Physics-based preconditioners for large-scale subsurface flow simulations

Kees Vuik

Delft University of Technology

Delft Institute of Applied Mathematics

Schlumberger EUREKA Fluid Mechanics Workshop

July 13 2016

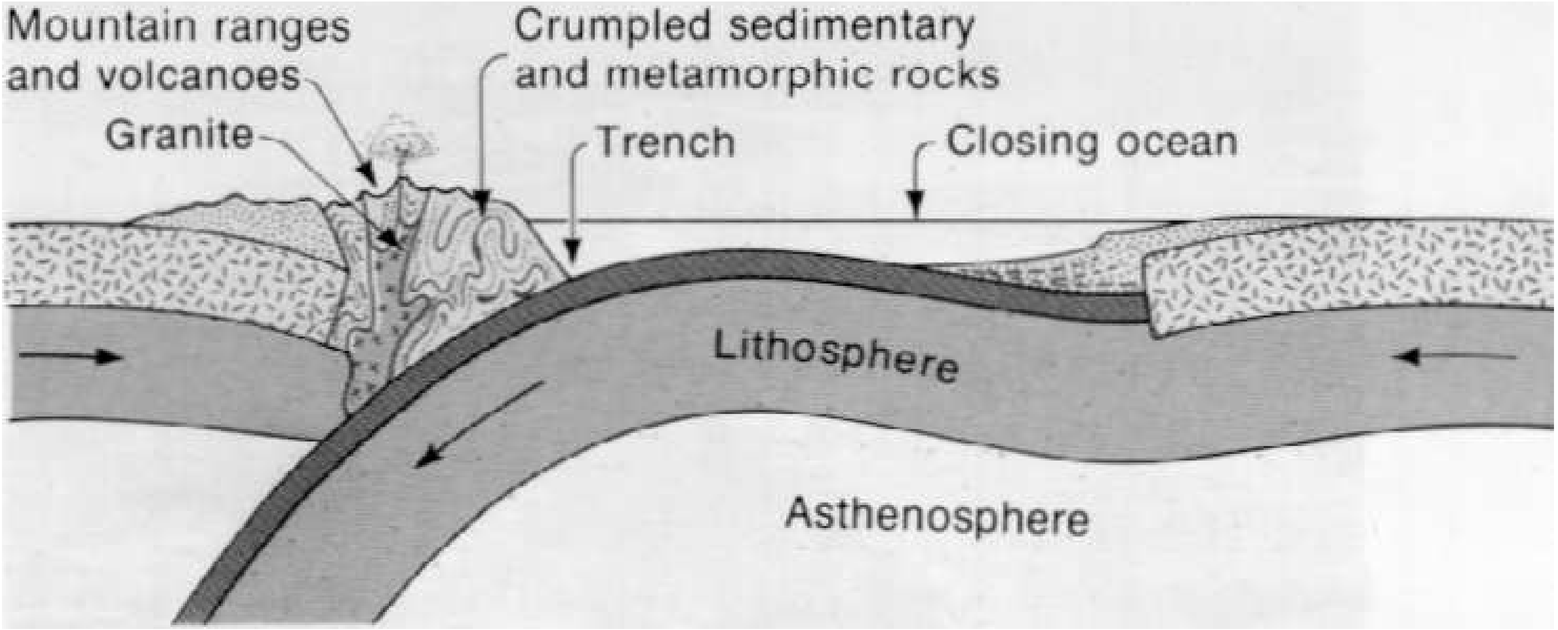
Boston, MA, USA

Contents

1. Introduction

Motivation

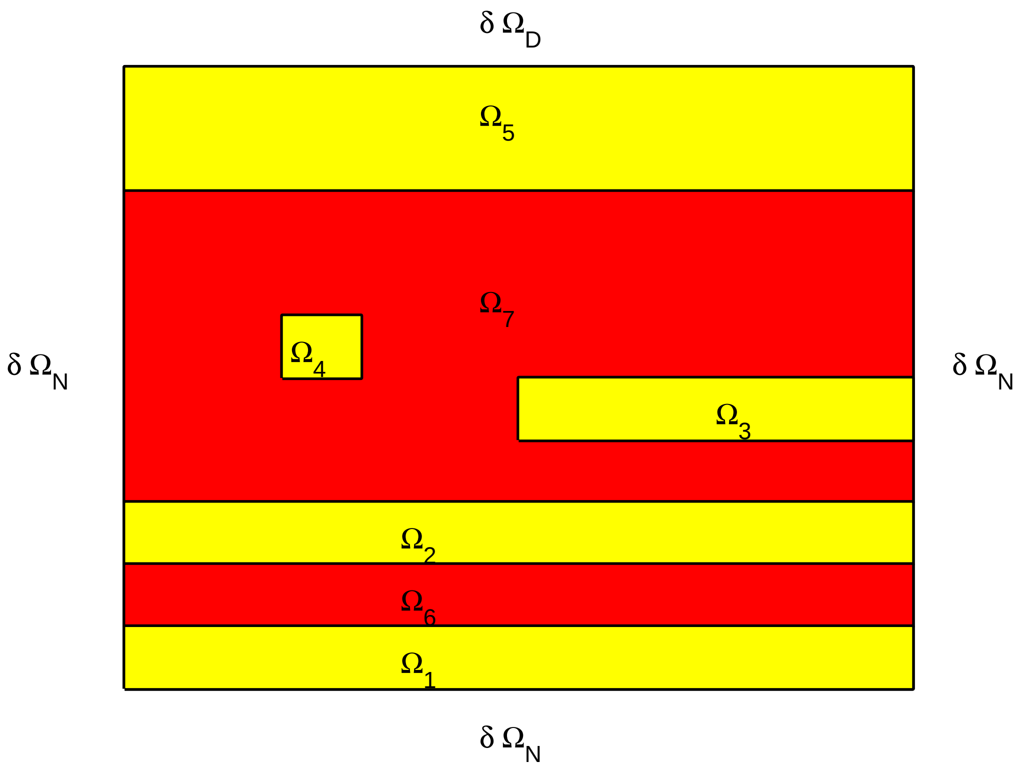
Knowledge of the fluid pressure in rock layers is important for an oil company to predict the presence of oil and gas in reservoirs.



The earth's crust has a layered structure

Mathematical model

Computation of fluid pressure $-\text{div}(\sigma \nabla p(x)) = 0$ on Ω, p fluid pressure, σ permeability



$\sigma_h = 1$ (sand) $\sigma_l = \varepsilon = 10^{-7}$ (shale)

Properties and Applications

$$Ax = b$$

A is sparse and SPD

Condition number of A is $O(10^7)$, due to large contrast in permeability

Applications

- reservoir simulations
- porous media flow
- electrical power networks
- semiconductors
- magnetic field simulations

