Discontinuous Galerkin
applied to a generic two-phase flow in a porous medium

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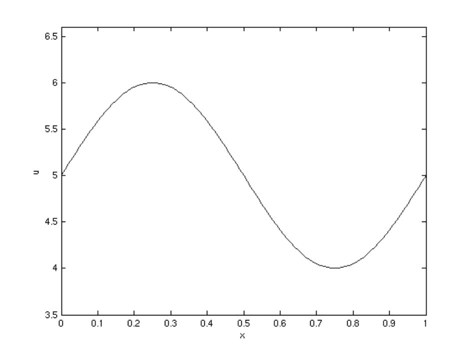
Delft University of Technology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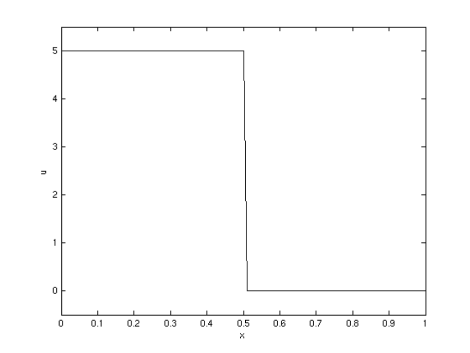
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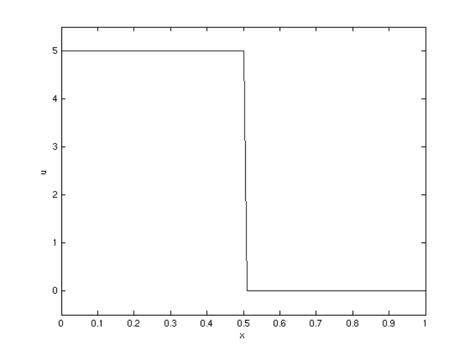
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x

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1. Discontinuous Galerkin

Discretization

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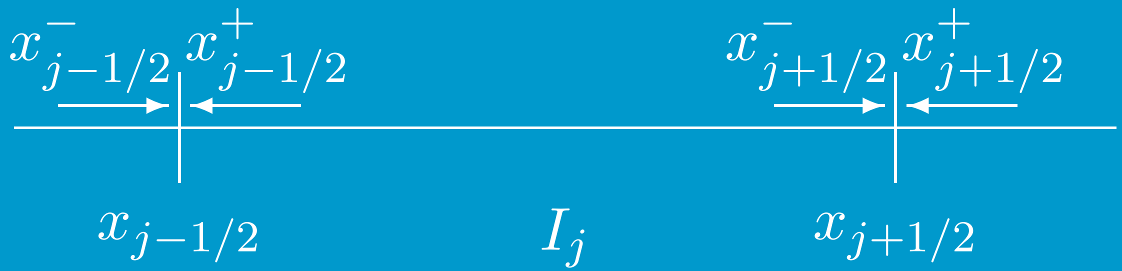
1. Discontinuous Galerkin

Discretization



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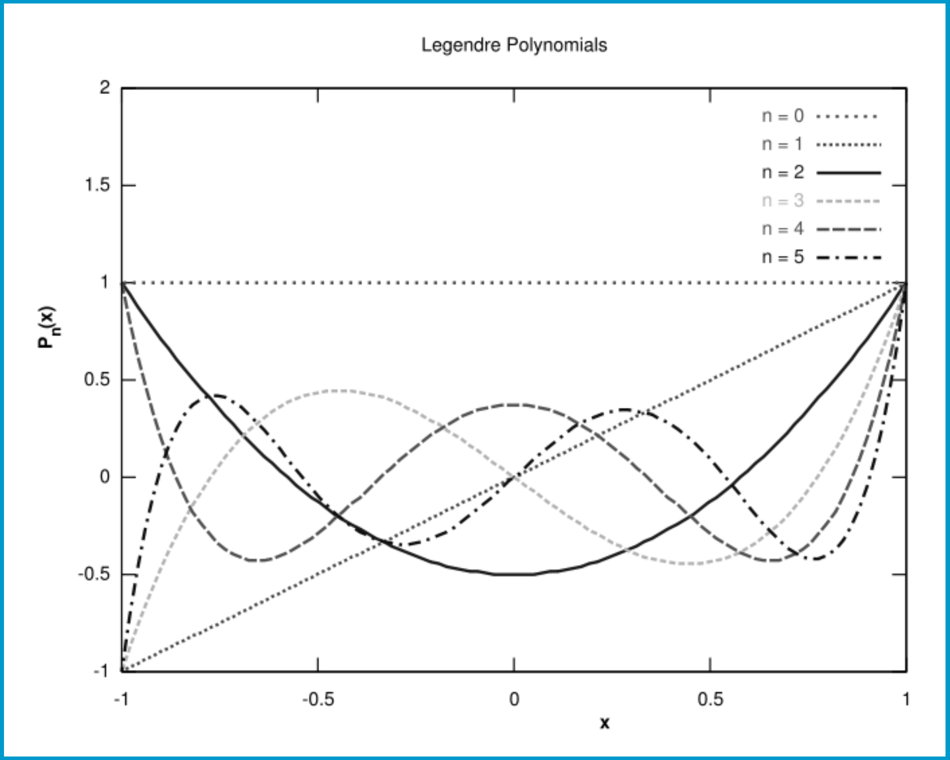
1. Discontinuous Galerkin

