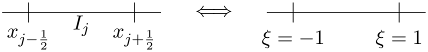
Discontinuous Galerkin: approximation space

$$u_h(x,t) = \sum_{\ell=0}^{k} u_j^{(\ell)}(t)\phi_\ell(\xi), \text{ on element } I_j,$$
$$\xi = \frac{2}{\Delta x}(x - x_j).$$





Global coordinates Local coordinates

$\int_{I_j}(u_t + u_x)vdx = 0,$
use $u_h(x,t)$ and $v_h(x) = \phi_m(\xi), m \in \{0, \dots, k\}$.

Discontinuous Galerkin: weak form

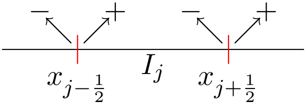
$$\int_{I_j} (u_{h,t} + u_{h,x}) v_h dx = 0.$$

- Integration by parts;
- Coordinate transformation to $\xi = \frac{2}{\Delta x}(x - x_j)$;
- Orthonormal property of scaled Legendre polynomials;

$$\frac{\Delta x}{2} \frac{du_j^{(m)}(t)}{dt} = \sum_{\ell=0}^{k} u_j^{(\ell)}(t) \int_{-1}^{1} \phi_\ell(\xi) \frac{d}{d\xi} \phi_m(\xi) d\xi +$$
$$- \hat{u}_h(x_{j+\frac{1}{2}}) v_h(x_{j+\frac{1}{2}}) + \hat{u}_h(x_{j-\frac{1}{2}}) v_h(x_{j-\frac{1}{2}}).$$

Discontinuous Galerkin: fluxes

Required: $\hat{u}_h(x_{j+\frac{1}{2}})v_h(x_{j+\frac{1}{2}})$ and $\hat{u}_h(x_{j-\frac{1}{2}})v_h(x_{j-\frac{1}{2}})$



Boundaries of element I_j

- u_h: exact solution $u(x,t) = u^0(x-t)$: initial condition advects to the right $\longrightarrow u_h(x^-_{j-\frac{1}{2}})$ and $u_h(x^-_{j+\frac{1}{2}})$;
- v_h: inside element $\longrightarrow v_h(x^+_{j-\frac{1}{2}})$ and $v_h(x^-_{j+\frac{1}{2}})$.

Discontinuous Galerkin: differential equation

$$\frac{\Delta x}{2}\frac{du_j^{(m)}}{dt} = \sum_{\ell=0}^{k} u_j^{(\ell)} \int_{-1}^{1} \phi_\ell(\xi)\frac{d}{d\xi}\phi_m(\xi)d\xi + \\ -\underbrace{\left(\sum_{\ell=0}^{k} u_j^{(\ell)}\phi_\ell(1)\right)\phi_m(1)}_{\hat{u}_h(x_{j+\frac{1}{2}})v_h(x_{j+\frac{1}{2}})} + \underbrace{\left(\sum_{\ell=0}^{k} u_{j-1}^{(\ell)}\phi_\ell(1)\right)\phi_m(-1)}_{\hat{u}_h(x_{j-\frac{1}{2}})v_h(x_{j-\frac{1}{2}})}.$$

Let $\quad \mathbf{u}_j = [u_j^{(0)}(t) \;\; u_j^{(1)}(t) \;\; \dots \;\; u_j^{(k)}(t)]^\top$, then

$$M\frac{d}{dt}\mathbf{u}_j = S_1\mathbf{u}_j + S_2\mathbf{u}_{j-1}, j = 0, \dots, N.$$

Discontinuous Galerkin: time stepping

- Time stepping: total variation diminishing RK, $\mathcal{O}(\Delta t)^3$;
- Approximation space with polynomials $\phi_0, \ldots, \phi_k$: $\mathcal{O}(\Delta x)^{k+1}$ if function is smooth.

