Solving instabilities at small timesteps in CONTACT

Interim literature report

by

Niels van der Wekken

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Supervisers:	Dr. ir. E. A. H. Vollebregt, VORtech
	Prof. dr. ir. C. Vuik, TU Delft

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Preface

This master thesis will be done at V$\mathcal{O}$Rtech under supervision of E.A.H. Vollebregt. Vollebregt, once a student of Prof. J.J. Kalker, is an expert in the field of (rolling) contact mechanics and has done much work on the software package `CONTACT`. Recently it has been discovered that for certain small values in the discretized timestep non-physical 'wiggles' show up in the results. My master thesis will focus on finding out what causes these wiggles to arise and how we can prevent this.

V$\mathcal{O}$Rtech is a scientific software engineering company that produces, maintains, and optimizes scientific software. They have long-term contracts with big corporations like Shell, Rijkswaterstaat, Deltares, and TNO as well as smaller projects with SMEs. The majority of its 25 employees has a mathematical or physical background. In the first place V$\mathcal{O}$Rtech works by letting clients hire V$\mathcal{O}$Rtech's mathematicians to be an addition to their team, backing them up with specific expertise. Secondly V$\mathcal{O}$Rtech develops and maintains software packages for clients that need certain software but do not develop software them self. Lastly V$\mathcal{O}$Rtech provides clients with mathematical consultancy.

The software package `CONTACT` is maintained by Vollebregt, co-founder of V$\mathcal{O}$Rtech. `CONTACT` is mainly used in railway simulation packages to calculate the forces and other physical parameters that occur in the wheel-rail contact. My master thesis will revolve around the forces calculated by this software package.

Niels van der Wekken
Delft, June 2016

Chapter 1

Introduction

Contact mechanics describes the way two (elastic) objects interact with each other when they touch, make contact. As the two bodies are pressed against each other, at the point of contact the bodies want to take up the same spot in space, resulting in each of them applying a repelling force on the other. This force will deform the bodies so that a contact area appears where the resulting contact forces balance the forces pressing the bodies together. In a wheel-rail system the two bodies (the wheel and the rail) are not just pressed together, but the wheel will be rolling over the rail. This rolling adds a dynamical component to the system, in the form of tangential friction.

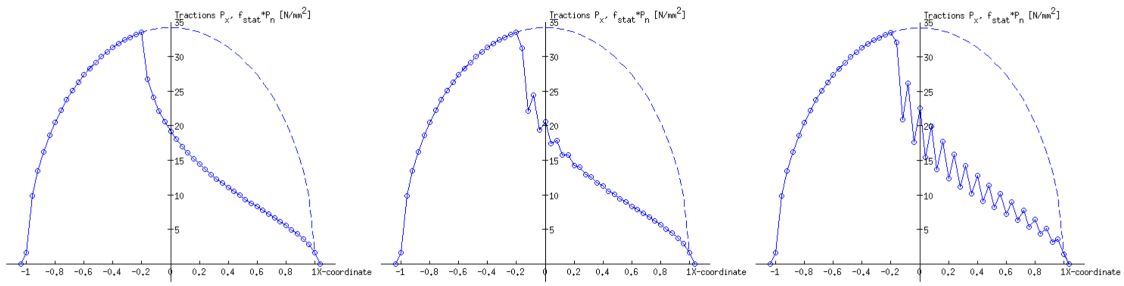
J.J. Kalker [17] first developed the software package `DUVOROL` in 1978 [10,11], this package assumed an elliptic contact area, as described in the Hertz theory [7], and was limited to steady-state rolling contact. In 1982 he finished its successor `CONTACT`. This package could determine the actual contact area, and could calculate for an instant frictional shift and transient rolling contact. Both programs were written in `FORTRAN IV`. The underlying theory and the framework of the algorithms are described in Kalker's most cited work [12]. The book describes the problem of rolling contact mechanics and gives both analytic solutions for certain geometries and the numerical scheme for more complex geometries. Altough the software version of 1990 is outdated, the core of the methodology has not changed.