Discretization



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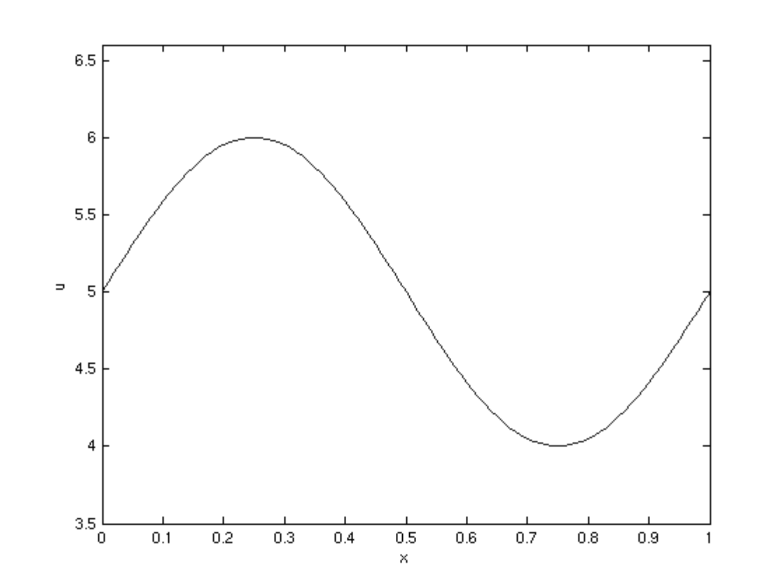
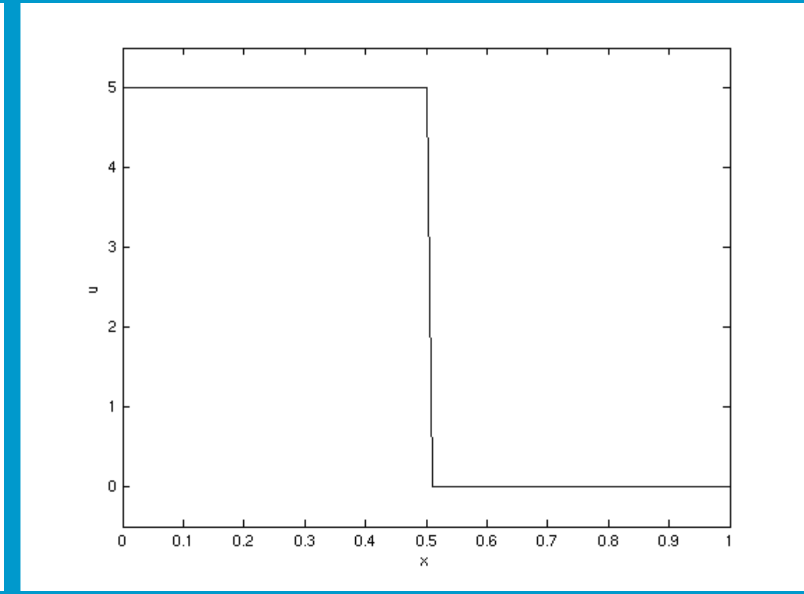
1. Discontinuous Galerkin

Application





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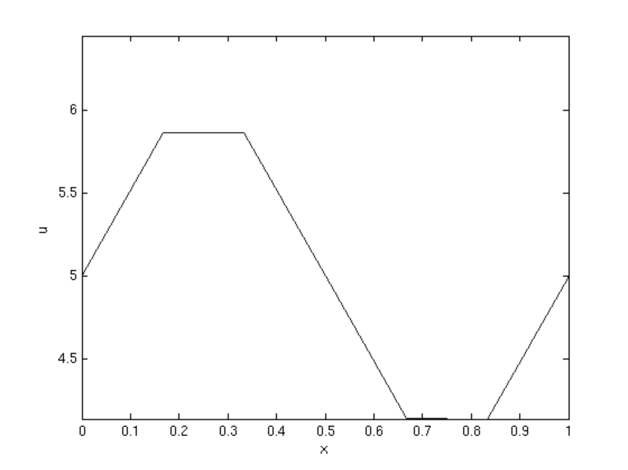
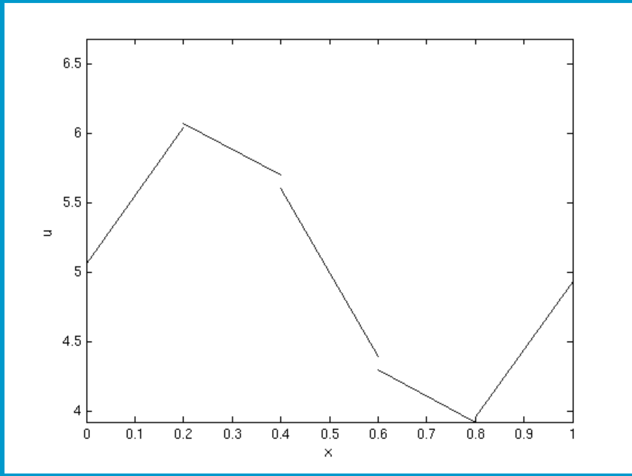
1. Discontinuous Galerkin

Comparison





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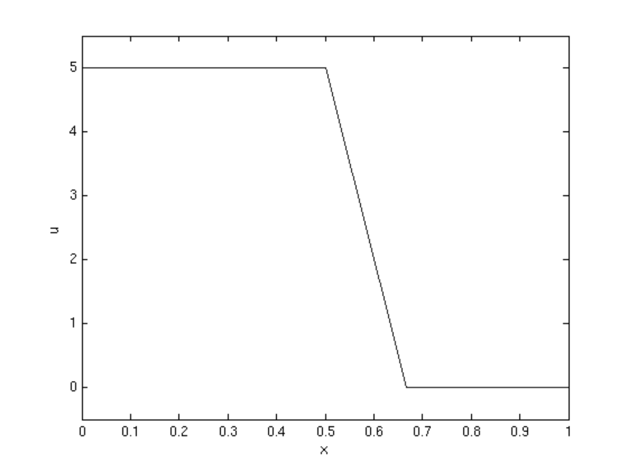
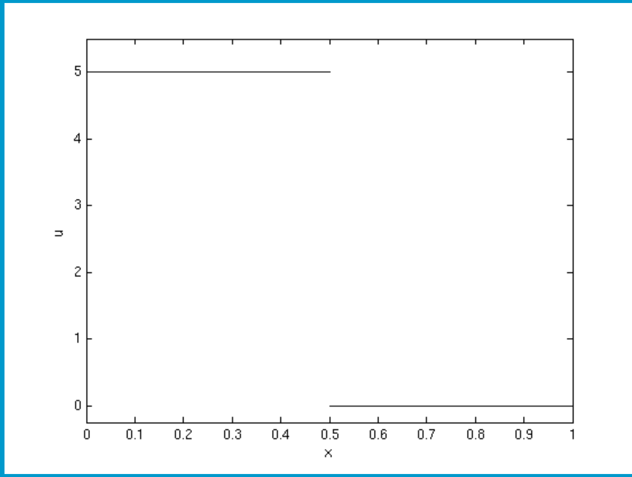
1. Discontinuous Galerkin

