Global Dynamics in the Leslie–Gower Model with the Allee Effect

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We complete the global bifurcation analysis of the Leslie–Gower system with the Allee effect which models the dynamics of the populations of predators and their prey in a given ecological or biomedical system. In particular, studying global bifurcations of limit cycles, we prove that such a system can have at most two limit cycles surrounding one singular point.

Keywords: Leslie–Gower model; Allee effect; field rotation parameter; bifurcation; singular point; limit cycle; Wintner–Perko Termination Principle.

1. Introduction

In this paper, we complete the global qualitative analysis of a predator–prey system derived from the Leslie–Gower type model, where the most common mathematical form to express the Allee effect in the prey growth function is considered; see [Aguirre et al., 2014; González-Olivares et al., 2006; González-Olivares et al., 2011]. The basis for analyzing the dynamics of such complex ecological or biomedical systems is the interactions between two species, particularly the dynamical relationship between predators and their prey [Li & Xiao, 2007]. From the classical Lotka–Volterra model, several alternatives for modeling continuous time consumer-resource interactions have been proposed [Turchin, 2003]. In our paper, a predator–prey model described by an autonomous two-dimensional differential system is analyzed considering the following aspects: (1) the prey population is affected by the Allee effect [Berec et al., 2007]; Courchamp et al., 1999; (2) the functional response is linear [Seo & Kol, 2008]; (3) the equation for predator is a logistic-type growth function as in the Leslie–Gower model [Aziz-Alaoui & Daher Okiye, 2003].

The main objective of the study in González-Olivares et al., 2011 was to describe the model behavior and to establish the number of limit cycles for the system under consideration. Such results are quite significant for the analysis of most applied mathematical models, thus facilitating the understanding of many real world oscillatory phenomena in nature. The problem of determining conditions which guarantee the uniqueness of a limit cycle or the global stability of the unique positive equilibrium in predator–prey systems has been extensively studied over the last decades. This question starts with the work [Chen, 1981], where it was proved for the first time the uniqueness of a limit cycle for a
specific predator–prey system with a Holling-type II functional response using the symmetry of prey iso-
cline. It is well-known that if a unique unstable posi-
tive equilibrium exists in a compact region, then,
according to the Poincaré–Bendixon theorem, at
least one limit cycle must exist. On the other hand,
if the unique positive equilibrium of a predator–
prey system is locally stable but not hyperbolic,
there might be more than one limit cycle created
via multiple Hopf bifurcations [Chacón 2000] and
the number of limit cycles must be established.

The studied system is defined in an open positive
invariant region and the Poincaré–Bendixon theo-
rem does not apply. Due to the existence of a het-
eroclinic curve determined by the equilibrium point
associated to the strong Allee effect, a subregion
in the phase plane is determined where two limit
cycles exist for certain parameter values, the inner-
most stable and the outermost unstable. Such result
has not been reported in previous papers and rep-
resents a significant difference with the Gause-type
predation models [González-Olivares et al. 2004].
In González-Olivares et al. [2011], it was proved
also the existence of parameter subsets for which the
system can have: a cusp point (Bogdanov–Takens
bifurcation), homoclinic curves (homoclinic bifur-
cation), Hopf bifurcation and the existence of two
limit cycles, the innermost stable and the outermost
unstable, in inverse stability as they usually appear
in the Gause-type models. However, the qualita-
tive analysis of González-Olivares et al. [2011] was
incomplete, since the global bifurcations of limit
cycles could not be studied properly by means of
the methods and techniques which were used earlier
in the qualitative theory of dynamical systems. Apply-
ing to the system new bifurcation methods and geo-
metric approaches developed in Broer & Gaiko
2007a, 2007b, 2007c, 2008, 2009a, 2009b, the fol-
lowing predator–prey model has been studied:
\begin{align}
\dot{x} &= x(a - \lambda x) - \beta x y (\text{prey}), \\
\dot{y} &= -\gamma y + \mu y x (\text{predator}).
\end{align}

The variables \(x > 0\) and \(y > 0\) denote the den-
sity of the prey and predator populations respec-
tively, while \(p(x)\) is a nonmonotonic response func-
tion given by

\[p(x) = \frac{m x}{\alpha x^2 + \beta x + 1},\]

where \(a, m\) are positive and where \(\beta > -2\sqrt{\alpha}.\)