- fictitious domain methods

2. IC preconditioned CG

Error estimate

$$Ax = b$$

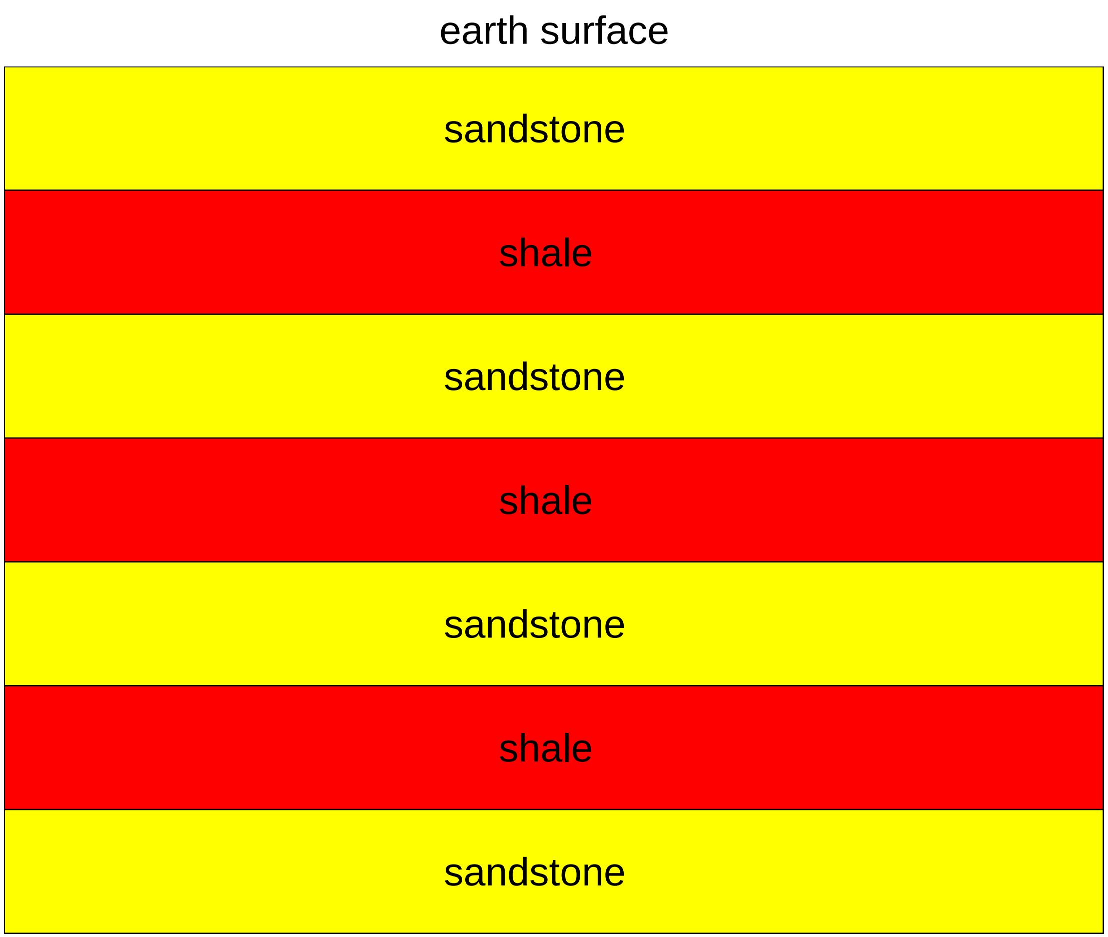
$$M^{-1}Ax = M^{-1}b$$

$$x - x_k = (M^{-1}A)^{-1}M^{-1}A(x - x_k)$$

$$\|x - x_k\|_2 \leq \frac{1}{\lambda_{min}}\|M^{-1}r_k\|_2$$

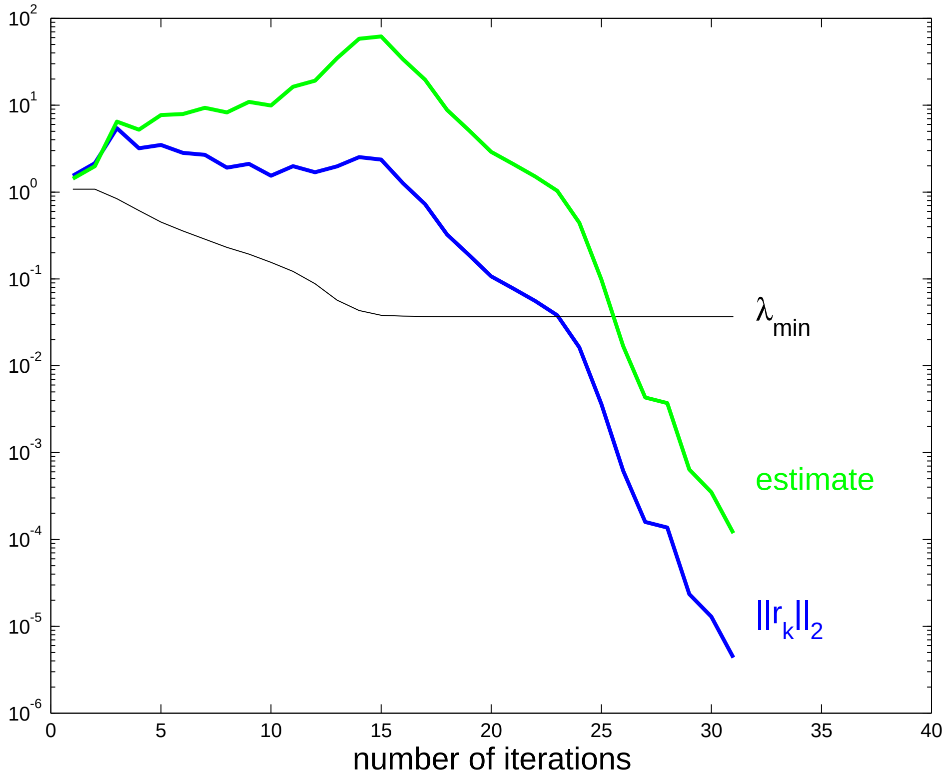
λ_{min}: smallest eigenvalue of $M^{-1}A$

Test problem



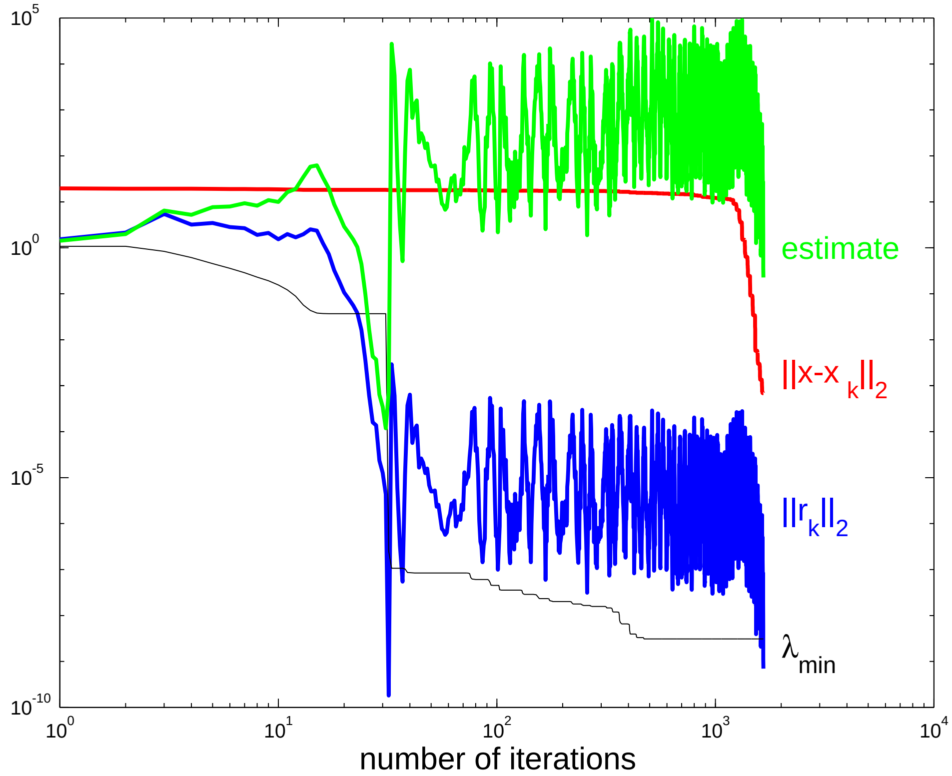
Configuration with 7 straight layers

Convergence CG



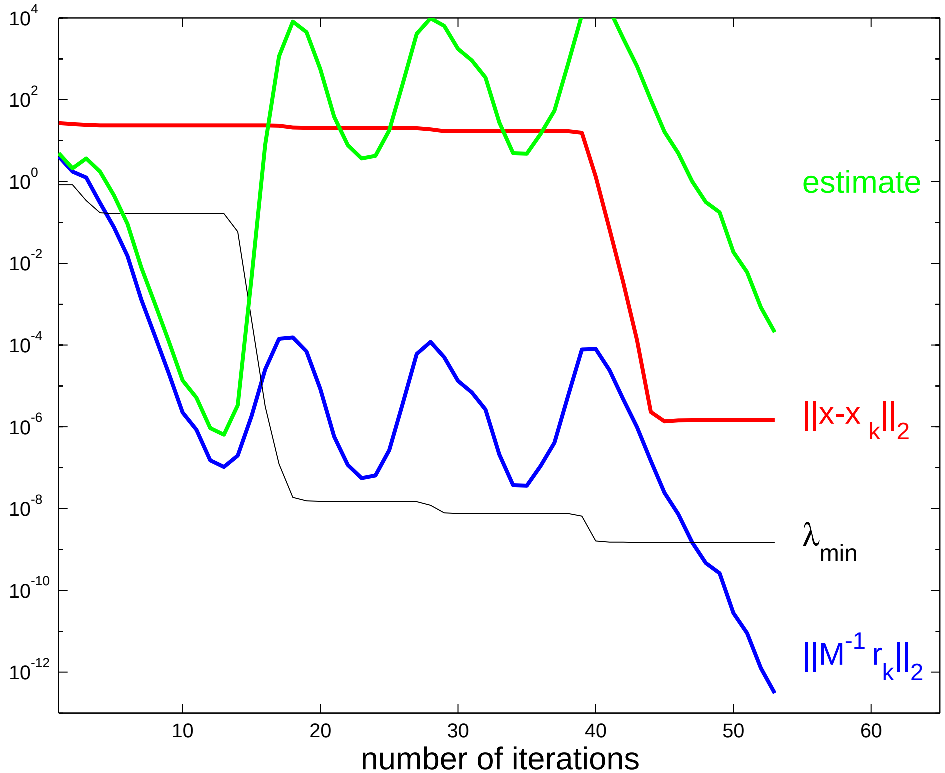
Convergence behavior of CG without preconditioning

Convergence CG



Convergence behavior of CG without preconditioning

Convergence ICCG



Convergence behavior of ICCG

Spectrum of IC preconditioned matrix

L is the Incomplete Cholesky factor of A

k^s is the number of high-permeability domains not connected to a Dirichlet boundary

D is a diagonal matrix $(d_{ii} > 0)$ and $\hat{A} = D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$

Theorem 1 (scaling invariance)
$L^{-1} A L^{-T}$ and $\hat{L}^{-1} \hat{A} \hat{L}^{-T}$ are identical.