Comparison





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1. Discontinuous Galerkin

Comparison

- Difference in spaces for the test- and basis functions
- Polynomial per element or defined on entire region
- Boundary condition versus flux functions

Computation cost

- Size mass matrix
- Solving per element
- Time integration

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1. Discontinuous Galerkin

Shock detector

$$
\begin{aligned}
\mathbf{I}_j &= \int_{\partial\Omega_j^-} \left(Q_j - Q_{nbj}\right)\, d\Gamma, \\
&= \int_{\partial\Omega_j^-} \left(Q_j - q\right)\, d\Gamma + \int_{\partial\Omega_{nbj}^+} \left(q - Q_{nbj}\right)\, d\Gamma, \\
\mathcal{I}_j &= \frac{\left|\int_{\partial\Omega_j^-} \left(Q_j - Q_{nbj}\right)\, d\Gamma\right|}{h^{(k+1)/2}|\partial\Omega_j^-|\,\|Q_j\|}, \\
&\Rightarrow \begin{cases} \mathcal{I}_j > 1 \quad \Rightarrow \quad q \text{ is discontinuous,} \\ \mathcal{I}_j < 1 \quad \Rightarrow \quad q \text{ is smooth.} \end{cases}
\end{aligned}
$$

1. Discontinuous Galerkin

Limiter

- Limit from highest order coefficient,
- Stop if order is zero or limited value is identical,
- Use forward and backward difference.

1. Discontinuous Galerkin

Limiter

- Limit from highest order coefficient,
- Stop if order is zero or limited value is identical,
- Use forward and backward difference.

$$\tilde{u}_j^l = \text{minmod}\left(u_j^l, u_{j+1}^{l-1} - u_j^{l-1}, u_j^{l-1} - u_{j-1}^{l-1}\right),$$

$$\text{minmod}(a,b,c) := \begin{cases} \text{sgn}(a)\min(|a|,|b|,|c|), & \text{equal signs,} \\ 0, & \text{otherwise.} \end{cases}$$

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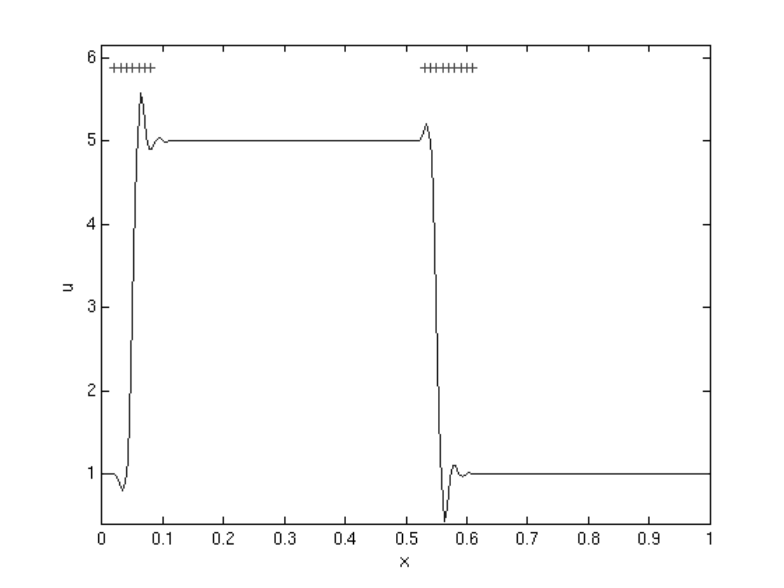
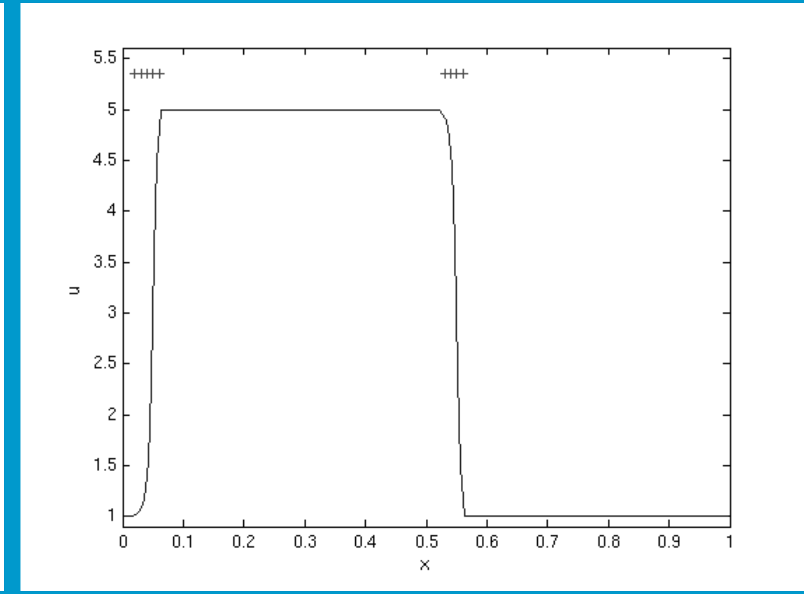
1. Discontinuous Galerkin

