Efficient Augmented Lagrangian-type Preconditioning for the Oseen Problem using Grad-Div Stabilization

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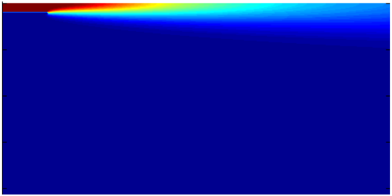
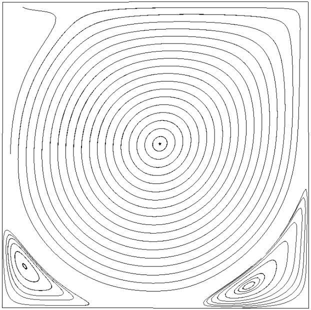
Setting

- Stationary, incompressible flow problems
- Need: efficient and robust linear solvers
- Examples:



Laminar flames
(chemically reacting)

(source: F. Bisetti/KAUST)



Lid-driven cavity
(prototype)

Grad-Div based Preconditioning

- Numerical analysis has two subfields:

Error Analysis

Error estimates
Stabilization methods

Numerical Linear Algebra

Solvers
Preconditioners

- Problem: often treated separately

$\rightsquigarrow$ Here: use Grad-Div stabilization to get efficient linear algebra

Heister and Rapin.
Efficient augmented Lagrangian-type preconditioning for the Oseen problem using Grad-Div stabilization.
Int. J. Numer. Meth. Fluids, 2013, 71: 118–134.

Introduction

Incompressible Navier-Stokes equations (instationary, nonlinear)

Find velocity $\boldsymbol{u}$ and pressure p in domain Ω with

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu \triangle \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = f,$$
$$\nabla \cdot \boldsymbol{u} = 0$$

Time discretization and linearization gives

Oseen Problem (stationary, linear)

$$c\boldsymbol{u} - \nu \triangle \boldsymbol{u} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{u} + \nabla p = f,$$
$$\nabla \cdot \boldsymbol{u} = 0$$

(viscosity ν, reaction coefficient c, convection $\boldsymbol{b}$)

$\rightsquigarrow$ Of interest: $c \ll 1$, $\nu \ll 1$, $\|\boldsymbol{b}\| \sim 1$ (convection dominated)

Introduction: Linear System

- Inf-sup stable finite element discretization, here Taylor-Hood Q_{k+1}-Q_k Lagrange elements[1]
- Gives linear saddle point problem:

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

- Krylov subspace method (flexible GMRES)
- Need preconditioner P:

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} P^{-1} \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}, \qquad P^{-1} \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} u \\ p \end{pmatrix}$$

[1] tensor-product polynomials of order $k+1$ for the velocity and k for the pressure

Grad-Div Stabilization

$$
\begin{aligned}
\text{find } (\boldsymbol{u}, p) \in \boldsymbol{V} \times Q := [H_0^1(\Omega)]^d \times L_*^2(\Omega) \text{ with} \\
(\nu \nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + ((\boldsymbol{b} \cdot \nabla)\boldsymbol{u} + c\boldsymbol{u}, \boldsymbol{v}) + (\gamma \nabla \cdot \boldsymbol{u}, \nabla \cdot \boldsymbol{v}) - (\nabla \cdot \boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}) \\
(\nabla \cdot \boldsymbol{u}, q) = 0 \\
\text{for all } (\boldsymbol{v}, q) \in \boldsymbol{V} \times Q.
\end{aligned}
$$

Grad-Div:

- Vanishes in the continuous case
- Discretized: penalty term for the divergence
- Why? Incompressibility
- How to choose parameter γ_K on each cell K?

Parameter Design: a-priori Analysis

Theorem (Olshanskii, Lube, Heister, Löwe)

Given a sufficiently smooth continuous solution $(\boldsymbol{u}, p)$, the optimal error is obtained with the choice:

$$\gamma_K \sim \max\left\{ \frac{|p|_{H^k(K)}}{|\boldsymbol{u}|_{H^{k+1}(\widetilde{K})}} - \nu, 0 \right\} \text{ on each cell } K.$$

Olshanskii, Lube, Heister, and Löwe.
Grad-div stabilization and subgrid pressure models for the incompressible Navier-Stokes equations.
Computer Methods in Applied Mechanics and Engineering, 198(49-52):3975 – 3988, 2009.