Proof:

$\hat{L} = D^{-\frac{1}{2}} L$ and $\hat{L}^{-1} \hat{A} \hat{L}^{-T} = L^{-1} D^{\frac{1}{2}} (D^{-\frac{1}{2}} A D^{-\frac{1}{2}}) D^{\frac{1}{2}} L^{-T} = L^{-1} A L^{-T}$.

Spectrum of IC preconditioned matrix

Take $D = \text{diag}(A)$

Theorem 2

$\hat{A}$ has k^s eigenvalues of $O(\varepsilon)$, where ε is the ratio between high and low permeability.

Theorem 3

The IC preconditioned matrix $L^{-1}AL^{-T}$ has k^s eigenvalues of $O(\varepsilon)$.

Proof: Scaling invariance (Theorem 1) implies

$$\text{spectrum}(L^{-1}AL^{-T}) = \text{spectrum}(\hat{L}^{-1}\hat{A}\hat{L}^{-T})$$

In [Vuik, Segal, Meijerink, Wijma, 2001] we have shown that the number and size of small eigenvalues of $\hat{A}$ and $\hat{L}^{-1}\hat{A}\hat{L}^{-T}$ are the same. The theorem is proven by using Theorem 2. ⊠

3. Deflated ICCG

Idea: remove the bad eigenvectors from the error/residual.

Krylov	Ar
Preconditioned Krylov	$M^{-1}Ar$
Block Preconditioned Krylov	$\sum_{i=1}^{m}(M_i^{-1})Ar$
Block Preconditioned Deflated Krylov	$\sum_{i=1}^{m}(M_i^{-1})PAr$

3. Deflated ICCG

Idea: remove the bad eigenvectors from the error/residual.

Various choices are possible:

- Projection vectors
  Physical vectors, eigenvectors, coarse grid projection vectors (constant, linear, ...)
- Projection method
  Deflation, coarse grid projection, balancing, augmented, FETI
- Implementation
  sparseness, with(out) using projection properties, optimized, ...

Literature

Deflated CG (start)

Nicolaides 1987, Mansfield 1990

Literature

Deflated CG (start)

Nicolaides 1987, Mansfield 1990

Deflated CG (further development)

Graham and Hagger 1997, 1999, Kolotilina 1998, Vuik, Segal and Meijerink 1999, Saad, Yeung, Erhel and Guyomarc'h 2000, Frank and Vuik 2001, Aksoylu, Rodriguez, Klie and Wheeler 2006, Nabben and Vuik 2004, 2006, Scheichl and Graham, 2006

Literature

Deflated CG (start)

Nicolaides 1987, Mansfield 1990

Deflated CG (further development)

Graham and Hagger 1997, 1999, Kolotilina 1998, Vuik, Segal and Meijerink 1999, Saad, Yeung, Erhel and Guyomarc'h 2000, Frank and Vuik 2001, Aksoylu, Rodriguez, Klie and Wheeler 2006, Nabben and Vuik 2004, 2006, Scheichl and Graham, 2006

Deflation and restarted GMRES

Morgan 1995, Erhel, Burrage and Pohl 1996, Chapman and Saad 1997, Morgan 2002, Giraud and Gratton 2005, Kilmer and De Sturler 2006

Deflated ICCG

A is SPD, Conjugate Gradients

$P = I - AZE^{-1}Z^T$ with $E = Z^T AZ$

and $Z = [z_1 ... z_m]$, where $z_1, ..., z_m$ are independent deflation vectors.

Properties

1. $P^T Z = 0$ and $PAZ = 0$
2. $P^2 = P$
3. $AP^T = PA$

Deflated ICCG

$$x = (I - P^T)x + P^T x$$

$$(I - P^T)x = ZE^{-1}Z^TAx = ZE^{-1}Z^Tb, \qquad AP^Tx = PAx = Pb$$

Deflated ICCG

$$x = (I - P^T)x + P^T x$$

$$(I - P^T)x = ZE^{-1}Z^TAx = ZE^{-1}Z^Tb, \qquad AP^Tx = PAx = Pb$$

DICCG

$k = 0,\ \hat{r}_0 = Pr_0,\ p_1 = z_1 = L^{-T}L^{-1}\hat{r}_0;$

while $\|\hat{r}_k\|_2 > \varepsilon$ **do**

$k = k + 1;$

$\alpha_k = \frac{(\hat{r}_{k-1}, z_{k-1})}{(p_k, PAp_k)};$

$x_k = x_{k-1} + \alpha_k p_k;$

$\hat{r}_k = \hat{r}_{k-1} - \alpha_k PAp_k;$

$z_k = L^{-T}L^{-1}\hat{r}_k;$

$\beta_k = \frac{(\hat{r}_k, z_k)}{(\hat{r}_{k-1}, z_{k-1})}; \qquad p_{k+1} = z_k + \beta_k p_k;$

end while

Convergence and termination criterion

Choose z_1 , z_2 , z_3 eigenvectors of $L^{-T}L^{-1}A$

Convergence

$$\|P^T x - P^T x_k\|_2 \leq 2\sqrt{K}\|P^T x - P^T x_0\|_2 \left(\frac{\sqrt{K}-1}{\sqrt{K}+1}\right)^k$$

where $K = \frac{\lambda_n}{\lambda_4}$

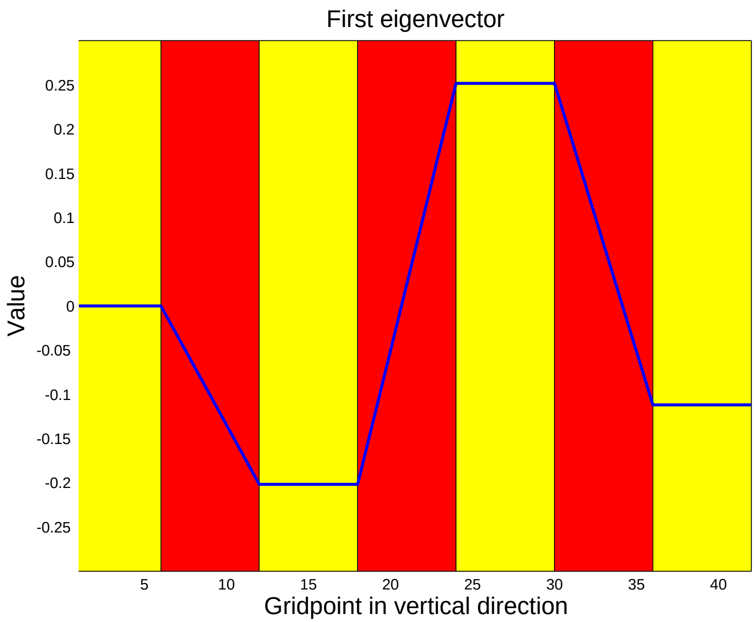
Termination criterion

$$\|L^{-T}L^{-1}Pb - L^{-T}L^{-1}PAx_k\|_2 \leq \frac{\delta}{\lambda_4} \text{ implies } \|P^T x - P^T x_k\|_2 \leq \delta$$