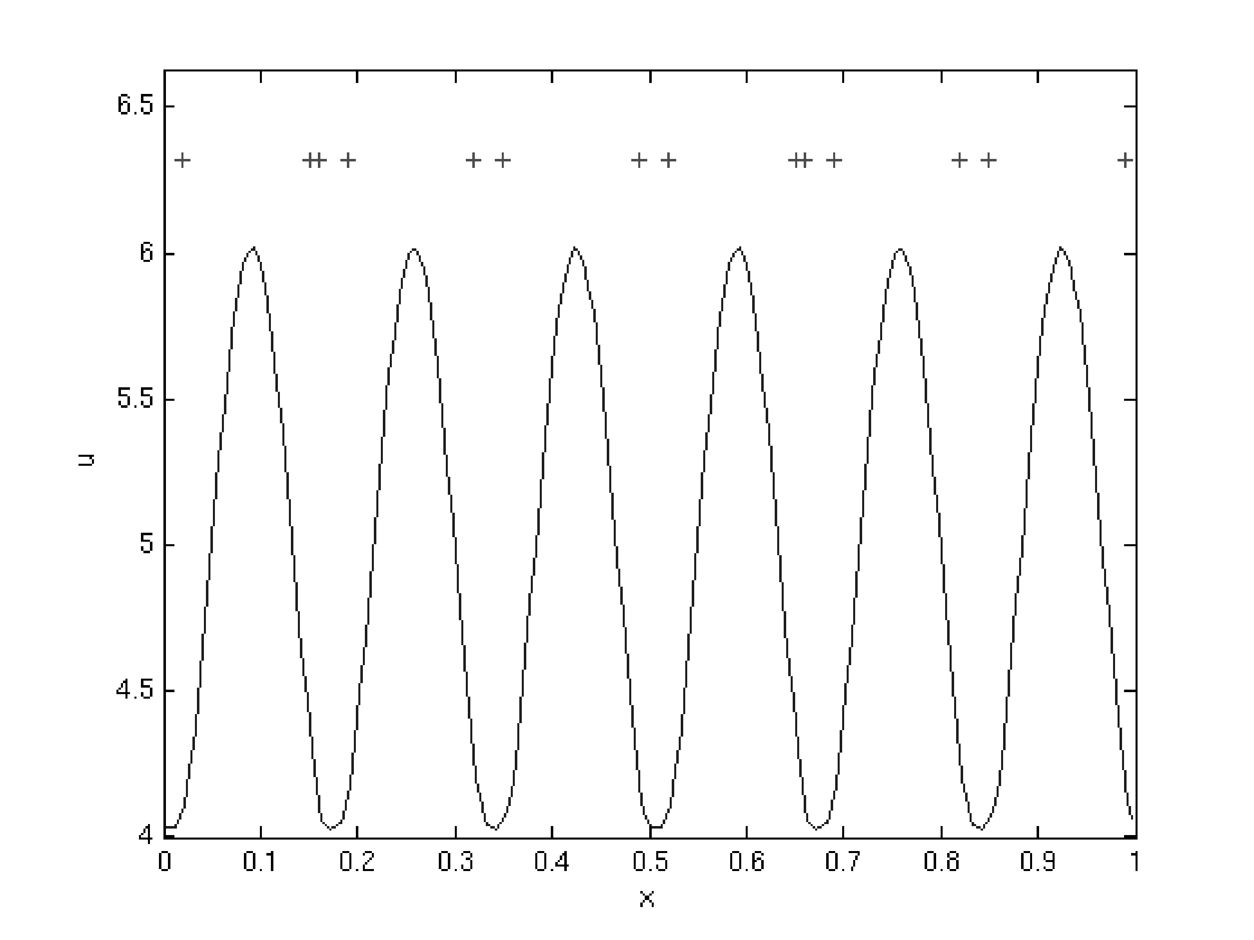
Results





1. Discontinuous Galerkin

Results



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2. Two-phase flow model

Construction

Model:

$$
\begin{aligned}
\varphi \frac{\partial S_1}{\partial t} + \nabla \cdot q_1(S_1) &= 0, \\
\varphi \frac{\partial S_2}{\partial t} + \nabla \cdot q_2(S_2) &= 0, \\
S_1 + S_2 &= 1.
\end{aligned}
$$

Boundary and initial condition:

$$
S_1(\mathbf{x}, t) = f(\mathbf{x}), \quad \forall x \in \Gamma_1, \ \forall t \in (0, T],
$$

$$
S_1(\mathbf{x}, 0) = S_0(\mathbf{x}), \quad \forall x \in \Omega.
$$

2. Two-phase flow model

Construction

Model:

$$
\begin{aligned}
\varphi \frac{\partial S_1}{\partial t} + \nabla \cdot q_1(S_1) &= 0, \\
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S_1 + S_2 &= 1.
\end{aligned}
$$

Darcy's Law:

$$
\begin{aligned}
q_1 &= -\lambda(S_1)\nabla(p_1 + \rho_1 g z), \\
q_2 &= -\lambda(S_2)\nabla(p_2 + \rho_2 g z).
\end{aligned}
$$

2. Two-phase flow model

Construction

Model:

$$\begin{aligned} \varphi \frac{\partial S_1}{\partial t} + \nabla \cdot q_1(S_1) &= 0, \\ \varphi \frac{\partial S_2}{\partial t} + \nabla \cdot q_2(S_2) &= 0, \\ S_1 + S_2 &= 1. \end{aligned}$$

Darcy's Law:

$$\begin{aligned} q_1 &= -\lambda(S_1)\nabla(p_1), \\ q_2 &= -\lambda(S_2)\nabla(p_2). \end{aligned}$$