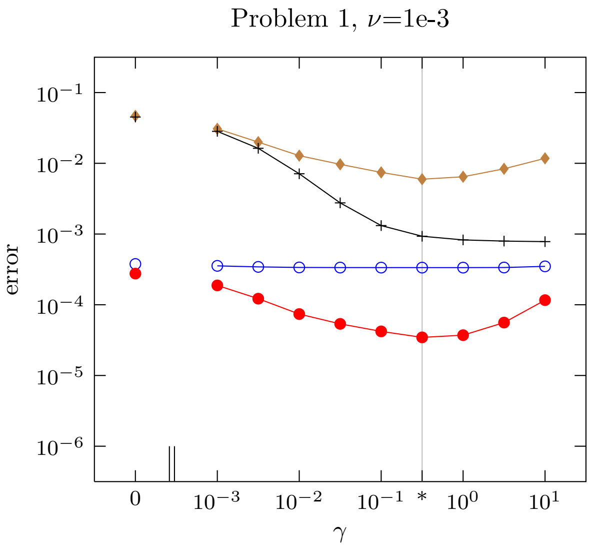
Parameter Design: In Practice

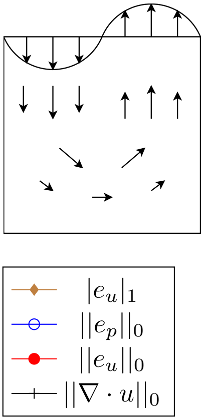
Parameter Design

$$\gamma_K \sim \max\left\{ \frac{|p|_{H^k(K)}}{|\boldsymbol{u}|_{H^{k+1}(\widetilde{K})}} - \nu, 0 \right\} \text{ on each cell } K.$$

- Evaluating γ_K is hard: non-linear, missing regularity, high order derivatives . . .
- For example $f = 0$, $k = 1$ gives $\gamma_K \sim \nu + C\|b\|_K$
- Often used: constant models $\gamma_K = \gamma$ (homogeneous flows)
- Experiments: $\gamma \in [0.1, 1]$ often good, better than $\gamma = 0$
- From now on: $\gamma_K = \gamma$

Parameter Design: an Example





Grad-Div Stabilization

Theorem (Heister,Rapin)

Let Π *be the* L^2 *orthogonal projector into pressure space* Q_h*.*
Define the fluctuation operator $\kappa := Id - \Pi$*.*
With velocity basis functions (φ_i) *we have:*

$$\begin{aligned}(\nabla\cdot\varphi_j,\nabla\cdot\varphi_i) &= (\Pi(\nabla\cdot\varphi_j),\Pi(\nabla\cdot\varphi_i)) &&+ (\kappa(\nabla\cdot\varphi_j),\kappa(\nabla\cdot\varphi_i))\\ &= \left(B^T M_p^{-1} B\right)_{ij} &&+ \text{Stab}\end{aligned}$$

first part:

- does not change solution, because $Bu = 0$
- algebraic influence
- known: augmented Lagrangian

second part:

- changes the solution
- "projection stabilization"
- adds dissipation on some scales
- vanishes for $h \to 0$

Heister and Rapin.
Efficient augmented Lagrangian-type preconditioning for the Oseen problem using Grad-Div stabilization.
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Augmented Lagrangian Preconditioner

- Add $\gamma B^T M_p^{-1} B$ to A:

$$\begin{pmatrix} A + \gamma B^T M_p^{-1} B & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

- Use Schur complement based block preconditioner
- Efficient approximation of Schur complement possible
- Problem: handling $A + \gamma B^T M_p^{-1} B$ numerically

$\rightsquigarrow$ Here: Grad-Div instead of $\gamma B^T M_p^{-1} B$

Benzi and Olshanskii.
An Augmented Lagrangian-Based Approach to the Oseen Problem.
SIAM J. Sci. Comput, 28:2095–2113, 2006.

The Preconditioner

- Discretized Oseen problem (with Grad-Div in A):

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

- Krylov method with block triangular preconditioner:

$$P^{-1} := \begin{pmatrix} \widetilde{A} & B^T \\ 0 & \widetilde{S} \end{pmatrix}^{-1}$$

with approximations for A, and Schur complement

$$S = -BA^{-1}B^T$$

(see Elman, Silvester, Wathen)

Schur Complement

- Approximate Schur complement:

$$\begin{aligned} S^{-1} &= -\left(BA^{-1}B^T\right)^{-1} \\ &= -\left(B\left[\nu L_u + N + cM_u + \gamma B^T M_p^{-1} B + \gamma Stab\right]^{-1} B^T\right)^{-1} \\ &= -\left(B\left[\nu L_u + N + cM_u + \gamma Stab\right]^{-1} B^T\right)^{-1} - \gamma M_p^{-1} \\ &\approx -\nu M_p^{-1} - cL_p^{-1} - \gamma M_p^{-1} \end{aligned}$$