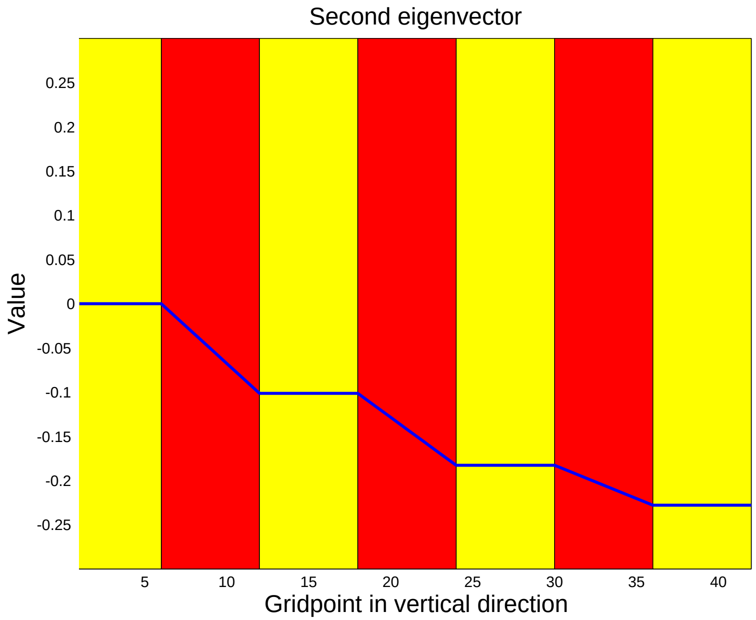
Deflation vectors

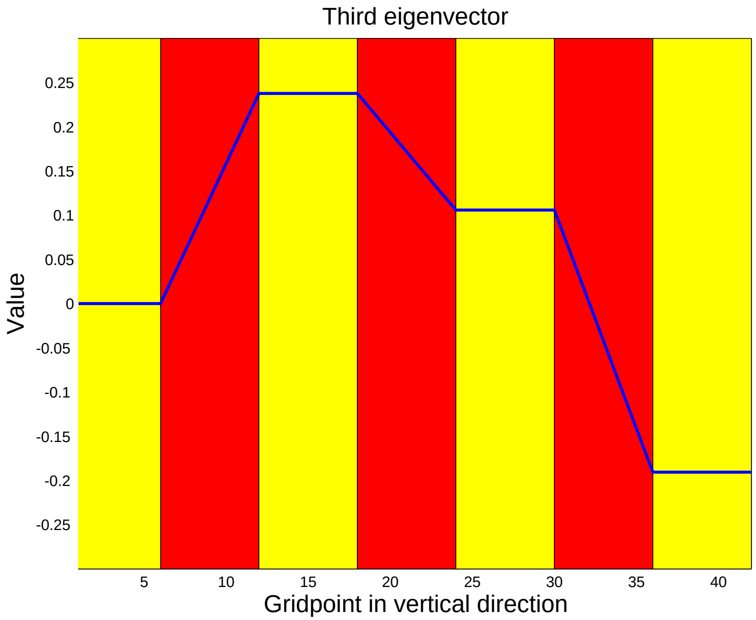
Choose eigenvectors of $L^{-T}L^{-1}A$. Properties of cross sections:

- a constant value in sandstone layers
- in shale layers their graph is linear



Eigenvectors of $L^{-T}L^{-1}A$





4. Physical deflation vectors

k is number of subdomains

$\Omega_i, i = 1, ..., k^s$ high-permeability subdomains without a Dirichlet B.C.;
$i = k^s + 1, ..., k^h$ remaining high-permeability subdomains

- define z_i for $i \in \{1, ..., k^s\}$
- $z_i = 1$ on $\bar{\Omega}_i$ and $z_i = 0$ on $\bar{\Omega}_j, j \neq i, j \in \{1, ..., k^h\}$
- z_i satisfies equation:

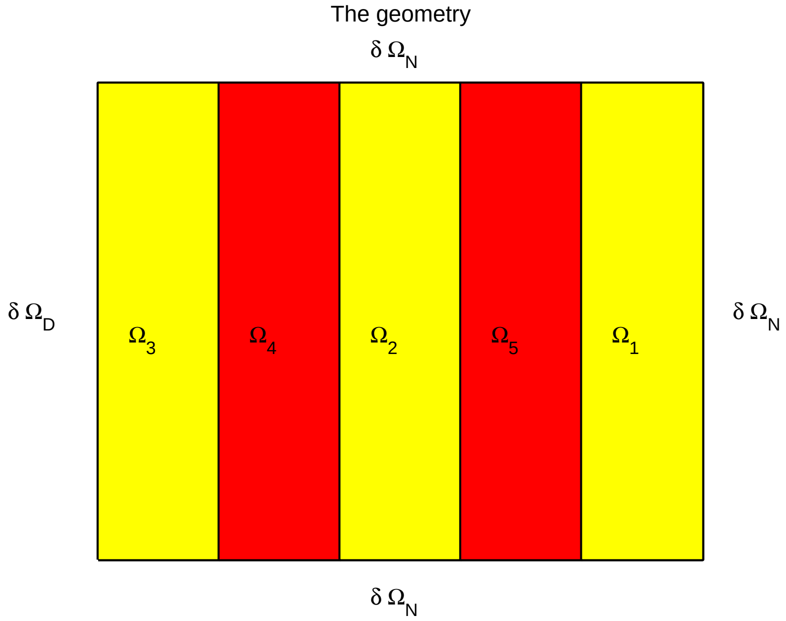
$$-\text{div}(\sigma_j \nabla z_i) = 0 \text{ on } \Omega_j, j \in \{k^h + 1, ..., k\},$$

with appropriate boundary conditions

Sparse vectors, subproblems are cheap to solve

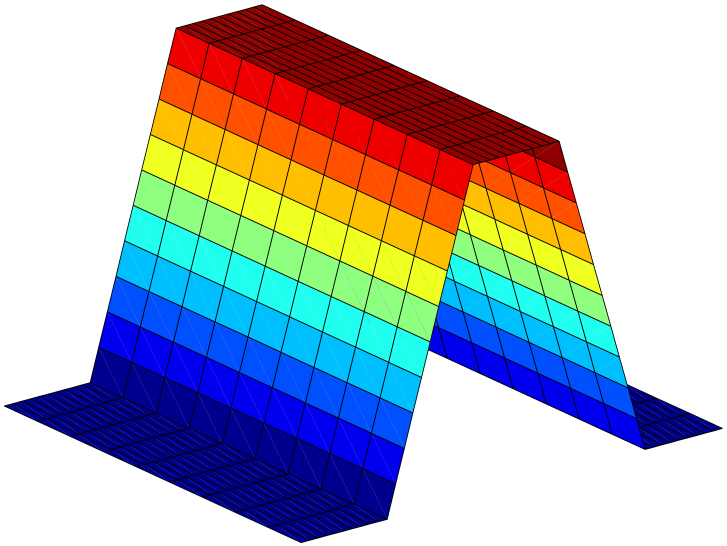
Physical deflation vectors

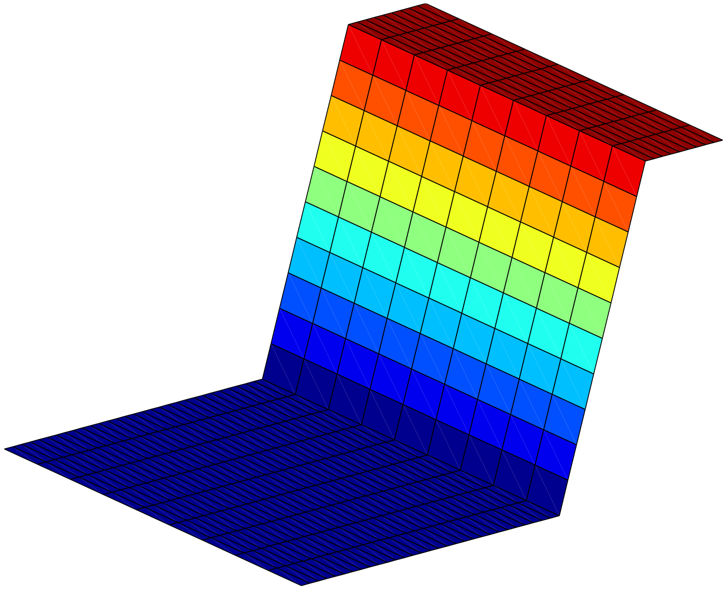
Example with $k_s = 2, k_h = 3$, and $k = 5$



Physical deflation vectors

Example with $k_s = 2, k_h = 3$, and $k = 5$





Properties

Theorem 4

The deflation vectors are such that for $D = \text{diag}(A)$

- $\|D^{-1}Az_i\|_\infty = O(\varepsilon)$
- $\|L^{-T}L^{-1}Az_i\|_2 = O(\varepsilon)$

Define $Z = [z_1 ... z_{k^s}]$ and $U = [u_1 ... u_{k^s}]$, where u_i are 'small' eigenvectors.