2. Two-phase flow model

Construction

$$p_{cap} = p_2 - p_1.$$

At fixed depth, justifiable $p_{cap} = 0$. So

$$p := p_2 = p_1.$$

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2. Two-phase flow model

Solving partial differential equations

Add

$$\varphi \frac{\partial S_1}{\partial t} + \nabla \cdot q_1(S_1) = 0,$$

to

$$\varphi \frac{\partial S_2}{\partial t} + \nabla \cdot q_2(S_2) = 0,$$

and use

$$S_1 + S_2 = 1,$$

to obtain

$$\nabla \cdot (q_1(S_1) + q_2(S_2)) = 0.$$

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2. Two-phase flow model

Solving partial differential equations

Substitute

$$q_1 = -\lambda(S_1)\nabla(p_1),$$
$$q_2 = -\lambda(S_2)\nabla(p_2).$$

with $p := p_1 = p_2$ in

$$\nabla \cdot (q_1(S_1) + q_2(S_2)) = 0.$$

to obtain

$$-\nabla \cdot ((\lambda(S_1) + \lambda(S_2))\nabla p) = 0.$$

2. Two-phase flow model

Solving model for the pressure

To solve

$$-\nabla \cdot ((\lambda(S_1) + \lambda(S_2))\nabla p) = 0.$$

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2. Two-phase flow model

Solving model for the pressure

To solve

$$-\nabla \cdot ((\lambda(S_1) + \lambda(S_2))\nabla p) = 0.$$

With boundary conditions

$$\begin{aligned} p(\mathbf{x}) &= p_0(\mathbf{x}), & \mathbf{x} \in \Gamma_1, \\ -\left(\lambda(S_1) + \lambda(S_2)\right) \frac{\partial p}{\partial n}(\mathbf{x}) &= Q_1(\mathbf{x}), & \mathbf{x} \in \Gamma_2, \\ -\left(\lambda(S_1) + \lambda(S_2)\right) \frac{\partial p}{\partial n}(\mathbf{x}) &= 0, & \mathbf{x} \in \Gamma_3. \end{aligned}$$

2. Two-phase flow model

Solving model for the pressure

Use finite elements

Basis functions: $\varphi_i(x,y) = \alpha_i + \beta_i x + \gamma_i y$

Determine gradient by:

$$\nabla p_{el_j} = \sum_{k=1}^{3} \begin{pmatrix} \beta_k \\ \gamma_k \end{pmatrix} p_k.$$

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2. Two-phase flow model

Solving model for the saturation

Knowing the gradient of the pressure, ∇p, we will solve:

$$\varphi \frac{\partial S_1}{\partial t} - \nabla \cdot \lambda(S_1) \nabla p = 0,$$

with discontinuous Galerkin.

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2. Two-phase flow model

Shock detector

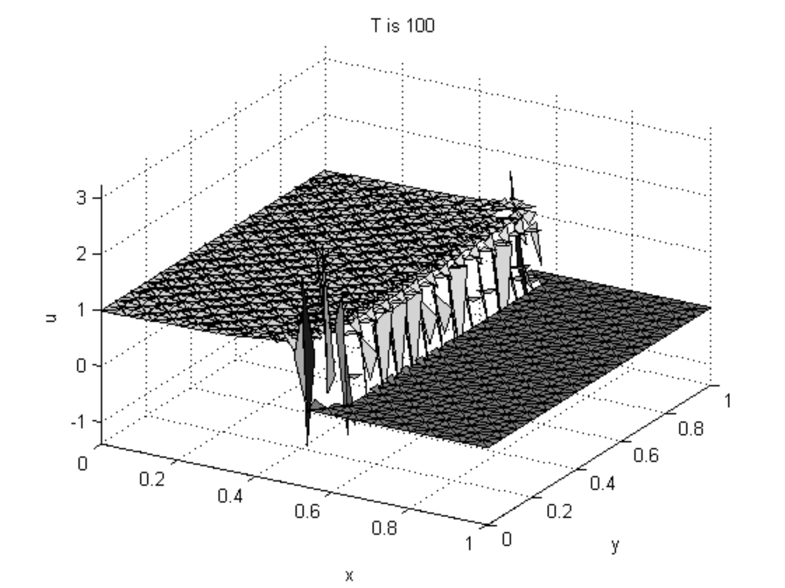
- No periodic boundary conditions, so not all edges can be limited,
- Flag all elements with edge on boundary.

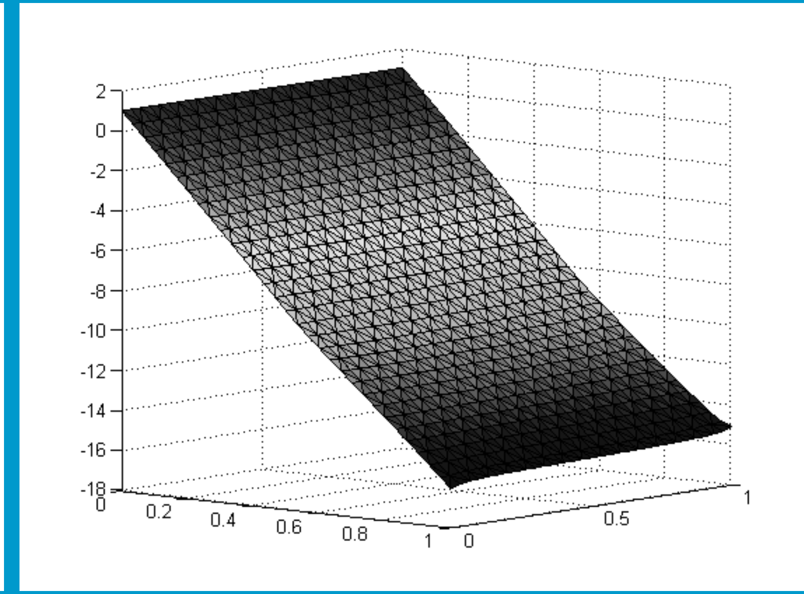
Limiter

- Limit only midpoints on edges not on the boundary.