	$k=1$		$k=2$	
$N+1$	$\lvert\lvert u-u_h\rvert\rvert_\infty$	order	$\lvert\lvert u-u_h\rvert\rvert_\infty$	order
10	0.1659	-	0.0244	-
20	0.0435	1.9329	0.0030	3.0146
40	0.0130	1.7420	3.9854e-04	2.9236
80	0.0035	1.8997	5.0421e-05	2.9826
160	8.9903e-04	1.9532	6.3179e-06	2.9965

Norms of errors and orders, $T = 0.5$, $u^0(x) = \sin(2\pi x)$, using $\phi_0, \ldots, \phi_k$

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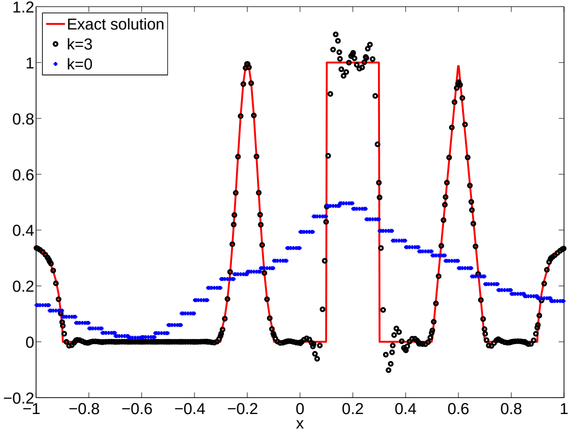
Limiters needed



Approximations, discontinuous initial condition, $\phi_0, \ldots, \phi_k$, $T = 0.5$

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Limiters: overview

Currently used limiters for DG are, for example:

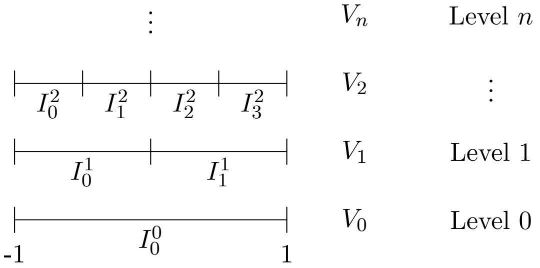
- Minmod limiters (Shu);
- Projection limiters (Cockburn and Shu);
- Moment limiters (Krivodonova);
- WENO limiters (Qiu and Shu);
- Multiwavelet limiters (Iacono, Hovhannisyan).

Drawbacks: reduce to low order around discontinuities, multidimensional case.

Research project focuses on new type of multiwavelet limiter.

Multiwavelets

Theory of multiwavelets uses several levels to approximate a function $f \in L^2(-1,1)$.



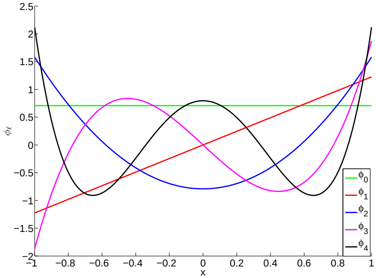
Subdividing intervals in their midpoint

Level n contains 2^n intervals.

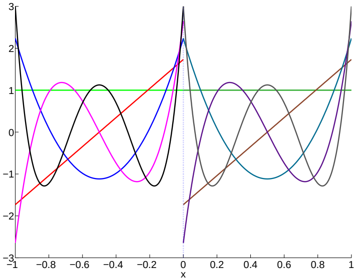
(Alpert, SIAM, 1993)

Multiwavelets: scaling functions

Scaling functions on level n : $\phi_{\ell j}^{n}, \ell = 0, \ldots, 4, j = 0, \ldots, 2^n - 1$,
ϕ_ℓ is scaled Legendre polynomial of degree ℓ.



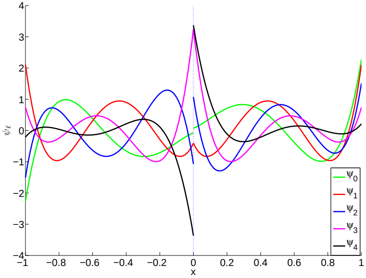
Scaling functions ϕ_ℓ, level 0 (V_0)



Scaling functions $\phi_{\ell j}^{1}$, level 1 (V_1)
Dilation and translation of ϕ_ℓ

Multiwavelets

$$\underbrace{V_0}_{\text{scaling functions}} \oplus \underbrace{W_0}_{\text{multiwavelets}} = V_1.$$



Multiwavelets ψ_ℓ, in W_0

$$V_n = V_{n-1} \oplus W_{n-1} = V_{n-2} \oplus W_{n-2} \oplus W_{n-1}$$
$$= \ldots = V_0 \oplus W_0 \oplus \ldots \oplus W_{n-1}.$$