Theorem 5

There is a matrix X such that $Z = UX + E$, with $\|E\|_2 = O(\sqrt{\varepsilon})$

Sensitivity of deflation vectors

- Random vector added in shale layers (amplitude $\alpha/2$)

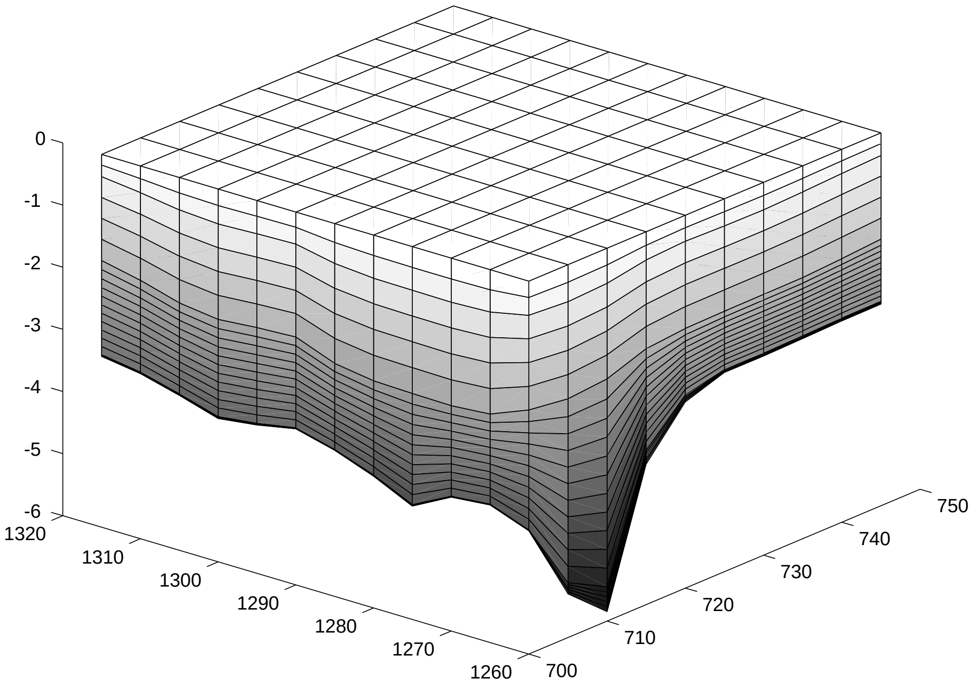
α	0	10^{-1}	1	ICCG
λ_{per}	0.164	0.164	$8.2 \cdot 10^{-3}$	$1.6 \cdot 10^{-9}$
iter	14	15	24	54

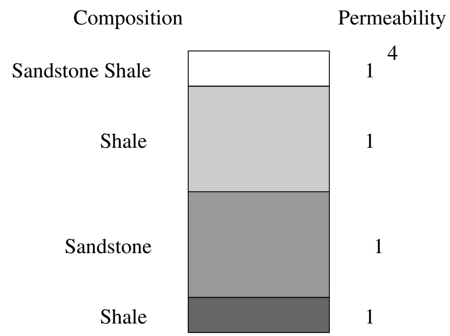
- Random vector added to the nonzero parts

α	0	10^{-3}	10^{-1}	ICCG
λ_{per}	0.164	$9 \cdot 10^{-4}$	$9 \cdot 10^{-8}$	$1.6 \cdot 10^{-9}$
iter	14	27	56	54

After perturbation the smallest eigenvalues remain exactly zero, however, the smallest non-zero eigenvalue can change considerably.

Geometry oil flow problem





Results oil flow problem

Varying σ_{shale}

σ	ICCG		DICCG	
	λ_{min}	iter	λ_{min}	iter
10^{-3}	$1.5 \cdot 10^{-2}$	26	$6.9 \cdot 10^{-2}$	20
10^{-5}	$2.2 \cdot 10^{-4}$	59	$7.7 \cdot 10^{-2}$	20
10^{-7}	$2.3 \cdot 10^{-6}$	82	$7.7 \cdot 10^{-2}$	20

Varying accuracy

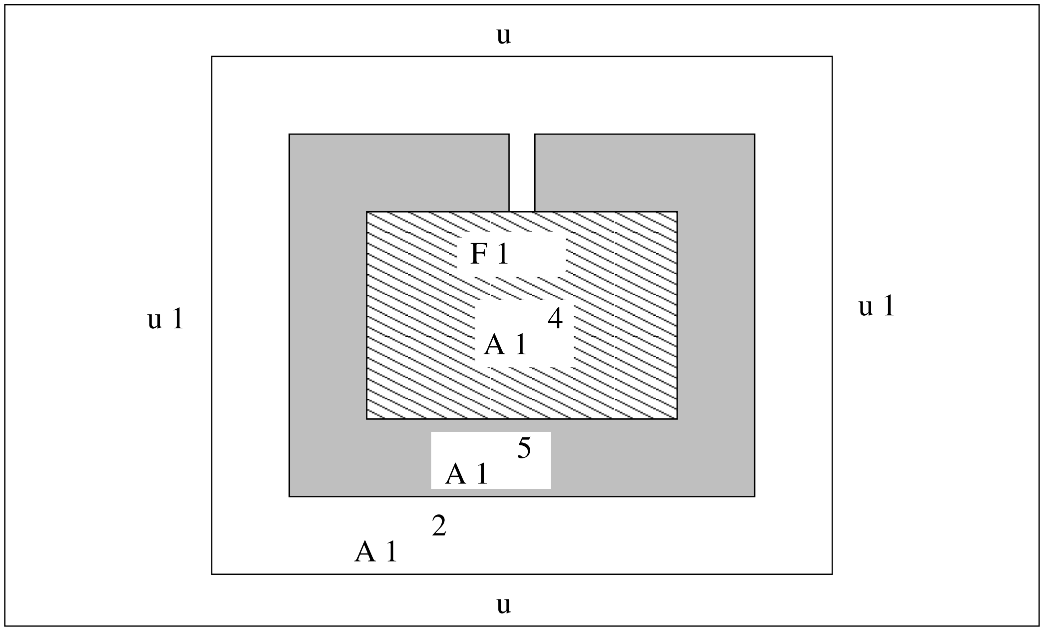
accuracy	ICCG		DICCG	
	iter	CPU	iter	CPU
10^{-5}	82	18.9	20	6.3
10^{-3}	78	18.0	12	4.1
10^{-1}	75	17.2	2	1.2

A groundwater flow problem

The pressure in groundwater satisfies the equation:

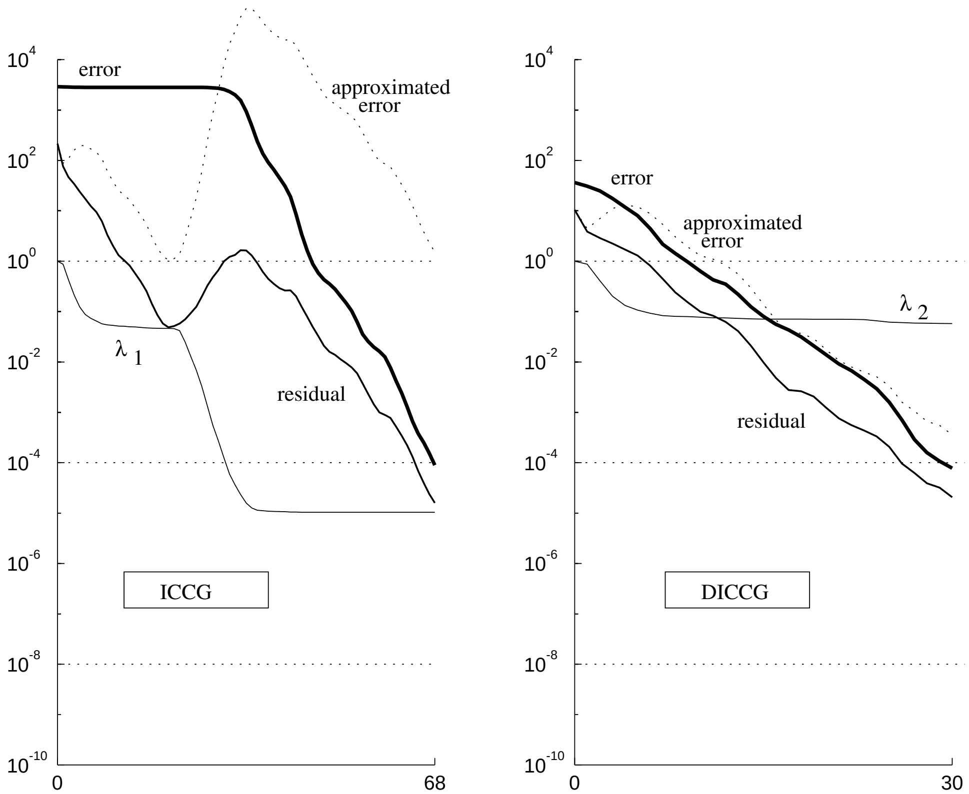
$$-\nabla \cdot (A\nabla u) = F, \quad (1)$$

where the coefficients and geometry of the problem are:



A groundwater flow problem

The low permeable layer ($A = 10^{-5}$) and the jump in permeabilities between the two sand sections lead to a 'small' eigenvalue.



5. Conclusions

- DICCG is a robust and efficient method to solve diffusion problems with discontinuous coefficients.
- The choice of the projection vectors is important for the success of a projection method.
- For layered problems the physical deflation vectors are the optimal choice for the projection vectors.
- For many problems a second level preconditioner (Deflation) saves a lot of CPU time.