2. Two-phase flow model

Results





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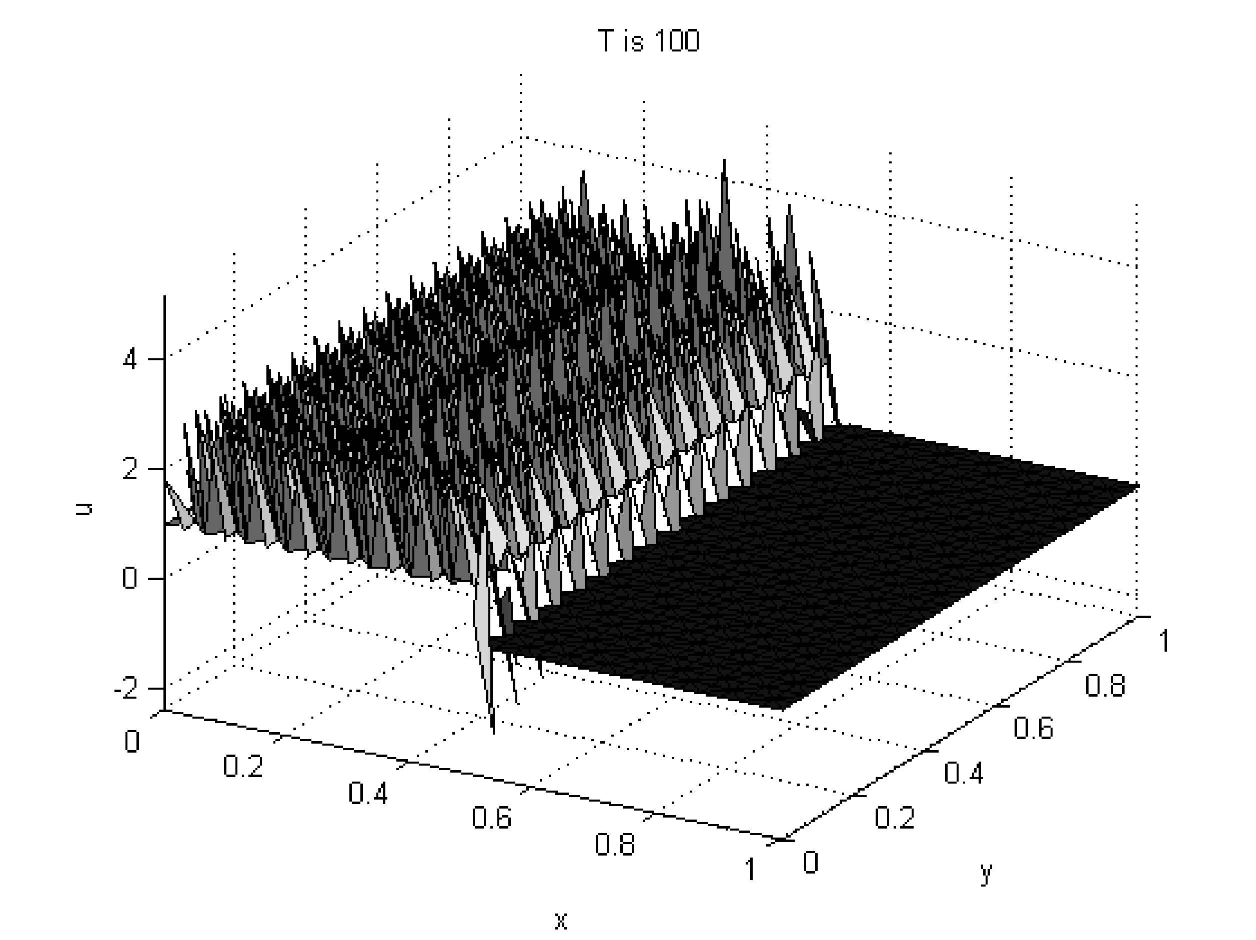
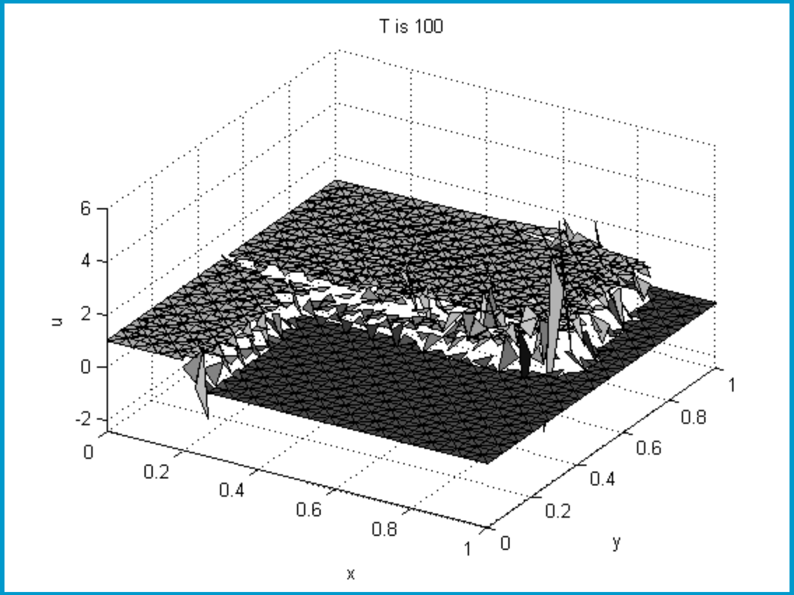
2. Two-phase flow model