(L_u,L_p: Laplacian, M_u,M_p: mass matrices)

- Neglect convection term N
  $\rightsquigarrow$ good approximation, if $\nu + c + \gamma \gtrsim \|b\|$

Summary

Augmented Lagrangian:

- Add $\gamma B^T M_p^{-1} B$ to A
- Does not change solution
- Free choice for γ
- Assembly: hard, dense matrix

Grad-Div preconditioner:

- Add Grad-Div to A, which is $\gamma B^T M_p^{-1} B + \gamma Stab$
- Changes solution
- No free choice for γ
- Easy to assemble, sparse

both:

- Schur complement:

$$S^{-1} \approx -(\nu + \gamma) M_p^{-1} - c L_p^{-1}$$

- Increasing γ:
  - improves approximation quality of S
  - makes solving for A harder
- Large enough γ: iteration numbers independent of h, ν, order

Dependency on h, Element Order, and Viscosity

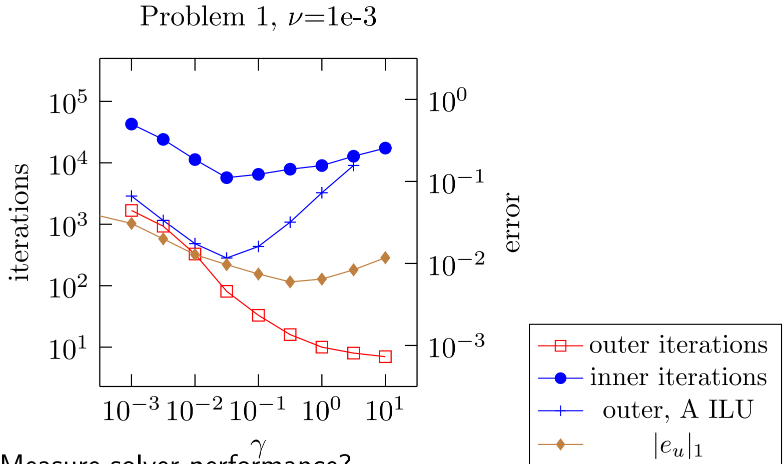
Number of Iterations:

	h: \ ν:	γ=1.0 1e-1	γ=1.0 1e-3	γ=1.0 1e-5	γ=0.31 1e-1	γ=0.31 1e-3	γ=0.31 1e-5	γ=0.1 1e-1	γ=0.1 1e-3	γ=0.1 1e-5
Q2Q1	1/16	13	13	13	19	19	20	28	38	38
	1/64	13	12	12	18	19	19	27	37	37
Q3Q2	1/16	13	13	13	19	20	20	29	38	38
	1/64	13	12	12	18	19	19	27	36	37
Q4Q3	1/16	13	13	13	19	20	20	28	37	38
	1/64	13	12	13	18	19	19	27	36	36

(high numbers due to very small stopping criterion: rel. res. 1e-10)

- As expected: dependent on γ
- Independent of h, element order, ν (this is really good!)

Solver vs. Accuracy: a Tradeoff

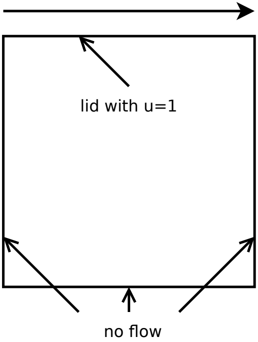


Measure solver performance?

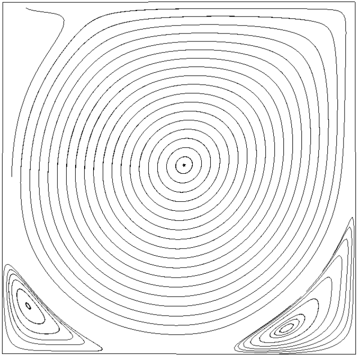
- total number of inner iterations (GMRES + ILU)
- or inner with just an ILU for A
- or factorization? (independent of γ)

Lid-driven Cavity

- Popular benchmark
- Stationary solutions if below critical reynolds number
- Here: treat as stationary Navier-Stokes (nonlinear iteration!), no wall adapted meshes