In 1992-4 E.A.H. Vollebregt rewrote `CONTACT` in `FORTRAN 77` [24], modernizing it in a way that is easier adaptable. After the publishing of Kalker's book in 1990 there has been much development on `CONTACT`. Adding more functionality, like extensions on the friction model [23, 26] and an extension towards conformal contact areas [27]. But also implementing faster numerical solvers, utilizing Conjugate Gradient methods and Fast Fourier Transformations [20, 22, 29] to improve the performance.

This thesis will focus on the occurrence of artificial numerical wiggles that arise when the timestep becomes small. More specifically, the problems arise when the traversed distance per timestep $\delta t \cdot V$ is small compared to the gridsize δx. For brevity this traversed distance per timestep will be simply called the "timestep" δq although δq is actually a distance. Its size compared to the gridsize is controlled by a parameter c given by $\frac{\delta t \cdot V}{\delta x} = \frac{\delta q}{\delta x}$. In the numerical scheme that arises the most influential factor is a matrix B. Without giving further details yet, there are a few things that happen to B when $c \ll 1$. When $c = 0$ the matrix is singular and no solution exists. As $c \downarrow 0$ the imaginary parts of the eigenvalues of B become bigger, relative to the real parts. Also the condition number grows rapidly as c decreases, meaning that the solution will be very sensitive to small deviations in the input, so little errors blow up in the results.

In [21] it is shown that the ratio $c = \frac{\delta q}{\delta x}$ is an important factor concerning the accuracy of the model. Whereas in other applications it is typically found that a lower value of c results in better accuracy here it is found that for $c < 0.55$ the accuracy decreases again.

An example of the wiggles is given in Figure 1.1.



(a) Case: $c = 1$ (b) Case: $c = 0.1$ (c) Case: $c = 0.025$

Figure 1.1: Computations of tractions in 2D in `CONTACT` choosing different values of δq. This is the 2D Carter/Fromm problem, see [24, section 5.2].

In chapter 2 the necessary theoretical framework will be given. This includes the physical background that lies at the basis of contact mechanics in sections 2.1 and 2.2, techniques used to simplify the problem in sections 2.3 - 2.6, the discretization 2.7, and implementation 2.8. Also remarks on the use of piece-wise functions 2.9 and the use of itterative solvers 2.10 is given.
Chapter 3 describes the problem that needs to be solved and the different cases that can be considered. In section 3.1 a short description of possible research is given, together with a set of goals to reach in the thesis.
Finally in section 3.2 the research questions for my master thesis will be given.

Chapter 2

Theory

In the following sections more details will be given concerning the equations that need to be solved, and the way they are currently solved. The basic theory of elasticity can be found in Love [15]. This book includes theory on elasticity and the concepts of stress and strain. A good basis for the theory on contact mechanics can be found in Johnson [8].

2.1 Elasticity

In order to work with elastic contact problems we need to understand what elastic deformation is. Deformation of a physical body is the transformation of the positions of particles in the original state to a their positions in the new state. When a particle had position $\mathbf{x}$ and after deformation has position $\mathbf{x} + \mathbf{u}$.

Now introduce the strain ϵ. The strain consists of longitudial strains $\epsilon_{ii} = \frac{\partial u_i}{\partial x_i}$ corresponding to stretching of the body, and shear strains $\epsilon_{ij} = \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}$ for $i \neq j$ corresponding to bending of the body. The strains describe the relative displacement of the particles in the body.

Strain in the body can be caused both by outside stress, forces like gravity or pressure acting on the body, as well as by internal elastic stress, resulting from the material resisting change. The strains are linked to (internal) stress σ, according to the generalized Hooke's Law through the elastic tensor C by $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$. We will use Einstein summation convention throughout this documents (as we just did with the elastic tensor), there will be no summation over the ϵ_{ii} though in the previous paragraph.

When the elastic behaviour of a material is independent on the direction of the stress- and strain-components the material is said to be isotropic. For example in the case of metals. An example where this is no longer valid is wood, because of the uniformly directed fibrous structure the wood reacts different under stress perpendicular or parallel to the direction of the fibres. The elastic tensor C_{ijkl} for an isotropic material reduces to two material properties λ and μ. Here μ is the rigidity and $\lambda + \frac{2}{3}\mu$ the modulus of compression. Furthermore, when the material is homogeneous the λ and μ are constant through the material. For an isotropic material we can give the stresses as functions of the strains through: $\sigma_{ii} = \lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33}) + 2\mu\epsilon_{ii}$ and $\sigma_{ij} = \mu\epsilon_{ij}$, $i \neq j$. Again do not use summation over σ_{ii} and ϵ_{ii}.