2. Predator–Prey Models

In Gaiko [2016, 2017], we considered a quartic fam-
ily of planar vector fields corresponding to a ratio-
nal Holling-type dynamical system which models
the dynamics of the populations of predators and
their prey in a system which is a variation on the
classical Lotka–Volterra one. For the latter system
the change of the prey density per unit of time
per predator called the response function is propor-
tional to the prey density. This means that there is
no saturation of the predator when the amount of
available prey is large. However, it is more realis-
tic to consider a nonlinear and bounded response
function, and in fact different response functions
have been used in the literature to model the preda-
tor response; see [Bailey 1965, Broer et al. 2007;
Broer & Gaiko 2010, Holling 1966, Lamontagne

For instance, in Zhu et al. [2002], the following
predator–prey model has been studied:

\begin{align}
\dot{x} &= x(a - \lambda x) - \gamma x (\text{prey}), \\
\dot{y} &= -\delta y + \mu y x (\text{predator}).
\end{align}

where \(\alpha, m > 0\), \(\lambda > 0\), \(\delta > 0\), \(\lambda > 0\), and \(\mu > 0\)
are parameters. Note that (3) is obtained from (1)
by adding the term $-\mu y^2$ to the second equation and after scaling $x$ and $y$, as well as the parameters and the time $t$. In this way, competition has been taken into account between predators for resources other than prey. The non-negative coefficient $\mu$ is the rate of competition amongst predators. Systems \[ and \] represent predator–prey models with generalized Holling response functions of type IV.

In \cite{Lamontagne2008}, the following generalized Gause predator–prey system has been considered:

\begin{equation}
\dot{x} = rx \left(1 - \frac{x}{K}\right) - yp(x), \quad (4)
\end{equation}

with a generalized Holling response function of type III:

\begin{equation}
p(x) = \frac{mx^2}{ax^2 + bx + 1} \quad (5)
\end{equation}

This system, where $x > 0$ and $y > 0$, has seven parameters: the parameters $a$, $c$, $d$, $k$, $m$, $r$ are positive and the parameter $b$ can be negative or non-negative. The parameters $a$, $b$, and $m$ are fitting parameters of response function. The parameter $d$ is the death rate of the predator while $c$ is the efficiency of the predator to convert prey into predators. The prey follows a logistic growth with a rate $r$ in the absence of predator. The environment has a prey capacity determined by $k$.

The case $b > 0$ has been studied earlier; see the references in \cite{Lamontagne2008}. The case $b < 0$ is more interesting: it provides a model for a functional response with limited group defence. In opposition to the generalized Holling function of type IV studied in \cite{Broer2001, Broer2002}, where the response function tends to zero as the prey population tends to infinity, the generalized function of type III tends to a nonzero value as the prey population tends to infinity. The functional response of type III with $b < 0$ has a maximum at some point; see \cite{Lamontagne2008}. When studying the case $b < 0$, one can find also a Bogdanov–Takens bifurcation of codimension three which is an organizing center for the bifurcation diagram of system \[ and \].

After scaling $x$ and $y$, as well as the parameters and the time $t$, this system can be reduced to a system with only four parameters \((\alpha, \beta, \delta, \rho)\) \cite{Lamontagne2008}.

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\begin{equation}
\dot{x} = px(1 - x) - yp(x), \quad (6)
\end{equation}

where

\begin{equation}
p(x) = \frac{x^2}{ax^2 + bx + 1} \quad (7)
\end{equation}

In \cite{González-Olivares2011}, we studied the system

\begin{equation}
\dot{x} = x \left(1 - \lambda x - \frac{xy}{ax^2 + bx + 1}\right), \quad (8)
\end{equation}

where $x > 0$ and $y > 0$; $\alpha \geq 0$, $-\infty < \beta < +\infty$, \( \delta > 0 \), \( \lambda > 0 \), and $\mu$ are parameters.