Further information

- http://ta.twi.tudelft.nl/nw/users/vuik/pub_it_def.html
- C. Vuik, A. Segal and J.A. Meijerink
  An efficient preconditioned CG method for the solution of a class of layered problems with extreme contrasts in the coefficients
  J. Comp. Phys., 152, pp. 385-403, 1999.
- J. Frank and C. Vuik
  On the construction of deflation-based preconditioners
  SIAM Journal on Scientific Computing, 23, pp. 442–462, 2001
- R. Nabben and C. Vuik
  A comparison of Deflation and Coarse Grid Correction applied to porous media flow
  SIAM J. on Numerical Analysis, 42, pp. 1631-1647, 2004
- R. Nabben and C. Vuik
  A comparison of Deflation and the Balancing preconditioner
  SIAM Journal on Scientific Computing, 27, pp. 1742-1759, 2006

Physics-based preconditioners for large-scale subsurface flow simulation Part 2

Kees Vuik [1], Gabriela B. Diaz Cortes [1], Jan Dirk Jansen [2].

[1]Department of Applied Mathematics, TU Delft.

[2]Department of Geoscience & Engineering, TU Delft.

Schlumberger EUREKA Fluid Mechanics Workshop, July 13 2016, Boston, MA, USA

Optimal Control

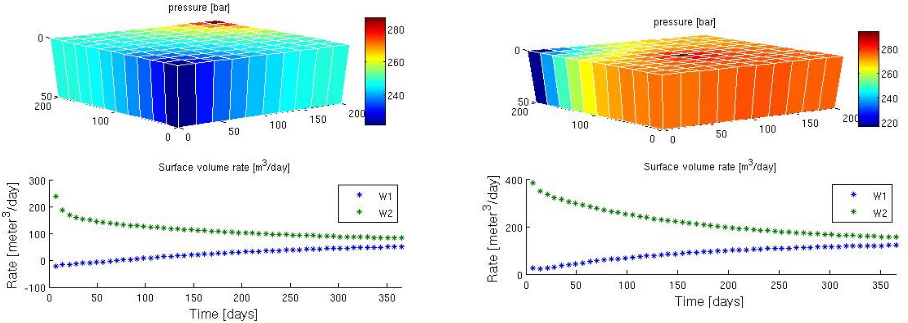


Figure : Optimal Control[1].

[1] MRST [1]

Reservoir Simulation

Single-phase flow through porous media [2]

Darcy's law + mass balance equation

$$-\nabla \cdot \left[\frac{\alpha\rho}{\mu} \vec{\mathbf{K}} (\nabla \mathbf{p} - \rho g \nabla d) \right] + \alpha\rho\phi c_t \frac{\partial \mathbf{p}}{\partial t} - \alpha\rho\mathbf{q} = 0.$$

$$c_t = (c_l + cr),$$

α a geometric factor

ρ fluid density

μ fluid viscosity

$\mathbf{p}$ pressure

$\vec{\mathbf{K}}$ rock permeability

g gravity

d depth

ϕ rock porosity

q sources

c_r rock compressibility

c_l liquid compressibility

Problem Definition

Discretization

2D case, isotropic permeability, small rock and fluid compressibilities, uniform reservoir thickness and no gravity forces.

$$-\frac{h}{\mu}\frac{\partial}{\partial x}\left(k\frac{\partial \mathbf{p}}{\partial x}\right)-\frac{h}{\mu}\frac{\partial}{\partial y}\left(k\frac{\partial \mathbf{p}}{\partial y}\right)-\frac{h}{\mu}\frac{\partial}{\partial z}\left(k\frac{\partial \mathbf{p}}{\partial z}\right)+h\phi_0 c_t\frac{\partial \mathbf{p}}{\partial t}-h\mathbf{q}=0.$$

$$\boxed{\mathcal{V}\dot{\mathbf{p}}+\mathcal{T}\mathbf{p}=\mathbf{q}.}$$

Accumulation matrix

$$\mathcal{V}=Vc_t\phi_0\mathcal{I},$$

$$V=h\Delta x\Delta y\Delta z.$$

Transmissibility matrix

$$\mathcal{T}_{i-\frac{1}{2},j,l}=\frac{\Delta y}{\Delta x}\frac{h}{\mu}k_{i-\frac{1}{2},j,l},$$

$$k_{i-\frac{1}{2},j}=\frac{2}{\frac{1}{k_{i-1,j,l}}+\frac{1}{k_{i,j,l}}}.$$

Problem Definition

Incompressible model

$$\boxed{\mathcal{T}\mathbf{p} = \mathbf{q}.}$$

Properties of $\mathcal{T}$

Eigenvalues

$$\mathcal{T}\mathbf{p} = \lambda\mathbf{p}$$

Condition number of a SPD matrix.

$$\kappa_2(\mathcal{T}) = \frac{\lambda_{max}(\mathcal{T})}{\lambda_{min}(\mathcal{T})}$$

$\mathbf{q}$: sources or wells in the reservoir.

Peaceman well model

$$\mathbf{q} = -J_{well}(\mathbf{p} - \mathbf{p}_{well})$$

J_{well} is the well index, negative sign is a production well.

POD

Proper Orthogonal Decomposition (POD)

POD: find an 'optimal' basis Φ for a given data set (Markovinović et al. 2009 [5], Astrid et al. 2011 [6])

$$\Phi = [\phi_1, \phi_2, ...\phi_l] \in \mathbb{R}^{n\times l} \qquad \phi_i, \text{basis functions.}$$

- Get the snapshots
$$\mathcal{X} = [\mathbf{x}_1, \mathbf{x}_2, ...\mathbf{x}_m].$$
- Compute $\mathcal{R}$
$$\mathcal{R} := \frac{1}{m}\mathcal{X}\mathcal{X}^T \equiv \frac{1}{m}\sum_{i=1}^{m}\mathbf{x}_i\mathbf{x}_i^T.$$
- Basis functions: eigenvectors of the maximal number (l) of eigenvalues satisfying [7]:
$$\frac{\sum_{j=1}^{l}\lambda_j}{\sum_{j=1}^{m}\lambda_j} \le \alpha, \qquad 0 < \alpha \le 1, \tag{1}$$
with α close to 1 (eigenvalues are ordered from large to small with λ_1 the largest eigenvalue of $\mathcal{R}$).

Deflation vectors

Recycling deflation (Clemens 2004, [8]).

$$\mathcal{Z} = [\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^{q-1}],$$

x^i's are solutions of the system.
Multigrid and multilevel (Tang 2009, [9]).
The matrices $\mathcal{Z}$ and $\mathcal{Z}^T$ are the restriction and prolongation matrices of multigrid methods.
Subdomain deflation (Vuik 1999,[10]).

Proposal

- Use solution of the system with various well configurations as deflation vectors (Recycling deflation).
- Use as deflation vectors the basis obtained from Proper Orthogonal Decomposition (POD).

Deflation vectors

Lemma 1. Let $\mathcal{A} \in \mathbb{R}^{n \times n}$ be a non-singular matrix, such that

$$\mathcal{A}\mathbf{x} = \mathbf{b}, \tag{2}$$

and $\mathbf{x}_i, \mathbf{b}_i \in \mathbb{R}^n$, $i = 1, ..., m$, $\mathbf{b}_i$ are linearly independent ($l.i.$) such that:

$$\mathcal{A}\mathbf{x}_i = \mathbf{b}_i, \tag{3}$$

The following equivalence holds

$$\mathbf{x} = \sum_{i=1}^{m} c_i \mathbf{x}_i \qquad \Leftrightarrow \qquad \mathbf{b} = \sum_{i=1}^{m} c_i \mathbf{b}_i. \tag{4}$$

Proof $\Rightarrow$ Substituting $\mathbf{x}$ from (4) into $\mathcal{A}\mathbf{x} = \mathbf{b}$, and using linearity of $\mathcal{A}$ and(3):

$$\begin{aligned}\mathcal{A}\mathbf{x} &= \sum_{i=1}^{m} \mathcal{A} c_i \mathbf{x}_i = \mathcal{A}(c_1\mathbf{x}_1 + ... + c_m\mathbf{x}_m) \\ &= \mathcal{A}c_1\mathbf{x}_1 + ... + \mathcal{A}c_m\mathbf{x}_m = c_1\mathbf{b}_1 + ... + c_m\mathbf{b}_m = \sum_{i=1}^{m} c_i \mathbf{b}_i.\end{aligned} \tag{5}$$