2.2 Tractions and slip

The following scalar and vector quantities are important in the contact model. Before the bodies are brought into contact we refer to them by their coordinates $\mathbf{x}$ in a right-handed coordinate system where the x-axis points in the rolling direction and the z-axis points upwards into the upper body. When the bodies are brought into contact stresses σ, strains ϵ and displacements $\mathbf{u}$ arise. We are particularly interested in the surface quantities where we call the surface stresses of body 1 $\mathbf{p}^{(1)}(\mathbf{x})$ and of body 2 $\mathbf{p}^{(2)}(\mathbf{x})$. Because these surface stresses work against each other they have the same amplitude but opposite sign: $\mathbf{p}^{(1)}(\mathbf{x}) = -\mathbf{p}^{(2)}(\mathbf{x})$, so in the model we will only consider the surface stress in body 1 and call this $\mathbf{p}(\mathbf{x}) = \mathbf{p}^{(1)}(\mathbf{x})$. After the deformation the displacement of body 1 at a point $\mathbf{x}$ is given by $\mathbf{u}^{(1)}(\mathbf{x})$. Now let the displacement difference be given by the difference in the displacements of bodies 1 and 2 at a given position: $\mathbf{u}(\mathbf{x}) = \mathbf{u}^{(1)}(\mathbf{x}) - \mathbf{u}^{(2)}(\mathbf{x})$. Furthermore we split the stress vector into a scalar value p_n for the normal stress, and the 2-vector $\mathbf{p}_\tau$ for the tangential stresses, called tractions. The normal distance between the undeformed surfaces is given by h. Now the normal distance between the bodies in the deformed state is given by:

$$e := h + u_n \tag{2.1}$$

Finally we have a relative slip $\mathbf{s}$ (also a 2-vector as there is no slip in the normal direction) which describes how fast the upper and lower body slide over each other compared to the rolling velocity given by [12, equation 1.25]:

$$\mathbf{s} = \mathbf{w} + \frac{\dot{\mathbf{u}}}{V} \tag{2.2}$$

V is the rolling velocity magnitude, because we choose the velocity to always be in the x-direction we have that $\mathbf{v} = [V, 0, 0]^T$. Here $\mathbf{w} = [\xi - \phi y, \eta + \phi x]^T$ is the relative rigid slip given as a function of the longitudinal and lateral creepage ξ and η and the spin creepage ϕ. The relative slip and relative rigid slip are relative compared to the magnitude of velocity. In rolling the build up of tractions is mostly governed by the velocity of creeping relative to the overall rolling velocity. Therefore we define the relative slip velocity $\mathbf{s}_{relative} = \mathbf{s}_{absolute}/V$ as a dimensionless slip velocity and abbreviate $\mathbf{s} := \mathbf{s}_{relative}$. The quantity $\dot{\mathbf{u}}$ is the material derivative of the displacement given by: $\dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial t} - V\frac{\partial \mathbf{u}}{\partial x}$. The minus in front of the spatial derivative is because as the upper body moves in positive x-direction, this means that the surface particles actually move in the negative x-direction through the contact area.

Now we can define the contact conditions:
In the normal problem:

$$\text{in exterior area E}: \quad e > 0,\ p_n = 0 \tag{2.3}$$

$$\text{in contact area C}: \quad e = 0,\ p_n \geq 0 \tag{2.4}$$

In the tangential problem:

$$\begin{aligned} \text{in exterior area E}: &\quad \mathbf{s} \text{ free},\ \mathbf{p}_\tau = \mathbf{0} && (2.5)\\ \text{in adhesion area H}: &\quad ||\mathbf{s}|| = 0,\ ||\mathbf{p}_\tau|| \leq g && (2.6)\\ \text{in slip area S}: &\quad ||\mathbf{s}|| > 0,\ \mathbf{p}_\tau = -g\frac{\mathbf{s}}{||\mathbf{s}||} && (2.7) \end{aligned}$$

This means that in the exterior area E the bodies do not touch each other so there is no stress and the bodies move freely from each other so the slip can be anything. In the adhesion area H the bodies touch each other without slipping, the stress between them is bound by an upper bound g, in the slip area S the bodies slide over each other and the stress reaches its upper bound and is directed in the opposite direction from the slip. This upper bound is called the traction bound and is given by Coulomb's friction law [4] as

$$g = \mu p_n \tag{2.8}$$

where μ is the coefficient of friction.