The Leslie–Gower predator–prey model incorporating the Allee effect phenomenon on prey is described by the Kolmogorov-type rational dynamical system \cite{González-Olivares2011}:

\begin{equation}
\dot{x} = x \left(1 - \frac{\alpha x}{K}\right) (x - m) - qy \quad (prey), \quad (9)
\end{equation}

\begin{equation}
\dot{y} = y \left(\beta - \mu y - \frac{\gamma x}{nx}\right) \quad (predator),
\end{equation}

where the parameters have the following biological meanings: $r$ and $s$ represent the intrinsic prey and predator growth rates, respectively; $K$ is the prey environment carrying capacity; $m$ is the Allee threshold or minimum of viable population; $q$ is the maximal per capita consumption rate, i.e. the maximum number of prey that can be eaten by a predator in each time unit; $n$ is a measure of food quality that the prey provides for conversion into predator births.

System \[ can be written in the form of a quartic dynamical system \cite{González-Olivares2011}:

\begin{equation}
\dot{x} = x^2((1 - x)(x - m) - s\gamma) + P, \quad (10)
\end{equation}

\begin{equation}
\dot{y} = y(\beta x - s\gamma y) \equiv Q.
\end{equation}

Together with \[, we will also consider an auxiliary system (see \cite{Bautin1966, Gaiko2004, Perko2002}):

\begin{equation}
\dot{x} = P - \delta Q, \quad \dot{y} = Q + \delta P, \quad (11)
\end{equation}

applying to these systems new bifurcation methods and geometric approaches developed in \cite{Broer2007, Gaiko2004, Gaiko2011, Gaiko2012}.
3. Basic Facts on Singular Points and Limit Cycles

The study of singular points of system (12) or (13) will use two Poincaré Index Theorems; see Bautin & Leontovich [1990]. But first let us define the singular point and its Poincaré Index.

Definition 3.1 (Bautin & Leontovich, 1990). A singular point of the dynamical system

\[ \dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \]

(12)

where \( P(x, y) \) and \( Q(x, y) \) are continuous functions (for example, polynomials), is a point at which the right-hand sides of (12) simultaneously vanish. \( f = 0 \) where \( f \) is a polynomial with \( f \) a polynomial vector function.

Theorem 3.1 (First Poincaré Index Theorem)

Let \( S \) be some point on the phase plane passing through a singular point of system (12) and \( M \) be some point on \( S \). If the point \( M \) goes around the curve \( S \) in positive direction (counter-clockwise) one time, then the vector coinciding with the direction of a tangent to the trajectory passing through the point \( M \) is rotated through the angle \( 2\pi j \) (\( j = 0, \pm 1, \pm 2, \ldots \)). The integer \( j \) is called the Poincaré Index of the closed curve \( S \) relative to the vector field of system (12) and has the expression

\[ j = \frac{1}{2\pi} \int_{S} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) ds. \]

According to this definition, the index of a node or a focus, or a center is equal to +1 and the index of a saddle is −1.

Theorem 3.2 (Second Poincaré Index Theorem)

Let \( S \) be a simple closed curve in the phase plane not passing through a singular point of system (12) and \( M \) be some point on \( S \). If the point \( M \) goes around the curve \( S \) in positive direction (counter-clockwise) one time, then the vector coinciding with the direction of a tangent to the trajectory passing through the point \( M \) is rotated through the angle \( 2\pi j \) (\( j = 0, \pm 1, \pm 2, \ldots \)). The integer \( j \) is called the Poincaré Index of the closed curve \( S \) relative to the vector field of system (12) and has the expression

\[ j = \frac{1}{2\pi} \int_{S} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) ds. \]

Then in some neighborhood \( |w| < \varepsilon, |z| < \delta \) of the points \( (0, 0) \) the function \( F(w, z) \) can be represented as

\[ F(w, z) = (w^k + A_1(z)w^{k-1} + \cdots + A_{k-1}(z)w + A_k(z))\Phi(w, z), \]

where \( \Phi(w, z) \) is an analytic function not equal to zero in the chosen neighborhood and \( A_1(z), \ldots, A_k(z) \) are analytic functions for \( |z| < \delta \).

From this theorem it follows that the equation \( F(w, z) = 0 \) in a sufficiently small neighborhood of the point \( (0, 0) \) is equivalent to the equation

\[ w^k + A_1(z)w^{k-1} + \cdots + A_{k-1}(z)w + A_k(z) = 0, \]

for which the left-hand side is a polynomial with respect to \( w \). Thus, the Weierstrass Preparation Theorem reduces the local study of the general case of implicit function \( w(z) \), defined by the equation

\[ w^k + A_1(z)w^{k-1} + \cdots + A_{k-1}(z)w + A_k(z) = 0, \]
\( F(w, z) = 0 \), to the case of implicit function, defined by the algebraic equation with respect to \( w \).