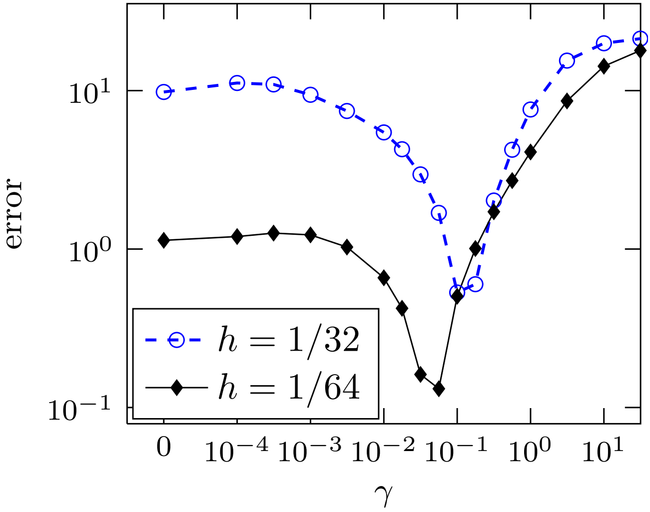


setup



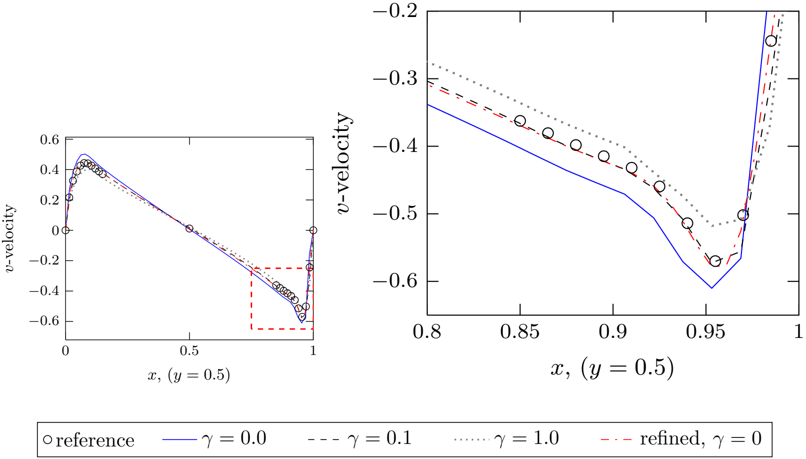
streamlines

Lid-driven Cavity and Grad-Div Stabilization



(error in the minimum of the stream function, Re$= 5000$)

Lid-driven Cavity



Lid-driven Cavity

		$h = 1/32$			$h = 1/64$		
ν	Grad-Div	PCD	GD	#nonlinear	PCD	GD	#nonlinear
1e-2	$\gamma = 0$	13	18	15	13	18	15
	$\gamma = 0.1$	17	**4**	15	16	**5**	15
1e-3	$\gamma = 0$	44	342	34	42	511	29
	$\gamma = 0.1$	91	**6**	31	109	**8**	29
2e-4	$\gamma = 0$	4822	-	104	1031	-	49
	$\gamma = 0.1$	1064	**7**	40	1249	**8**	43

- PCD: state of the art preconditioner (Elman, Silvester, Wathen)
- Number of non-linear and average number of linear iterations per non-linear step
- Regular mesh; $Re = 100$, $Re = 1000$ and $Re = 5000$
- Optimal γ from the error point of view, always $\gamma = 0.1$

Implementation and Parallelization

easy:

- Grad-div stabilization is just another term in the PDE
- Block preconditioner consists of
  matrix multiplications and inner solvers
- Schur complement can be assembled
- No difficulties with boundary conditions
- Also no difficulties in parallel (no mat-mat needed)

Disadvantages

- Mostly useful for stationary problems
- Needs Grad-div stabilization
- Solving for A
- Good parameter γ? Compromise?
- Equal-order elements

source: http://sparklette.net/

Equal order elements?

- Algebraic term in the splitting

$$(\nabla \cdot u, \nabla \cdot v) = (\Pi(\nabla \cdot u), \Pi(\nabla \cdot v)) + (\kappa(\nabla \cdot u), \kappa(\nabla \cdot v))$$

  does no longer vanish:

$$\gamma B^T M_p^{-1} B u = \gamma B^T M_p^{-1} C p \neq 0$$

  because of the (2,2)-block from stabilization C

$\rightsquigarrow$ that means Grad-Div gives feedback from pressure?

- Possible with Augmented Lagrangian
- Not easy with the Grad-Div preconditioner: ☹
- Also: theory gives $\gamma_{EO} = h \cdot \gamma$, too small to be useful?

- "Just a different discretization of Augmented Lagrangian"?
- Competitive(?) alternative to known preconditioners
- Uses and profits from Grad-div stabilization
- Detects regime (diffusion/reaction/convection dominant)
- Implementation/parallelization is easy

Thanks for your attention!