The displacements at $\mathbf{x}$ can now be determinted by integrating the product of the stress at the contact area C with an influence function using:

$$u_i(\mathbf{x}) = \int_C A_{ij}(\mathbf{x}, \mathbf{y}) p_j(\mathbf{y}) dS \tag{2.9}$$

This equation describes how $\mathbf{u}$ is a function of the tractions $\mathbf{p}$ at the contact surface C. The indices i, j in the function A_{ij} run over $1, 2, 3$ and $A_{ij}(\mathbf{x}, \mathbf{y})$ tells us how the traction $p_j(\mathbf{y})$ influences the the displacement $u_i(\mathbf{x})$. Note that we use Einstein summation convention over the index j.

2.3 Boundary Element Methods

The Boundary Element Method (BEM) is a numerical approximation method for solving partial differential equations (PDEs). The BEM is derived by rewriting the PDE into an integral equation defined on a boundary. Discretizing this integral equation gives the BEM. An advantage of BEMs over finite element methods, finite volume methods, and finite difference methods is that the BEM only solves the equations on the boundary of the domain of interest. Especially when Greens functions are known and the influence can be calculated analytically it is interesting to solve the system on the boundary using a BEM. Only calculating for the boundary means that we have one dimension less to work with, so with the same grid coarseness as in other methods, less grid points need to be considered, beware though that the system of equations coming from a BEM is dense while a FEM generally gives a sparse system of equations. Another advantage is gained when the body of interest is unbounded but has a bounded boundary, thus resulting in a finite domain for the BEM.

The major drawback when using a BEM is of course the fact that you do not calculate anything that happens below the boundary. When you are interested in internal parameters that can not be calculated in a straightforward way from properties on the boundary (when there are no Greens functions or comparable relations) you introduce extra errors, usually in the form of extrapolation errors.

2.4 Halfspace approximation

Although the full stress-strain equations are very complex it is possible to make some assumptions to simplify the model. In our model we make use of the half-space approach. This means that we assume that the two contacting bodies are infinite half spaces. This approach can be made if the bodies look like the halfspaces in a zone where the elastic field is significant and only begin to differ significantly from the half spaces where the elastic field is very small. This means that the contact area must be small compared to the typical dimension of the bodies such that the radius of curvature is large near the contact area. See Figure 2.1. Results in Kalker [12, Figure 5.20] *"support the statement that the half-space approximation is justified, when the diameter of contact is less than 1/3 of the diameter of the contacting bodies"*.

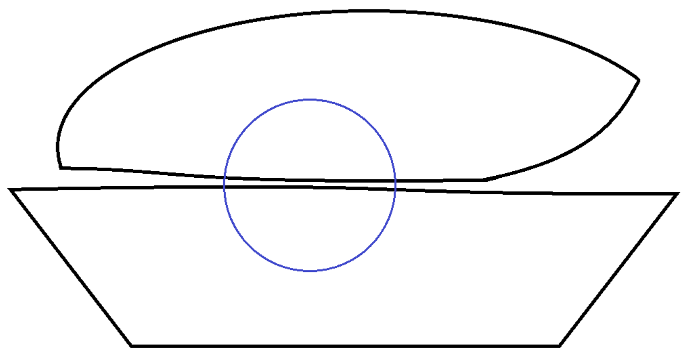


Figure 2.1: When the upper and lower body make contact the stresses and strains will only be 'felt' inside the blue circle because the influence of stresses and strains decreases as the distance increases away from the contact zone. So the geometry of the bodies outside the blue circle is unimportant and we may assume the bodies are two half spaces.

The strenght of the halfspace approximation lies in the fact that the formula's for the influence functions A_{ij} used in equation (2.9) are derived by Boussinesq [2] and Cerruti [3] for the halfspace. So explicit Greens functions can be used in the BEM.