**Theorem 3.4 (Implicit Function Theorem)** [Galko 2003, Perko 2002] Let \( F(w, z) \) be an analytic function in the neighborhood of the points \((0, 0)\) and \( F(0, 0) = F_w'(0, 0) \neq 0 \).

Then there exist \( \delta > 0 \) and \( \varepsilon > 0 \) such that for any \( z \) satisfying the condition \(|z| < \delta\) the equation \( F(w, z) = 0 \) has the only solution \( w = f(z) \) satisfying the condition \(|f(z)| < \varepsilon\). The function \( f(z) \) is expanded into the series on positive integer powers of \( z \) which converge for \( |z| < \delta \), i.e., it is a single-valued analytic function of \( z \) which vanishes at \( z = 0 \).

Assume that system (13) has a limit cycle \( L_0 \) of minimal period \( T_0 \) at some parameter value \( \mu = \mu_0 \in \mathbb{R}^1 \); see Fig. 1. [Galko 2003, Perko 2002].

Let \( l \) be the straight line normal to \( L_0 \) at the point \( p_0 = \varphi_0(0) \) and \( s \) be the coordinate along \( l \) with a positive exterior of \( L_0 \). It then follows from the Implicit Function Theorem that there is a \( \delta > 0 \) such that the Poincaré map \( h(s, \mu) \) is defined and analytic for \(|s| < 0\) and \(|\mu - \mu_0| < \delta\), which is a mapping from \( l \) to itself obtained by following trajectories from one intersection of \( l \) to the next [Galko 2003, Perko 2002]. Besides, the displacement function for system (15) along the normal line \( l \) to \( L_0 \) is defined as the function

\[
d(s, \mu) = h(s, \mu) - s.
\]

In terms of the displacement function, a multiple limit cycle can be defined as follows.

**Definition 3.3** [Galko 2003, Perko 2002] A limit cycle \( L_0 \) of (13) is a multiple limit cycle if \( d(0, \mu_0) = d_s(0, \mu_0) = 0 \) and it is a simple limit cycle (or hyperbolic limit cycle) if it is not a multiple limit cycle; furthermore, \( L_0 \) is a limit cycle of multiplicity \( m \) iff

\[
d(0, \mu_0) = d_s(0, \mu_0) = \cdots = d_s^{(m-1)}(0, \mu_0) = 0,
\]

where \( d_s(0, \mu_0) \) and \( d_s^{(j)}(0, \mu_0), j = 2, \ldots, m, \) are partial derivatives of the displacement function \( d(s, \mu) \) with respect to \( s \) for \( s = 0 \) and \( \mu = \mu_0 \).

Note that the multiplicity of \( L_0 \) is independent of the point \( p_0 \) through which we take the normal line \( l \).

Let us write down also the following formulas which have already become classical ones and determine the derivatives of the displacement function in terms of integrals of the vector field \( f \) along the periodic orbit \( \varphi_0(t) \) [Galko 2003, Perko 2002]:

\[
d_s(0, \mu_0) = \exp \int_0^{T_0} \nabla \cdot f(\varphi_0(t), \mu_0) \, dt - 1
\]

and

\[
d_s^{(j)}(0, \mu_0) = \frac{-\omega_0}{\|f(\varphi_0(0), \mu_0)\|} \times \int_0^{T_0} \exp \left( -\int_0^t \nabla \cdot f(\varphi_0(\tau), \mu_0) \, d\tau \right) \times \mathbf{f} \wedge \mathbf{f}_s(\varphi_0(t), \mu_0) \, dt
\]

for \( j = 1, \ldots, n \), where \( \omega_0 = \pm 1 \) according to whether \( L_0 \) is positively or negatively oriented, respectively, and where the wedge product of two vectors \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \) in \( \mathbb{R}^2 \) is defined as

\[
x \wedge y = x_1y_2 - x_2y_1.
\]

Similar formulas for \( d_s(0, \mu_0) \) and \( d_s^{(j)}(0, \mu_0) \) can be derived in terms of integrals of the vector field.

