Deflation with POD vectors for Porous Media Flow

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The Seventh International Conference on Numerical Algebra and Scientific Computing (NASC 2019), October 19 till 23, 2019, Nanjing, P.R. China
Single-phase flow, grid size 60 x 220 x 85 grid cells.

<table>
<thead>
<tr>
<th>Method</th>
<th>Number of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>ICCG</td>
<td>1011</td>
</tr>
<tr>
<td>DICCG</td>
<td>1</td>
</tr>
</tbody>
</table>

Table: Number of iterations for the SPE 10 benchmark (85 layers) for the ICCG and DICCG methods, tolerance of $10^{-7}$. 
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Reservoir Simulation

Single-phase flow through a porous media [1]

Darcy’s law + mass balance equation

\[- \nabla \cdot \left[ \frac{\alpha \rho}{\mu} \vec{K}(\nabla p - \rho g \nabla d) \right] + \alpha \rho \phi c_t \frac{\partial p}{\partial t} - \alpha \rho q = 0.\]

\[c_t = (c_l + c_r),\]

\(\alpha\) a geometric factor
\(\rho\) fluid density
\(\mu\) fluid viscosity
\(p\) pressure
\(\vec{K}\) rock permeability
\(g\) gravity
\(d\) depth
\(\phi\) rock porosity
\(q\) sources
\(c_r\) rock compressibility
\(c_l\) liquid compressibility
Problem Definition

Discretization

3D case, isotropic permeability, small rock and fluid compressibilities, uniform reservoir thickness and no gravity forces.

\[ -\frac{h}{\mu} \frac{\partial}{\partial x} \left( k \frac{\partial \mathbf{p}}{\partial x} \right) - \frac{h}{\mu} \frac{\partial}{\partial y} \left( k \frac{\partial \mathbf{p}}{\partial y} \right) - \frac{h}{\mu} \frac{\partial}{\partial z} \left( k \frac{\partial \mathbf{p}}{\partial z} \right) + h\phi_0 c_t \frac{\partial \mathbf{p}}{\partial t} - h \mathbf{q} = 0. \]

\[ \mathbf{V}\dot{\mathbf{p}} + \mathbf{T}\mathbf{p} = \mathbf{q}. \]

\( \mathbf{q} \): sources or wells in the reservoir, Peaceman well model, \( \mathcal{I}_{\text{well}} \) is the well index

\[ \mathbf{q} = -\mathcal{I}_{\text{well}} (\mathbf{p} - \mathbf{p}_{\text{well}}) \]

Accumulation matrix

\[ \mathbf{V} = V c_t \phi_0 \mathcal{I}, \]

\[ \mathbf{V} = h \Delta x \Delta y \Delta z. \]

Transmissibility matrix

\[ \mathbf{T}_{i-\frac{1}{2},j,l} = \frac{\Delta y}{\Delta x \Delta z} \frac{h}{\mu} k_{i-\frac{1}{2},j,l}, \]

\[ k_{i-\frac{1}{2},j} = \frac{2}{\frac{1}{k_{i-1,j,l}} + \frac{1}{k_{i,j,l}}} . \]
**Problem Definition**

*Incompressible model*

\[ T \mathbf{p} = \mathbf{q}. \]

*Compressible model*

\[ \nu^{n+1} \left( \frac{p^{n+1} - p^n}{\Delta t^n} \right) + T^{n+1} p^{n+1} = q^{n+1}. \]

Or:

\[ \mathcal{F}(p^{n+1}; p^n) = 0. \]  (1)
Using Newton-Raphson (NR) method, the system for the \((k + 1)\)-th NR iteration is:

\[
\mathcal{J}(p^k)\delta p^{k+1} = -\mathcal{F}(p^k; p^n), \quad p^{k+1} = p^k + \delta p^{k+1},
\]

where \(\mathcal{J}(p^k) = \frac{\partial \mathcal{F}(p^k; p^n)}{\partial p^k}\) is the Jacobian matrix, and \(\delta p^{k+1}\) is the NR update at iteration step \(k + 1\).

\[
\mathcal{J}(p^k)\delta p^{k+1} = b(p^k). \tag{2}
\]
Conjugate Gradient Method (CG)

Successive approximations to obtain a more accurate solution $\mathbf{x}$ [2]

$A\mathbf{x} = \mathbf{b},$

$\mathbf{x}^0$, initial guess $\mathbf{r}^{k-1} = \mathbf{b} - A\mathbf{x}^{k-1}.$

$\min_{\mathbf{x}^k \in \kappa_k(A, r^0)} \| \mathbf{x} - \mathbf{x}^k \|_A, \quad \| \mathbf{x} \|_A = \sqrt{\mathbf{x}^T A \mathbf{x}}.$

Convergence

$\| \mathbf{x} - \mathbf{x}^k \|_A \leq 2\| \mathbf{x} - \mathbf{x}^0 \|_A \left( \frac{\sqrt{\kappa(A)} - 1}{\sqrt{\kappa(A)} + 1} \right)^k.$

Preconditioning

Improve the spectrum of $A.$

$M^{-1}A\mathbf{x} = M^{-1}\mathbf{b}.$

Convergence

$\| \mathbf{x} - \mathbf{x}^k \|_A \leq 2\| \mathbf{x} - \mathbf{x}^0 \|_A \left( \frac{\sqrt{\kappa(M^{-1}A)} - 1}{\sqrt{\kappa(M^{-1}A)} + 1} \right)^k, \quad \kappa(M^{-1}A) \leq \kappa(A).$
**DPCG history**