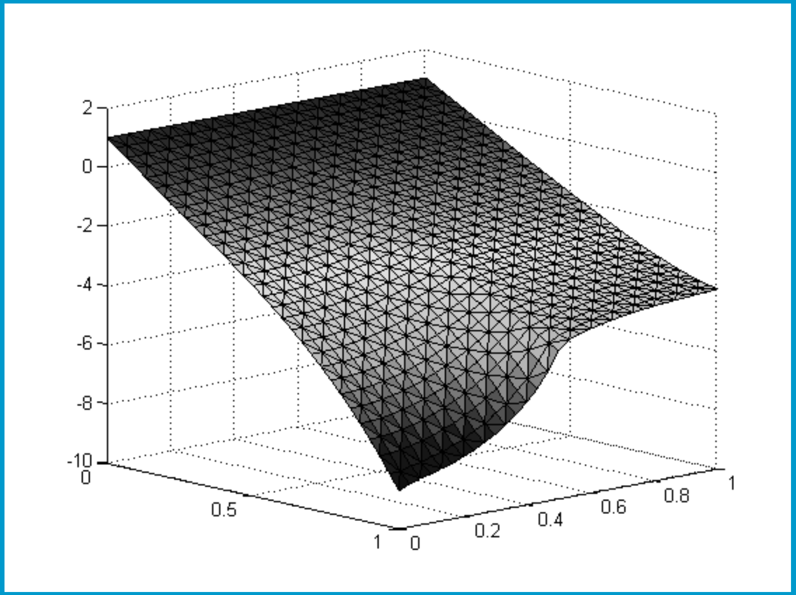
Results



2. Two-phase flow model

Results on permeability





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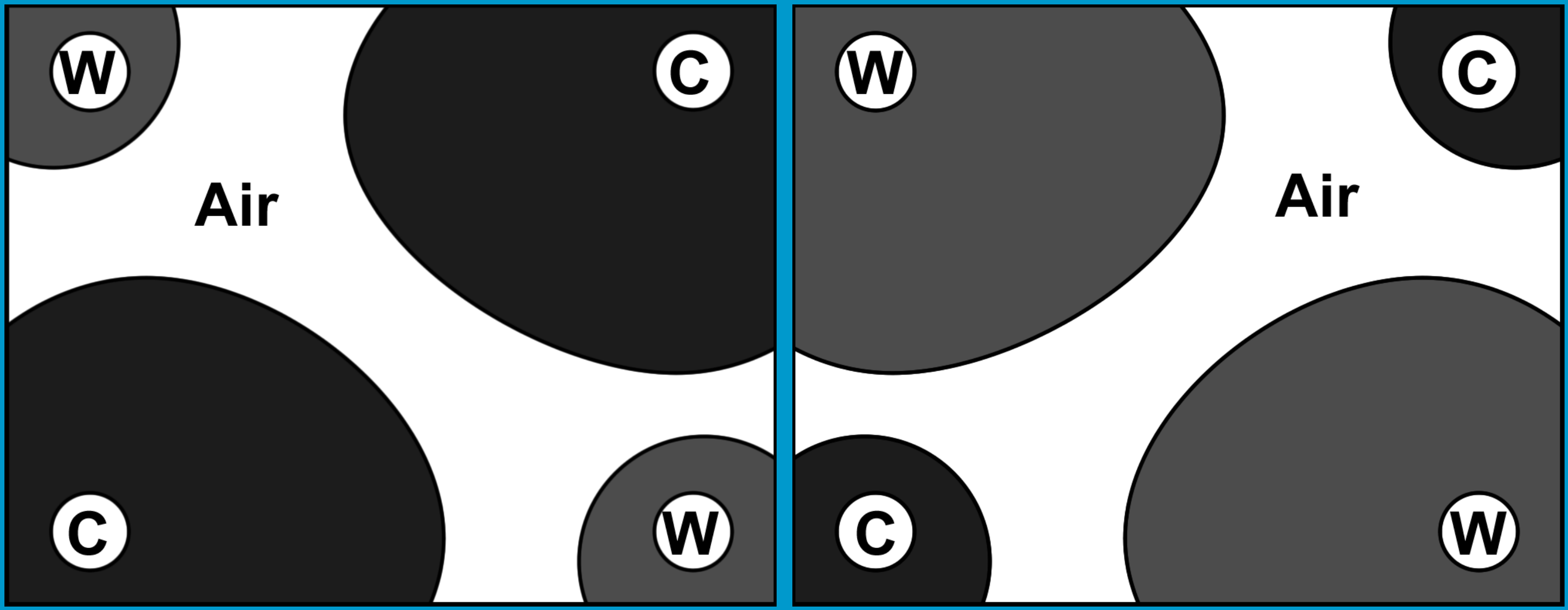
3. Application of two-phase flow

Background



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3. Application of two-phase flow

Upscaling

- Reality three dimensional,
- So we average over the height.