2.5 Quasi-identical behaviour

Another simplification occurs when the two bodies of contact are quasi-identical. This is the case when both the geometry and the elastic behaviour of the bodies are similar. The first is automatically the case when we already have applied the half-space approach, both bodies are halfspaces, and thus identical. The second condition is fulfilled when the Young's Modulus E and Poisson's Ratio ν are equal. Stresses in one direction influence deformations in all three directions, when the two contacting bodies are not quasi-identical the normal and tangential problem influence each other. When the bodies are quasi-identical however the tangential displacements are the same for both bodies so there is no relative tangential displacement and we can separate the normal and tangential problem.

Applying a BEM to solve the contact between two quasi-identical half spaces greatly simplifies the model. Because now the normal and tangential problems are separated we fist solve the normal problem on the contact surface and use the solution for the normal problem when solving the tangential problem on the contact surface.

2.6 Problem dependent simplifications

Based on which assumptions we make we can adjust the equations to make them easier to solve. We will make distinctions between solving the system in 2D and 3D and between transient and steady-state rolling.

2.6.1 2D vs 3D

When we make the half-space approximation and the objects of contact are quasi-identical the normal and tangential problems are already separated. So in both the 2D and 3D case we will first solve the normal problem. In the 3D case the tangential contact area is 2D and equation (2.7) is a quadratic equation. In the 2D case however the tangential contact area becomes a 1D line and equation (2.7) is linear, thus easy to solve. This means that the 2D case is much easier to solve. As the wiggles occur both in the 2D and 3D problem a solution is first sought for the 2D case since this case is easier to study and a solution in 2D might be applicable in the 3D case as well.

2.6.2 Time-dependency

Equation (2.2) contains a time derivative. When we have a transient system we need an initial state. Once this initial state is known the full solution has to be determinded by evolving the solution in time.

A steady-state solution can be found in two ways. The first is simply applying the algorithm to solve the transient rolling case and stop once the solution of the current time instance is the same as the solution of the previous time instance. This is also what was originally implemented in Kalker's `DUVOROL` software.
A more sophisticated approach is going back to the equation for the stress (2.2) and set the time-derivative to zero. Now $\mathbf{p}_\tau$ and thereby $\mathbf{u}$ can be calculated directly. This method will be called the direct approach.

2.7 Discretization

Now that we have simplified the physical model into a mathematical description we can discretize the problem. As stated in [12,21] in `CONTACT` the potential contact area is discretized using identical rectangles I with size $\Delta x \times \Delta y$. The centre of rectangle I is denoted by (x_I, y_I), note that there is no z-component because the contact area lies in the x-y plane so $z = 0$ is a constant on the whole contact area. Equation (2.3) through (2.7) are discretized by using piece-wise constant functions

ϕ_I that are 1 on rectangle I and 0 everywhere else. The true solution $\mathbf{p}$ is then approximated by the piece-wise constant function $\sum_i \phi_i \cdot \mathbf{p_i}$.

The slip will be discretized by rewriting the slip equation. When looking at steady-state the time derivative in the material derivative in equation (2.2) drops out and we are left with:

$$\mathbf{s} = \mathbf{w} - \frac{\partial \mathbf{u}}{\partial x} \tag{2.10}$$

Before we can discretize this equation we need to approximate the spatial derivative. The slip equation is an advection equation, these equations are generally solved using an upwind-scheme. What happened upstream is known, so using data from upstream will generally give good results. Because the velocity is in the negative direction this upwind-scheme gives us a forward differentiation for the spatial derivative. A first order scheme is used because of its robustness. We may now approximate with a small enough Δx the derivative to x by using: $\frac{\partial \mathbf{u}}{\partial x} \approx \frac{\mathbf{u}(\mathbf{x}+[\Delta x,0]^T)-\mathbf{u}(\mathbf{x})}{\Delta x}$. This will give us the expression:

$$\mathbf{s} = \mathbf{w} - \frac{\mathbf{u}(\mathbf{x} + [\Delta x, 0]^T) - \mathbf{u}(\mathbf{x})}{\Delta x} \tag{2.11}$$