![Fig. 1. The Poincaré return map in the neighborhood of a multiple limit cycle.](image-url)
field $f$ and its first and second partial derivatives along $\varphi_t(t)$.

Now we can formulate the Wintner–Perko Termination Principle [Perko 2002] for polynomial system (13).

**Theorem 3.5** (Wintner–Perko Termination Principle) [Perko 2002]. Any one-parameter family of multiplicity-$m$ limit cycles of relatively prime polynomial system (13) can be extended in a unique way to a maximal one-parameter family of multiplicity-$m$ limit cycles of (13) which is either open or cyclic.

If it is open, then it terminates either as the parameter or the limit cycles become unbounded; or, the family terminates either at a singular point of (13), which is typically a fine focus of multiplicity-$m$, or on a (compound) separatrix cycle of (13) which is also typically of multiplicity-$m$.

The proof of this principle for general polynomial system (13) with a vector parameter $\mu \in \mathbb{R}^n$ parallels the proof of the planar termination principle for the system

$$\begin{align*}
\dot{x} &= P(x, y, \lambda), \\
\dot{y} &= Q(x, y, \lambda)
\end{align*}$$

with a single parameter $\lambda \in \mathbb{R}$ (see Perko 2002), since there is no loss of generality in assuming that system (13) is parameterized by a single parameter $\lambda$; i.e. we can assume that there exists an analytic mapping $\mu(\lambda)$ of $\mathbb{R}$ into $\mathbb{R}^n$ such that (13) can be written as (14) and then we can repeat everything that had been done for system (13) in Perko 2002.

In particular, if $\lambda$ is a field rotation parameter of (13), the following Perko’s theorem on monotonic families of limit cycles is valid; see Perko 2002.

**Theorem 3.6** [Perko 2002]. If $L_0$ is a nonsingular multiple limit cycle of (13) for $\lambda = \lambda_0$, then $L_0$ belongs to a one-parameter family of limit cycles of (13) (14).

Furthermore:

1. If the multiplicity of $L_0$ is odd, then the family either expands or contracts monotonically as $\lambda$ increases through $\lambda_0$.
2. If the multiplicity of $L_0$ is even, then $L_0$ bifurcates into a stable and an unstable limit cycle as $\lambda$ varies from $\lambda_0$ in one sense and $L_0$ disappears as $\lambda$ varies from $\lambda_0$ in the opposite sense; i.e. there is a fold bifurcation at $\lambda_0$.

## 4. Global Bifurcation Analysis

Consider system (10). This system has two invariant straight lines: $x = 0$ (double) and $y = 0$. Its finite singularities are determined by the algebraic system

$$\begin{align*}
x^2((1 - x)(x - m) - o y) &= 0, \\
y(3x - \gamma y) &= 0.
\end{align*}$$

From (13), we have got: three singular points $(0, 0)$, $(m, 0)$, $(1, 0)$ (suppose that $m < 1$) and at most two points defined by the system

$$\begin{align*}
(1 - x)(x - m) - o y &= 0, \\
\beta x - \gamma y &= 0.
\end{align*}$$

According to the Second Poincaré Index Theorem (Theorem 3.6), the point $(0, 0)$ is a double (saddle-node), $(m, 0)$ is a node, and $(1, 0)$ is a saddle (for $m < 1$); see also Gonzalez-Olivares et al. 2011.

In addition, a double singular point (saddle-node) may appear in the first quadrant and bifurcate into two singular points. If there exist exactly two simple singular points in the open first quadrant, then the singular point on the left with respect to the $x$-axis is a saddle and the singular point on the right is an anti-saddle [Gonzalez-Olivares et al. 2011].

If a singular point is not in the first quadrant, in consequence, it has no biological significance.

To study singular points of (13) at infinity, consider the corresponding differential equation

$$\frac{dy}{dx} = \frac{y(3x - \gamma y)}{x^2((1 - x)(x - m) - o y)}$$

Dividing the numerator and denominator of the right-hand side of (17) by $x^4 \gamma y$ and denoting $x/y$ by $u$ (as well as $dy/dx$), we will get the equation

$$u = 0, \quad \frac{du}{dx} = \frac{y}{x}$$

for all infinite singularities of (13) except when $x = 0$ (the “ends” of the $y$-axis); see Bautin & Leontovich 1990, Gaiko 2003. For this special case we can divide the numerator and denominator of the right-hand side of (17) by $y^4 (y \neq 0)$ denoting $x/y$ by $v$ (as well as $dy/dx$) and consider the equation

$$v^4 = 0, \quad \frac{dv}{dx} = \frac{y}{x}$$

According to the Poincaré Index Theorems (Theorems 3.3 and 3.4), Eqs. (15) and (16) give two
singular points at infinity for a simple node on the “ends” of the x-axis and a quartic saddle-node on the “ends” of the y-axis.