Similar proof for $\Leftarrow$

Deflation vectors

Lemma 2. If the the deflation matrix $\mathcal{Z}$ is constructed with a set of m vectors

$$\mathcal{Z} = \begin{bmatrix} \mathbf{x}_1 & \dots & \dots & \mathbf{x}_m \end{bmatrix},$$

such that $\mathbf{x} = \sum_{i=1}^{m} c_i \mathbf{x}_i$, with $\mathbf{x}_i$ $l.i.$, then the solution of system $\mathcal{A}\mathbf{x} = \mathbf{b}$ is achieved within one iteration of DCG.
Proof.
The relation between $\hat{\mathbf{x}}$ and $\mathbf{x}$ is given as:

$$\mathbf{x} = \mathcal{Q}\mathbf{b} + \mathcal{P}^T \hat{\mathbf{x}}. \tag{6}$$

For the first term $\mathcal{Q}\mathbf{b}$, taking $\mathbf{b} = \sum_{i=1}^{m} c_i \mathbf{b}_i$ we have:

$$\mathcal{Q}\mathbf{b} = \mathcal{Z}\mathcal{E}^{-1}\mathcal{Z}^T \left(\sum_{i=1}^{m} c_i \mathbf{b}_i \right) = \mathcal{Z}(\mathcal{Z}^T \mathcal{A}\mathcal{Z})^{-1}\mathcal{Z}^T \left(\sum_{i=1}^{m} c_i \mathcal{A}\mathbf{x}_i \right) = \qquad \text{Lemma 1}$$
$$= \mathcal{Z}(\mathcal{Z}^T \mathcal{A}\mathcal{Z})^{-1}\mathcal{Z}^T (\mathcal{A}\mathbf{x}_1 c_1 + \dots + \mathcal{A}\mathbf{x}_m c_m) = \mathcal{Z}(\mathcal{Z}^T \mathcal{A}\mathcal{Z})^{-1}\mathcal{Z}^T \mathcal{A}\mathcal{Z}\mathbf{c}$$
$$= \mathcal{Z}\mathbf{c} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 + c_4\mathbf{x}_4 + c_5\mathbf{x}_5 = \sum_{i=1}^{m} c_i \mathbf{x}_i = \mathbf{x}$$

Deflation vectors

Lemma 2 (second part).
For the second term of Equation (6), $\mathcal{P}^T\hat{\mathbf{x}}$, we compute $\hat{\mathbf{x}}$ from the deflated system:

$$\begin{aligned}
\mathcal{P}\mathcal{A}\hat{\mathbf{x}} &= \mathcal{P}\mathbf{b} \\
\mathcal{A}\mathcal{P}^T\hat{\mathbf{x}} &= (\mathcal{I} - \mathcal{A}\mathcal{Q})\mathbf{b} \qquad \text{using } \mathcal{A}\mathcal{P}^T = \mathcal{P}\mathcal{A} \text{ [4] and definition of } \mathcal{P}, \\
\mathcal{A}\mathcal{P}^T\hat{\mathbf{x}} &= \mathbf{b} - \mathcal{A}\mathcal{Q}\mathbf{b} \\
\mathcal{A}\mathcal{P}^T\hat{\mathbf{x}} &= \mathbf{b} - \mathcal{A}\mathbf{x} = 0 \qquad \text{taking } \mathcal{Q}\mathbf{b} = \mathbf{x} \text{ from above,} \\
\mathcal{P}^T\hat{\mathbf{x}} &= 0 \qquad \text{as } \mathcal{A} \text{ is invertible.}
\end{aligned}$$

Then we have achieve the solution $\mathbf{x}$ in one step of DICCG.

Case 1. Heterogeneous permeability.

The experiments were performed for single-phase flow, with the following characteristics:

Grid size $nx \times ny$ grid cells, $nx = ny = 64$.

Permeability $\sigma_1 = 1mD$, σ_2 variable.

W1 = W2 = W3 = W4 = -1 bars.

W5 = +4 bars.

Neumann boundary conditions.

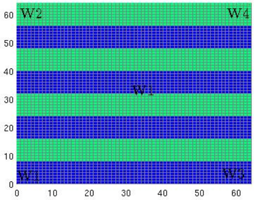


Figure : Model.

Snapshots

$\mathbf{z}_1$: W1 = 0 bars, W2 = W3 = W4 = -1 bars, W5 = b5 = +3 bars.
$\mathbf{z}_2$: W2 = 0 bars, W1 = W3 = W4 = -1 bars, W5 = b5 = +3 bars.
$\mathbf{z}_3$: W3 = 0 bars, W1 = W3 = W4 = -1 bars, W5 = b5 = +3 bars.
$\mathbf{z}_4$: W4 = 0 bars, W1 = W2 = W3 = -1 bars, W5 = b5 = +3 bars.
$\mathbf{z}_5$: W1 = W2 = W3 = W4 = -1 bars, W5 = b5 = +4 bars.

Results

σ_2 (mD)	10^{-1}	10^{-3}	10^{-5}	10^{-7}
ICCG	90	131	65*	64*
DICCG_4	1	1	1*	1*
DICCG_5	1	500*	500*	500*

Table : Number of iterations for different contrast in the permeability of the layers ($\sigma_1 = 1mD$) for the ICCG and DICCG methods, tolerance of 10^{-11}, snapshots 10^{-11}. DICCG_4 is the method with 4 deflation vectors and DICCG_5 is the method with 5 deflation vectors.

Numerical experiments (Heterogeneous permeability), POD

Snapshots

$\mathbf{z}_1$: W1 = 0 bars, W2 = W3 = W4 = -1 bars, W5 = b5 = +3 bars.
$\mathbf{z}_2$: W2 = 0 bars, W1 = W3 = W4 = -1 bars, W5 = b5 = +3 bars.
$\mathbf{z}_3$: W3 = 0 bars, W1 = W3 = W4 = -1 bars, W5 = b5 = +3 bars.
$\mathbf{z}_4$: W4 = 0 bars, W1 = W2 = W3 = -1 bars, W5 = b5 = +3 bars.

We use 4 snapshots and 2 POD basis vectors as deflation vectors.
Results

σ_2 (mD)	10^{-1}	10^{-3}	10^{-5}	10^{-7}
ICCG	90	131	65*	64*
DICCG	1	1	1*	1*
DICCG_{POD}	1	1	1*	1*

Table : Table with the number of iterations for different contrast in the permeability of the layers ($\sigma_1 = 1mD$), for the ICCG, DICCG and DICCG_{POD} methods, tolerance of solvers and snapshots 10^{-11}.

Numerical experiments (Heterogeneous permeability)

Condition number of a SPD matrix.

$$\kappa_2(\mathcal{A}) = \frac{\lambda_{max}(\mathcal{A})}{\lambda_{min}(\mathcal{A})}$$

σ_2 (mD)	10^{-1}	10^{-3}	10^{-5}	10^{-7}
$\kappa(A)$	2.6×10^3	2.4×10^5	2.4×10^7	2.4×10^9
$\kappa(M^{-1}A)$	206.7	8.3×10^3	8.3×10^5	8.3×10^7
$\kappa_{eff}(M^{-1}PA)$	83.27	6×10^3	1×10^6	6×10^7

Table : Condition number for various permeability contrasts between the layers, grid size of 32 x 32, $\sigma_1 = 1mD$.

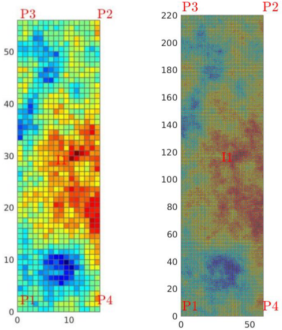
Relative error, $e = \frac{||\mathbf{x}-\mathbf{x}^k||_2}{||\mathbf{x}||_2} \leq \kappa_2(A)\epsilon$, $\mathbf{x}$: true solution, $\mathbf{x}^k$: approximation.
Taking $e = 10^{-7}$,

σ_2 (mD)	10^{-1}	10^{-3}	10^{-5}	10^{-7}
$tol = \frac{e}{\kappa_2(M^{-1}A)} = \frac{10^{-7}}{\kappa_2(M^{-1}A)}$	5×10^{-9}	1×10^{-10}	1×10^{-12}	1×10^{-14}
$tol = \frac{e}{\kappa_{eff}(M^{-1}PA)} = \frac{10^{-7}}{\kappa_{eff}(M^{-1}PA)}$	1×10^{-8}	2×10^{-10}	1×10^{-12}	2×10^{-14}

Table : Tolerance needed for various permeability contrast between the layers, grid size of 32 x 32, $\sigma_1 = 1mD$, for an error of $e = 10^{-7}$.

Numerical experiments (SPE 10)

SPE 10 model, 2nd layer



(a) 16 x 56 (b) 60 x 220

Figure : Permeability field, 16 x 56 and 60 x 220 grid cells.

Grid size	16 x 56	30 x 110	46 x 166	60 x 220
Contrast ($\times 10^7$)	1.04	2.52	2.6	2.8

Table : Contrast in permeability for different grid sizes ($\sigma_{max}/\sigma_{min}$).

Condition number	value
$\kappa(A)$	2.2×10^6
$\kappa(M^{-1}A)$	377
$\kappa_{eff}(M^{-1}PA)$	82.7

Table : Table with the condition number of the SPE10 model, grid size of 16 x 56.