If we now use $\Delta x = V \cdot dt = dq$ and write $\mathbf{x} + [dq, 0]^T = \mathbf{x}'$ then a particle that has position $\mathbf{x}$ at time t had position $\mathbf{x}'$ at time $t' = t - dt$. This turns the slip equation into:

$$\mathbf{s} = \mathbf{w} + \frac{\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}')}{dq} \tag{2.12}$$

Now only looking at the values of $\mathbf{s}$ and $\mathbf{w}$ in the centres of the rectangles I we get the discretized form:

$$\mathbf{s}_I = \mathbf{w}_I + \frac{\mathbf{u}(\mathbf{x}_I) - \mathbf{u}(\mathbf{x}'_I)}{dq} \tag{2.13}$$

The solution for the displacement do not need to be discretized, $\mathbf{u}(\mathbf{x})$ can be determined using equation (2.9) for any $\mathbf{x}$. Because $\mathbf{p}$ has been replaced by a piece-wise constant function the integral in equation (2.9), it can be solved analytically using the results by Boussinesq [2] and Cerruti [3]. The influence functions are discretized by calculating the displacements felt in element I as a result of tractions in element J. This means that influence coefficient A_{IiJj} indicates how the displacement in rectangle I in the i direction is influenced by tractions in rectangle J in the j direction. So we can write:

$$A_{IiJj} = \iint_S A_{ij}(\mathbf{z}) dS \tag{2.14}$$

Where S is the surface of a rectangle that has centre $\mathbf{x_J} - \mathbf{x_I}$ with width Δx and height Δy. The integral can thus be written as:

$$A_{IiJj} = \int_{x_J - x_I - \frac{\Delta x}{2}}^{x_J - x_I + \frac{\Delta x}{2}} \int_{y_J - y_I - \frac{\Delta y}{2}}^{y_J - y_I + \frac{\Delta y}{2}} A_{ij}(\mathbf{z}) dz_2 dz_1 \tag{2.15}$$

Explicit expressions for the influence functions A_{ij} and solutions to the integrals for A_{IiJj} are given in [12, section 4.3.2].

When the tractions $\mathbf{p}_J$ and influence functions A_{IiJj} are know we can now determine the values of $\mathbf{u}(\mathbf{x}_I)$ by using:

$$u_i(\mathbf{x}_I) = \sum_{Jj} A_{IiJj} p_{Jj} \tag{2.16}$$

Using this in equation (2.13) we find for the discretized slip equation in case of steady-state:

$$s_{Ii} = w_{Ii} + \frac{(A_{IiJj} - A'_{IiJj})p_{Jj}}{dq} \tag{2.17}$$

where A'_{IiJj} is calculated similar to A_{IiJj} but then with the centres of the rectangles I shifted a distance dq to the right.

2.8 Implementation of CONTACT

In CONTACT the equations arising from the contact mechanical theory are modified into a minimization problem. This is done through Kalker's variational theory [9], using the principals of minimizing the virtual work and maximizing the virtual complementary energy [12, §4.1-4.2]. This minimization problem turns out to be a strictly convex quadratic problem, opening it up to a rich theory around Quadratic Programming on a convex objective function constrained to a convex feasible region [1, 5, 13, 16, 19].

Kalker used an algorithm that solves the convex minimization problem to solve the contact problem and calculate the tractions. This algorithm, KOMBI, computes both the normal and tangential tractions. In the case of quasi identical elastic bodies the KOMBI algorithm can be split into first solving the normal stresses p_{In} using NORM and then using the normal solution to solve the tangential tractions $p_{I\tau}$ using TANG [12, §4.3].
These algorithms are active-set algorithms. Solving the equations for the tractions, either in adhesion or slip, and restoring discretized elements of the contact area to the slip or the adhesion region when the constraints (2.6) or (2.7) are exceeded.

Because of the convex quadratic programming approach, existence, uniqueness, and finite determinability of the active set algorithm are given.

2.9 Piecewise linear approximation

The use of a piecewise constant approximation for the tractions $\mathbf{p}$ might be a cause of the appearance of wiggles. To prevent or reduce these wiggles we will look at using piecewise linear approximations. One direct advantage these solutions will give is the fact that while the derivative at any location of a piecewise constant function is either zero or does not exist (it could be seen as a sum of Dirac delta functions), the derivative of a piecewise linear function is a piecewise constant function. In 1D the shape of a piecewise linear function is trivial, however, in 2D the function can increase in the x-direction, y-direction or in both the x- and y-direction at the same time. The last case is called bi-linear.