Using the obtained information on singular points and applying a geometric approach developed in [Broer & Gaiko 2014; Gaiko 2003, 2011; 2012, 2012a, 2012b, 2014; 2014; 2014; 2017], we can study the limit cycle bifurcations of system (10). The sense of this approach consists of constructing canonical systems with field rotation parameters by means of Erugin’s Two-Isocline Method, using geometric properties of the trajectories, and applying the Wintern–Perko Termination Principle connecting all local bifurcations of limit cycles.

Our study will use some results obtained in [González-Olivares et al. 2011], in particular, the results on the cyclicity of a singular point of (10). However, it is surely not enough to have only these results to prove the main theorem of this paper concerning the maximum number of limit cycles of system (10).

Applying the definition of a field rotation parameter [Bautin & Leontovich 1994; Gaiko 2003; Perko 2001], i.e. a parameter which rotates the field in one direction, to system (10), let us calculate the corresponding determinants for the parameters α, β, and γ, respectively:

\[ \Delta_\alpha = PQ_x' - QP_x' = x^2y^2(\beta x - \gamma y), \]  
\[ \Delta_\beta = PQ_y' - QP_y' = x^2y^2(1 - x)(x - m) - \alpha y, \]  
\[ \Delta_\gamma = PQ_x' - QP_x' = -x^2y^2((1 - x)(x - m) - \alpha y). \]  

It follows from (20) that in the first quadrant the sign of \( \Delta_\alpha \) depends on the sign of \( \beta x - \gamma y \) and from (21) and (22) that the sign of \( \Delta_\beta \) or \( \Delta_\gamma \) depends on the sign of \( (1 - x)(x - m) - \alpha y \) on increasing (or decreasing) the parameters \( \alpha, \beta, \) and \( \gamma \), respectively.

Therefore, to study limit cycle bifurcations of system (10), it makes sense together with (10) to consider also an auxiliary system (11) with a field rotation parameter \( \delta \) for which

\[ \Delta_\delta = P^2 + Q^2 \geq 0. \]

System (11) is more general than (10), but the introduced rotation parameter \( \delta \) does not change the location and the indexes of the finite singularities of (10) and, as we will see below, does not give additional limit cycles: see also [Broer & Gaiko 2014; Gaiko 2003, 2011, 2012, 2012a, 2012b, 2014; 2014; 2014; 2017]. Using system (11) and applying Perko’s results [Perko 2002], we will prove the following theorem.

**Theorem 4.1.** The Leslie–Gower system with the Allee effect can have at most two limit cycles surrounding one singular point.

**Proof.** In [González-Olivares et al. 2011], it was proved that system (10) can have at least two limit cycles. Let us prove now that this system has at most two limit cycles. The proof is carried out by contradiction applying Catastrophe Theory; see [Gaiko 2003; Perko 2001].

Suppose that system (10) with two finite singularities in the first quadrant, a saddle S and an anti-saddle A, has three limit cycles surrounding A. Consider system (11) with four parameters: \( \alpha, \beta, \gamma, \) and \( \delta \) (we can fix the parameter \( m \) fixing the position of the node on the x-axis). The field rotation parameter \( \delta \) does not change the location and the indexes of the finite singularities of (10) [Bautin & Leontovich 1994; Gaiko 2003]. Besides, it is a rough parameter. If we vary this parameter in one sense, the smallest limit cycle will disappear in the focus A (the Andronov–Hopf bifurcation) and two other limit cycles will combine into a semi-stable limit cycle which will then disappear in a “trajectory concentration” surrounding A [Bautin & Leontovich, 1994; Gaiko 2003]. If we vary the parameter \( \delta \) in the opposite sense, the largest limit cycle will disappear in a separatix loop of the saddle S and two others combining a semi-stable limit cycle will also disappear in a “trajectory concentration” [Bautin & Leontovich 1994; Gaiko 2003]. A possibility of the appearance of an additional semi-stable limit cycle surrounding A under the variation of \( \delta \), as we will see now, can also be excluded.