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Multiwavelets: approximation of functions

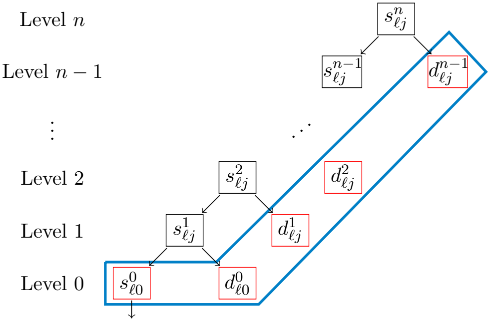
$$V_n = V_0 \oplus W_0 \oplus \ldots \oplus W_{n-1}.$$

Orthogonal projection of $f \in L^2(-1,1)$ on V_n, using a combination of scaling functions and multiwavelets:

$$P_n^k f(x) = \underbrace{\sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_\ell(\xi)}_{\text{projection on } V_0} + \sum_{m=0}^{n-1} \underbrace{\sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d_{\ell j}^m \psi_{\ell j}^m(\xi)}_{\text{projection on } W_m}.$$

Multiwavelets: decomposition

$$P_n^k f(x) = \sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_\ell(\xi) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d_{\ell j}^m \psi_{\ell j}^m(\xi)$$

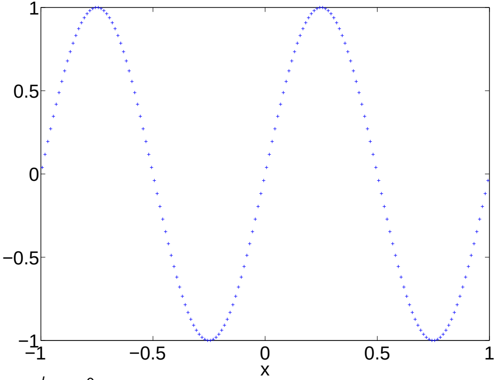


$P_0^k f(x) = \sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_\ell(\xi) \rightarrow$ standard DG approximation

Approximation on level n, decomposed into coefficients $s_{\ell 0}^0, d_{\ell 0}^0, \ldots, d_{\ell j}^{n-1}$

Multiwavelets: $f(x) = \sin(2\pi x), k = n = 3$

$$P_n^k f(x) = \sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_\ell(\xi) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d_{\ell j}^m \psi_{\ell j}^m(\xi)$$



$\sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_\ell(\xi)$

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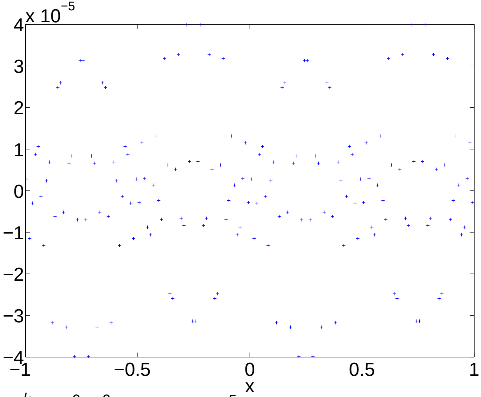
Multiwavelets: $f(x) = \sin(2\pi x), k = n = 3$

$$P_n^k f(x) = \sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_\ell(\xi) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d_{\ell j}^m \psi_{\ell j}^m(\xi)$$



$m = 0$: $\sum_{\ell=0}^{k} d_{\ell j}^0 \psi_{\ell j}^0(\xi), \mathcal{O}(10^{-5})$

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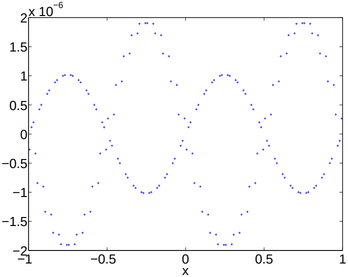
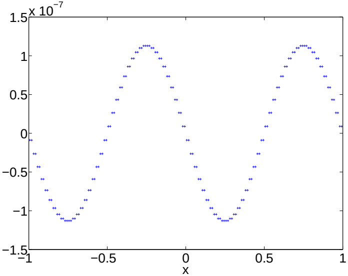
Multiwavelets: $f(x) = \sin(2\pi x), k = n = 3$

$$P_n^k f(x) = \sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_\ell(\xi) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d_{\ell j}^m \psi_{\ell j}^m(\xi)$$