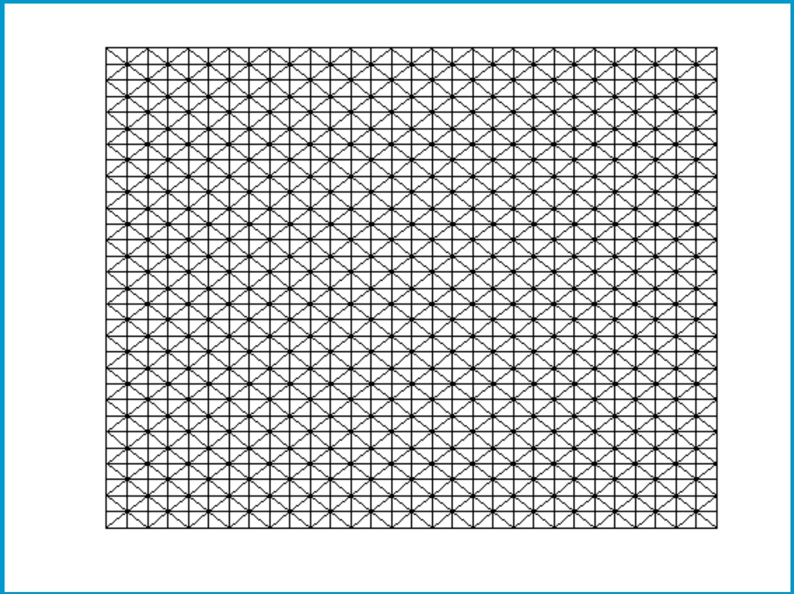
3. Application of two-phase flow

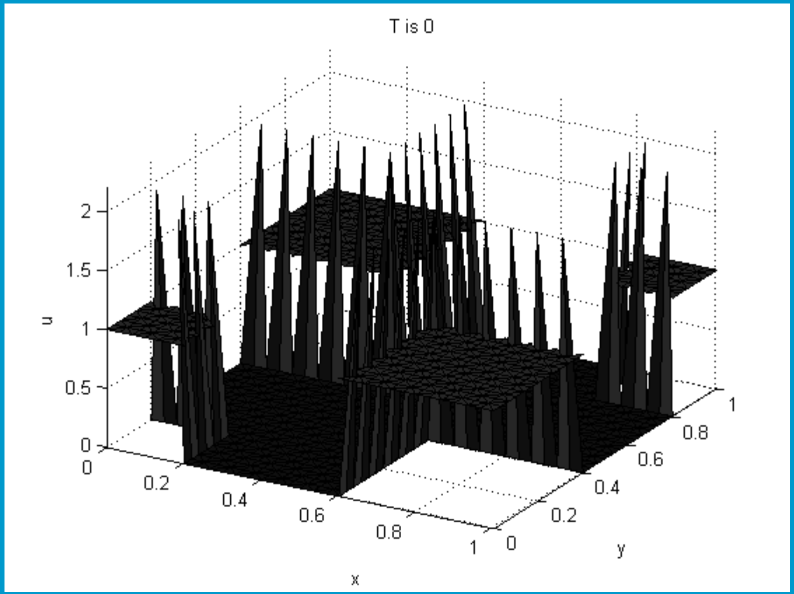
Upscaling

$$
\begin{aligned}
\bar{S}_1 &= \frac{1}{|z_2 - z_1|} \int_{z_1}^{z_2} S_1 dz, \\
\bar{q}_{1,x} &= \frac{1}{|z_2 - z_1|} \int_{z_1}^{z_2} -\lambda(S_1) \frac{\partial p}{\partial x} dz, \\
&= \frac{1}{|z_2 - z_1|} \int_{z_1}^{z_2} -\lambda(S_1) dz \frac{\partial p}{\partial x}, \\
&= -\lambda(\bar{S}_1) \frac{\partial p}{\partial x}, \\
\bar{q}_{1,y} &= -\lambda(\bar{S}_1) \frac{\partial p}{\partial y}, \\
\bar{q}_{1,z} &= 0.
\end{aligned}
$$

3. Application of two-phase flow

Results





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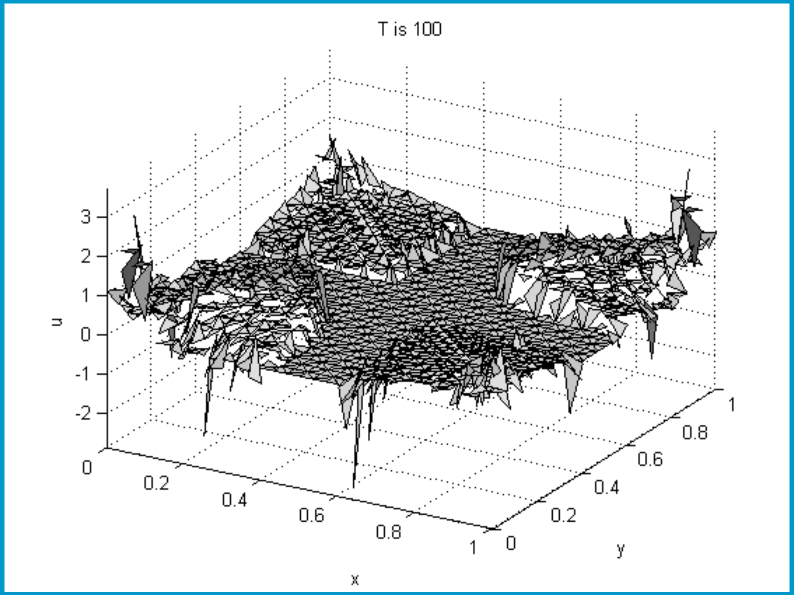
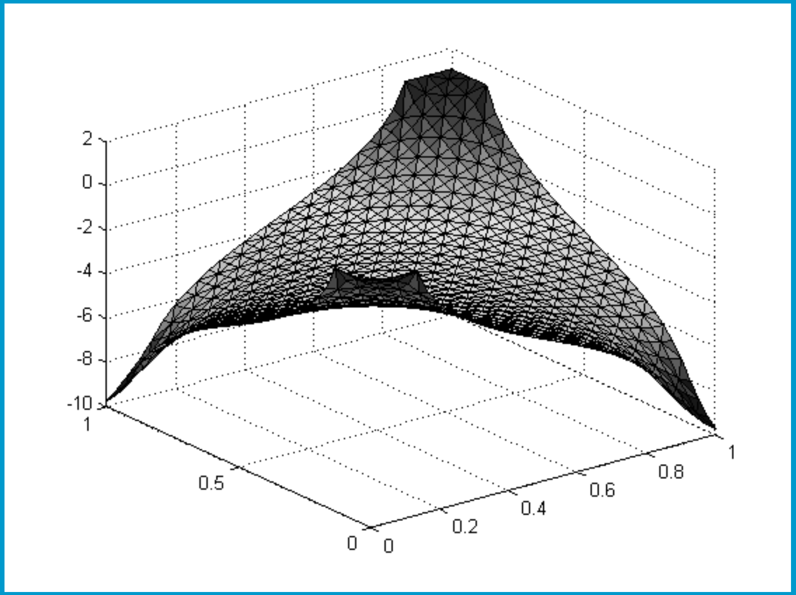
3. Application of two-phase flow

Results





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4. Conclusions

- Illustrated the application of the cooling and heating of EEMCS,
- Investigated having a non-constant permeability,
- Applied discontinuous Galerkin to saturation level, extra difficulty added.

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4. Conclusions

Future work

- Making model more realistic,
- Using a higher order approximation,
- Experimenting with the limiters,
- Making a comparison in computational cost.

Questions?