Numerical experiments (SPE 10)

SPE 10 model, 2nd layer

4 and 5 snapshots used as deflation vectors

Tol (snap-shots)	Method	16 x 56	30 x 110	46 x 166	60 x 220
	ICCG	34	73	126	159
10^{-1}	$DICCG_4$	33	72	125	158
	$DICCG_5$	500*	500*	500*	500*
10^{-3}	$DICCG_4$	18	38	123	151
	$DICCG_5$	18	35	123	150
10^{-5}	$DICCG_4$	11	21	27	55
	$DICCG_5$	9	22	23	54
10^{-7}	$DICCG_4$	1	1	1	1
	$DICCG_5$	1	1	1	1

Table : Number of iterations for ICCG and DICCG, diverse tolerance for the snapshots, different grid sizes. $DICCG_4$ is computed with 4 deflation vectors, $DICCG_5$ with 5.

Numerical experiments (SPE 10)

SPE 10 model, 2nd layer, POD

4 snapshots and 2 POD vectors used as deflation vectors

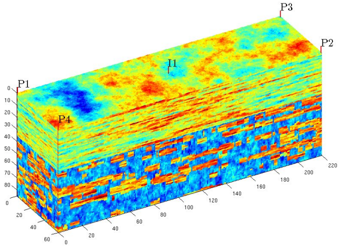
Tol	Method	16 x 56	30 x 110	46 x 166	60 x 220
	ICCG	34	73	126	159
10^{-1}	DICCG	33	72	125	158
	DICCG_{2POD}	33	72	125	158
10^{-3}	DICCG	18	38	123	151
	DICCG_{2POD}	21	40	123	153
10^{-5}	DICCG	11	21	27	55
	DICCG_{2POD}	11	21	27	48
10^{-7}	DICCG	1	1	1	1
	DICCG_{2POD}	1	1	1	1

Table : Table with the number of iterations for ICCG, DICCG and DICCG_{POD}, various tolerance for the snapshots, various grid sizes.

Numerical experiments (SPE 10)

SPE 10 model, 85 layers

Single-phase flow, grid size 60 x 220 x 85 grid cells.

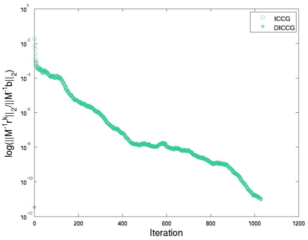


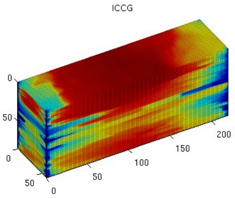
Tol. snapshots	Method	Iterations
	ICCG	1029
10^{-2}	$DICCG_4$	1029
	$DICCG_{2POD}$	1029
10^{-5}	$DICCG_4$	878
	$DICCG_{2POD}$	872
10^{-8}	$DICCG_4$	546
	$DICCG_{2POD}$	475
10^{-11}	$DICCG_4$	1
	$DICCG_{2POD}$	1

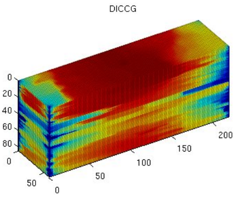
Table : Number of iterations for ICCG and DICCG, diverse tolerance for the snapshots. $DICCG_4$ is computed with 4 deflation vectors, $DICCG_{2POD}$ with 2 basis vectors of POD. Tolerance of the solvers 10^{-11}

Numerical experiments

SPE 10 model, 85 layers







Method	Number or iterations
ICCG	1029
DICCG	1

Table : Number of iterations for the SPE10 benchmark (85 layers) for the ICCG and DICCG methods, tolerance 10^{-11}.

Numerical experiments (SPE 10)

SPE 10 model, 85 layers

	W1 (bars)	W2 (bars)	W3 (bars)	W4 (bars)	W5 (bars)
z_1	-1	-1	-1	-1	4
z_2	0	-1	-1	-1	3
z_3	-1	0	-1	-1	3
z_4	-1	-1	0	-1	3
z_5	-1	-1	-1	0	3
z_6	0	0	-1	-1	2
z_7	-1	0	0	-1	2
z_8	-1	-1	0	0	2
z_9	0	-1	0	-1	2
z_{10}	-1	0	-1	0	2
z_{11}	0	-1	-1	0	2
z_{12}	-1	0	0	0	1
z_{13}	0	-1	0	0	1
z_{14}	0	0	-1	0	1
z_{15}	0	0	0	-1	1

Table : Values of the bhp for the wells.

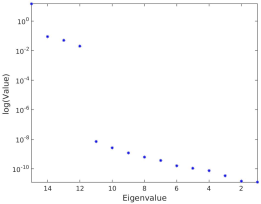


Figure : Eigenvalues of the snapshot correlation matrix $\mathcal{R} = \mathcal{X}\mathcal{X}^T$, 15 snapshots used.

ICCG	1029
DICCG_{15}	2000
DICCG_{4POD}	2

Table : Table with the number of iterations for different contrast in the permeability of the layers for the ICCG, DICCG_{15} and DICCG_{4POD} methods, tolerance of solvers and snapshots 10^{-11}.

Compressible problem, heterogeneous layered problem, contrast between layers 10

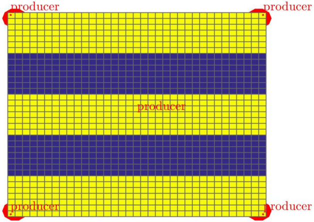


Figure : Heterogeneous permeability.

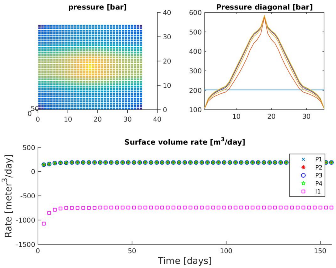


Figure : Solution, well fluxes

Compressible problem, heterogeneous layered problem, contrast between layers 10

Snapshots: 5 first time steps.

Deflation vectors: 3 POD basis vectors.

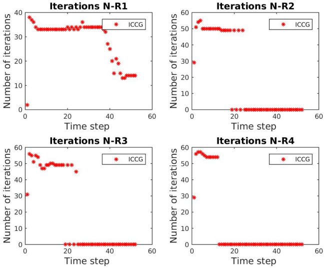


Figure : Number of iterations ICCG method.

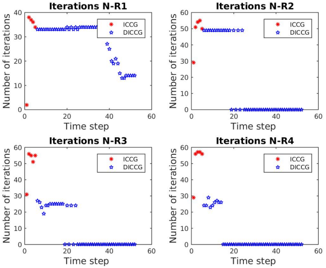


Figure : Number of iterations ICCG and DICCG methods.

Conclusions

- Solution is reached in 1 iteration for DICCG method.
- Number of iterations for the DICCG method does not depend on the contrast between the coefficients (Heterogeneous permeability example).
- Number of iterations for the DICCG method does not depend on the grid size (SPE 10 example).
- The choice of deflation vectors is important for a good performance of DICCG.

References I

K.A. Lie.
An Introduction to Reservoir Simulation Using MATLAB: User guide for the Matlab Reservoir Simulation Toolbox (MRST).
SINTEF ICT, 2013.

J.D. Jansen.
A systems description of flow through porous media.
Springer, 2013.

Y. Saad.
Iterative Methods for Sparse Linear Systems.
Society for Industrial and Applied Mathematics Philadelphia, PA, USA. 2nd edition, 2003.

J. Tang.
Two-Level Preconditioned Conjugate Gradient Methods with Applications to Bubbly Flow Problems.
PhD thesis, Delft University of Technology, 2008.

R. Markovinović.
System-Theoretical Model Reduction for Reservoir Simulation and Optimization.
PhD thesis, Delft University of Technology, 2009.

J. C Vink J.D. Jansen P. Astrid, G. Papaioannou.
Pressure Preconditioning Using Proper Orthogonal Decomposition.
In *2011 SPE Reservoir Simulation Symposium, The Woodlands, Texas, USA*, January 2011.

J. D. Jansen R. Markovinović.
Accelerating iterative solution methods using reduced-order models as solution predictors.
International journal for numerical methods in engineering, 68(5):525–541, 2006.

References II

R. Schuhmann T. Weiland M. Clemens, M. Wilke.
Subspace projection extrapolation scheme for transient field simulations.
IEEE Transactions on Magnetics, 40(2):934–937, 2004.

C. Vuik J.M. Tang, R. Nabben and Y. Erlangga.
Comparison of two-level preconditioners derived from deflation, domain decomposition and multigrid methods.
Journal of scientific computing, 39(3):340–370, 2009.

A. Segal C. Vuik and J. A. Meijerink.
An Efficient Preconditioned CG Method for the Solution of a Class of Layered Problems with Extreme Contrasts in the Coefficients.
Journal of Computational Physics, 152:385–403, 1999.