$m = 1 : \sum_{j=0}^{1} \sum_{\ell=0}^{k} d_{\ell j}^1 \psi_{\ell j}^1(\xi)$

$\mathcal{O}(10^{-6})$



$m = 2 : \sum_{j=0}^{3} \sum_{\ell=0}^{k} d_{\ell j}^2 \psi_{\ell j}^2(\xi)$

$\mathcal{O}(10^{-7})$

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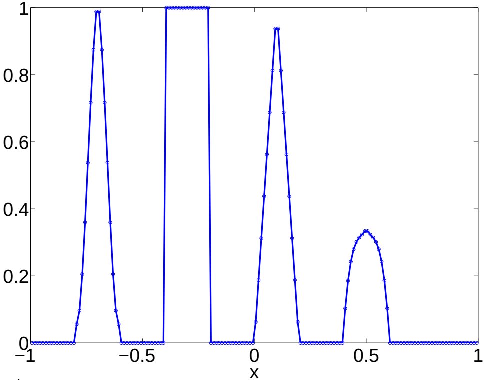
Discontinuous function, $k = n = 3$

$$P_n^k f(x) = \sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_\ell(\xi) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d_{\ell j}^m \psi_{\ell j}^m(\xi)$$



$\sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_\ell(\xi)$

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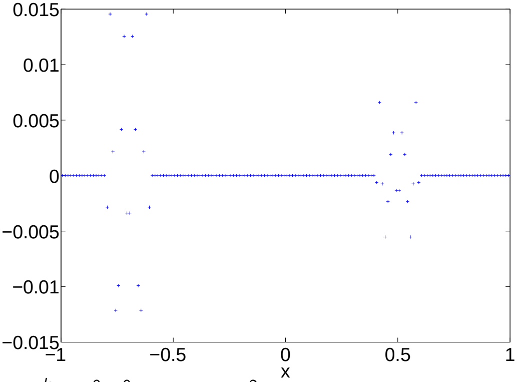
Discontinuous function, $k = n = 3$

$$P_n^k f(x) = \sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_\ell(\xi) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d_{\ell j}^m \psi_{\ell j}^m(\xi)$$



$m = 0:$ $\sum_{\ell=0}^{k} d_{\ell j}^0 \psi_{\ell j}^0(\xi), \mathcal{O}(10^{-2})$

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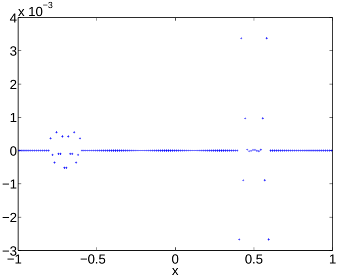
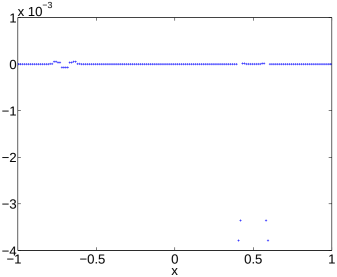
Discontinuous function, $k = n = 3$

$$P_n^k f(x) = \sum_{\ell=0}^{k} s_{\ell 0}^{0} \phi_\ell(\xi) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d_{\ell j}^{m} \psi_{\ell j}^{m}(\xi)$$



$$m = 1 : \sum_{j=0}^{1} \sum_{\ell=0}^{k} d_{\ell j}^{1} \psi_{\ell j}^{1}(\xi)$$

$$\mathcal{O}(10^{-3})$$



$$m = 2 : \sum_{j=0}^{3} \sum_{\ell=0}^{k} d_{\ell j}^{2} \psi_{\ell j}^{2}(\xi)$$

$$\mathcal{O}(10^{-3})$$

Summary and further research

Combine DG with multiwavelet limiter:

- Detect and manage discontinuities;
- Relation between degree of multiwavelet basis, k, and number of levels, n;
- High-order and multidimensional case.