Multiple groups have published on Boussinesq-Cerruti solutions for (piece-wise) linear functions instead of using the (piece-wise) constant function as applied in `CONTACT`.
Svec and Gladwell [18] give solutions of normal deformation due to polynomial normal pressure distributions on a triangular surface area. Li and Berger [14] extend this by giving the full solutions for constant and (bi)linear pressure loads over a triangular surface area.
The advantage of dividing the contact domain in triangles is its high adaptation level to curved edges of the contact domain compared to rectangles. However, using rectangles simplifies the computational model. Also, when the edges of the contact area are unknown in advance a rectangular grid has the advantage over a triangular grid. `CONTACT` also uses rectangles to describe the contact domain, so for this thesis we will focus on rectangular domains. Dydo and Busby [6] give solutions to constant and (bi)linear pressure loads over a rectangular surface area.

2.10 Iterative solvers

In the two dimensional case constraints (2.6) and (2.7) are linear, in the three dimensional case however these are not linear anymore. This means that the system of equations to solve the discretized slip is not linear either. In the early versions of `CONTACT` this system was solved using a Newton-Raphson method [28].
In 1993 an improvement was made by implementing a variation on the Gauss-Seidel method, later enhanced and stabylised by application of Successive Over Relaxation [20, 25].

Although these iterative solvers are used in the software, they are outside the area of interest of this thesis.

Chapter 3

Problem statement

Problems arise when the factor $c = \frac{dq}{dx}$ becomes small. Physically this means that the traversed distance per timestep is small compared to the gridsize. Mathematically this means that the discretized influence functions $A(\mathbf{x}) - A'(\mathbf{x} + \mathbf{V} \cdot dt)$ have increasingly smaller eigenvalues. In [20] Vollebregt remarks that the coefficients in steady-state depend on the time-step, and below a certain point there will be either slower convergence or no convergence at all, *"the restriction on the time step is sometimes quite severe, so that physically attractive values cannot be used"*.

A simple solution to this is never choosing a too small factor c. This is a nice solution when only running the program `CONTACT` by itself, however `CONTACT` has been integrated in larger train simulation packages, in such cases the input parameters by `CONTACT` are no longer handpicked but fed to the software by the overall package. Ideally it would be best if `CONTACT` can be adjusted so it converges properly for any value of c. Finding out why the wiggles occur can also help solve this problem, because in that case `CONTACT` can check it's input parameters and determine beforehand whether the parameters will result in a converging solution and give feedback (and adjust the critical parameters) when this is not the case.

3.1 Plan of action

A possible cause for the appearance of wiggles is that in the numerical scheme a piecewise constant approximation is used. When taking the difference between two of such approximations, for example when approximating a derivative or determining the influence coefficients $A - A'$, the middle part completely drops out and only the edges are left. When instead a piecewise linear approximation is used there will be a net result over the full interval, see Figure 3.1. Moving to linear approximations not only gives a 'smoother' solution but might also increase accuracy or rate of convergence.

Although this might be a possible solution first it is important to determine what exactely causes the wiggles to appear. From which factor c do the wiggles appear and why?

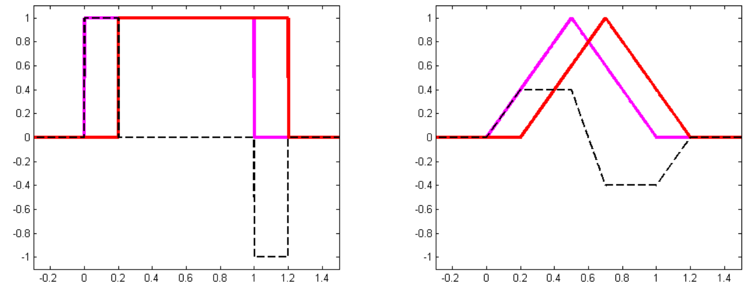


Figure 3.1: The magenta graph is the constant (left) or linear (right) basis function on $(0, 1)$. The red graph is the same basis function shifted slightly to the right, the black graph is the difference between them.