Note that, if we vary the parameter \( \delta \) in one sense, two limit cycles of system (11) will combine into a semi-stable (multiplicity-two) limit cycle; if we vary this parameter in the opposite sense, we will get another semi-stable (multiplicity-two) limit cycle; and the inner limit cycle, which is between the largest and the smallest ones, will be common for the formed semi-stable cycles. Therefore, varying the other parameters of (11), \( \alpha, \beta, \) and \( \gamma \), we will obtain two field bifurcation surfaces of multiplicity-two limit cycles forming a cusp bifurcation surface.
of multiplicity-three limit cycles in the space of the parameters $\alpha, \beta, \gamma$, and $\delta$; see Fig. 2, where $C^+_2$ and $C^-_2$ are the bifurcation curves of multiplicity-two limit cycles [Gaiko 2003; Perko 2002].

The corresponding maximal one-parameter family of multiplicity-three limit cycles cannot be cyclic, otherwise there will be at least one point corresponding to the limit cycle of multiplicity four (or even higher) in the parameter space; see Fig. 3, where $C^+_3$, $C^+_2$, and $C^-_3$ are the bifurcation surfaces of multiplicity-two limit cycles; $C^+_4$ and $C^-_4$ are the bifurcation curves of multiplicity-three limit cycles; $C_4$ is the bifurcation point of a multiplicity-four limit cycle [Gaiko 2003; Perko 2002].

Extending the bifurcation curve of multiplicity-four limit cycles through this point and parameterizing the corresponding maximal one-parameter family of multiplicity-four limit cycles by the field rotation parameter $\delta$, according to Theorem 3.6 we will get again two monotonic curves of multiplicity three and one, respectively, which, by the Wintner–Perko Termination Principle (Theorem 3.5), terminate either at the point $A$ or on a separatrix loop surrounding this point. Since we know at least the cyclicity of the singular point which is equal to two (see Gonzalez-Olivares et al. 2011), we have got a contradiction with the Termination Principle (Theorem 3.5), see Fig. 3 where $C^+_3(C^+_4)$ and $C^-_3(C^-_4)$ are the bifurcation surfaces of multiplicity-two (or multiplicity-three) limit cycles; $C_4(C_1)$ is the bifurcation curve of multiplicity-three (or multiplicity-four) limit cycles [Gaiko 2003; Perko 2002].

If the maximal one-parameter family of multiplicity-three limit cycles is not cyclic, using the same Principle, this again contradicts the cyclicity of $A$ (see Gonzalez-Olivares et al. 2011) not admitting the multiplicity of limit cycles higher than two.

On the same reasons, we can exclude a possibility of the appearance of an additional semi-stable limit cycle surrounding $A$ under the variation of $\delta$. If the maximal one-parameter family of multiplicity-three limit cycles is not cyclic, using the same Principle, this again contradicts the cyclicity of $A$ (see Gonzalez-Olivares et al. 2011) not admitting the multiplicity of limit cycles higher than two.

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Moreover, it also follows from the Termination Principle that a separatrix loop cannot have the multiplicity (cyclicity) higher than two in this case.

Thus, we conclude that system (11) (and system (10) as well) cannot have either a multiplicity-three limit cycle or more than two limit cycles surrounding a singular point which proves the theorem.

5. Conclusions

In this paper, we have completed the global bifurcation analysis of the Leslie–Gower system with the Allee effect which models the dynamics of the populations of predators and their prey in a given ecological or biomedical system. Studying global bifurcations of limit cycles, we have proved that such a system can have at most two limit cycles surrounding one singular point.

The mathematical tools used in this paper may also be helpful in the qualitative analysis of any two-dimensional model of species interactions in a biological system, in particular, in the contexts of conservation and biological control. Another line of research could be directed, for instance, towards studying the interaction of the Allee effect with random environmental conditions such as alien species invasions or other catastrophic events, which may increase the amplitude of population fluctuations and even drive a population to extinction; see [Aguirre et al. 2014; González-Olivares et al. 2006; González-Olivares et al. 2011] and the references therein.

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References