3.1.1 Slip-free 2D steady-state rolling

When solving the tangential problem we already have the solution of the normal problem. Instead of using a realistic normal solution, like one obtained by the Hertzian theory or the `NORM` algorthm, we can assume an infinite traction limit g. This means that constraint (2.6) is always met and as a consequence we have adhesion over the full contact area. This is a usefull assumption because in the slip region the traction reaches the traction bound and especially in two dimentions this solution becomes rather trivial. In the adhesion area however problems with wiggles can arise, so it is the adhesion area we are interested in. The advantage of assuming an infinite traction bound is that in adhesion we have no slip and thus solving equation (2.2) is much easier to solve by simply setting $\mathbf{s} = \mathbf{0}$. The first testcase will be:

- solving the wiggle-problem on 2D steady-state rolling while assuming the full contact area will be adhesion.

3.1.2 2D steady-state rolling

When a solution has been found in the case where we simplified the slip equation by setting the slip to zero the next step will be trying to implement this solution to the case where we do have a traction bound. So the second testcase will be:

- solving the wiggle-problem on 2D steady-state rolling in the precense of a slip-region.

3.1.3 2D transient rolling

Up until now focus has always been on steady-state rolling while using the direct approach. This was because the matrix-operator used in this situation is given by $A_{IiJj} - A'_{IiJj}$. We can calculate this matrix and use this to try and see what might cause the wiggles. The matrix-operator used in the transient situation seems to be just the matrix A_{IiJj}. But also in this situation wiggles are detected. These wiggles do not appear in the initial states of the solution but progress over time.

This has to do with the fact that the solution at a later time is the result of repeated application of these matrix-operatos. What makes this problem harder is that the problem is not caused by a single matrix operation but by an accumulation of operations. It would be convenient if a solution to the steady-state problem also solves the transient problem, this however has to be checked and possibly an extra solution has to be found. Just like for steady-state, also in the transient case we want a solution to hold when the traction bound is added to the equation. This gives the third test case:

- solving the wiggle-problem on 2D transient rolling while assuming the full contact area will be adhesion.
- solving the wiggle-problem on 2D transient rolling in the precense of a slip-region.

3.1.4 3D rolling

When everything has been solved in two dimensions it is time to move on to real life, the three dimensional world. A solution that works in the 2D case probably also works in the 3D case. This is because even when we consider 2D the influence matrices A_{IiJj} used are already calculated over squares, in order to stay in 2D we just used a grid with only one point in the y-direction. Adding more gridpoints in the y-direction will not significanly influence the behaviour of the influence matrices. Nevertheless, again we have to test if the solutions really fix the problem in 3D, giving us the fourth test case:

- solving the wiggle-problem in the 3 dimensional situation.

3.1.5 Implementing the solution in `CONTACT`

Finally, when a solution has been found for both steady-state and transient rolling contact in three dimensions, the last step is implementing this solution into the `CONTACT` software. Which brings us to the fifth test case:

- implementing the solution so that it works in the `CONTACT` software package in a algorithmic efficienct way.

3.2 Research Questions

As a final closure I will here state the research questions for my master thesis.

1. What causes the wiggles that arise when the factor $c = \frac{dq}{dx}$ becomes small?
2. Does replacing the piecewise constant basis functions by piecewise (bi)linear basis functions solve this problem?
    - If this is not the case, how can we solve it?
3. How does replacing the piecewise constant basis functions by piecewise (bi)linear basis functions influence the rate of convergence of the algorithm?

Chapter 4

Conclusion

The theory treated deals with the physical aspects concerned with rolling contact mechanics. After stating the physical laws a mathematical interpretation is made, catching the laws of nature in formula's and adding assumptions and simplifications. This mathematical model is then discretized in space (and time) in order to be able to solve it using a numerical scheme.

A plan of action has been made, together with a set of increasingly challenging testcases to test a solution against.

The final goal of this thesis is to find out what exacely causes the wiggles to arise when the timestep becomes small. If piecewise-linear approximations can solve this, and if a solution to the wiggles also influences the order of the rate of convergence of the `CONTACT` software package.

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