- 1987 Nicolaides and Dostal  
  First versions of DPCG
- 1999 Vuik, Meijerink, Segal  
  DPCG applied to reservoir simulations (Shell)
- 2004 Nabben, Vuik  
  Theory and porous media flow
- 2008 Nabben, Tang, Vuik, ...  
  Theory comparison: DPCG, MG and Domain Decomposition, bubbly flow
Deflated PCG

DPCG history

- 2008 Nabben Erlangga
  Convection diffusion, Helmholtz, MLK method
- 2010 Jönsthövel, Vuik
  Mechanical problems, parallel computing
- 2014 Nabben, Sheikh, Lahaye, Vuik, Garcia
  MLK/ADEF method Helmholtz equation
- 2016 Diaz, Jansen, Vuik
  Porous media flow, Model Order Reduction (MOR)
**Deflation**

\[ \mathcal{P} = \mathcal{I} - \mathcal{A} \mathcal{Q}, \quad \mathcal{P} \in \mathbb{R}^{n \times n}, \quad \mathcal{Q} \in \mathbb{R}^{n \times n}, \]
\[ \mathcal{Q} = \mathcal{Z} \mathcal{E}^{-1} \mathcal{Z}^T, \quad \mathcal{Z} \in \mathbb{R}^{n \times k}, \quad \mathcal{E} \in \mathbb{R}^{k \times k}, \]
\[ \mathcal{E} = \mathcal{Z}^T \mathcal{A} \mathcal{Z} \] (Tang 2008, [3]).

**Convergence**

**Deflated system**

\[ \|x - x^k\|_A \leq 2\|x - x^0\|_A \left( \frac{\sqrt{\kappa_{\text{eff}}(\mathcal{P} \mathcal{A})} - 1}{\sqrt{\kappa_{\text{eff}}(\mathcal{P} \mathcal{A})} + 1} \right)^k. \]

**Deflated and preconditioned system**

\[ \|x - x^k\|_A \leq 2\|x - x^0\|_A \left( \frac{\sqrt{\kappa_{\text{eff}}(\mathcal{M}^{-1} \mathcal{P} \mathcal{A})} - 1}{\sqrt{\kappa_{\text{eff}}(\mathcal{M}^{-1} \mathcal{P} \mathcal{A})} + 1} \right)^k. \]

\[ \kappa_{\text{eff}}(\mathcal{M}^{-1} \mathcal{P} \mathcal{A}) \leq \kappa_{\text{eff}}(\mathcal{P} \mathcal{A}) \leq \kappa(\mathcal{A}). \]
Recycling deflation (Clemens 2004, [4]).

\[ \mathcal{Z} = [\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^{q-1}] , \]

\( \mathbf{x}^i \)'s are solutions of the system.

Multigrid and multilevel (Tang 2009, [5]). The matrices \( \mathcal{Z} \) and \( \mathcal{Z}^T \) are the restriction and prolongation matrices of multigrid methods.

Subdomain deflation (Vuik 1999, [6]).
Deflation Vectors

**Model Order Reduction (MOR)**
Many modern mathematical models of real-life processes pose challenges when used in numerical simulations, due to complexity and large size.

Model order reduction aims to lower the computational complexity of such problems by a reduction of the model’s associated state space dimension or degrees of freedom, an approximation to the original model is computed. (Vuik 2005, [7])

- Proper Orthogonal Decomposition (POD)
- Reduced Basis Method (RBM)
- Principal Component Analysis (PCA)
- Singular Value Decomposition (SVD)
Proposal

Use solution of the system with diverse well configurations ‘snapshots’ as deflation vectors (Recycling deflation).

Use as deflation vectors the basis obtained from Proper Orthogonal Decomposition (POD).
Proper Orthogonal Decomposition (POD)

POD: find an 'optimal' basis for a given data set (Markovinović 2009 [8], Astrid 2011, [9])

- Get the snapshots
  \[ \mathcal{X} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m]. \]

- Form \( \mathcal{R} \)
  \[ \mathcal{R} := \frac{1}{m} \mathcal{X} \mathcal{X}^T \equiv \frac{1}{m} \sum_{i=1}^{m} \mathbf{x}_i \mathbf{x}_i^T. \]

- Then
  \[ \Phi = [\phi_1, \phi_2, \ldots, \phi_l] \in \mathbb{R}^{n \times l} \]
  are the \( l \) eigenvectors corresponding to the largest eigenvalues of \( \mathcal{R} \) satisfying:
  \[ \frac{\sum_{j=1}^{l} \lambda_j}{\sum_{j=1}^{m} \lambda_j} \leq \alpha, \quad 0 < \alpha \leq 1. \]
Lemma 1

Let $A \in \mathbb{R}^{n \times n}$ be a non-singular matrix, and $x$ is a solution of:

$$Ax = b.$$  \hfill (3)

Let $x_i, b_i \in \mathbb{R}^n$, $i = 1, \ldots, m$, be vectors linearly independent (l.i.) and

$$Ax_i = b_i.$$  \hfill (4)

The following equivalence holds

$$x = \sum_{i=1}^{m} c_i x_i \quad \Leftrightarrow \quad b = \sum_{i=1}^{m} c_i b_i.$$  \hfill (5)
Lemma 1 (proof)

Proof \( \Rightarrow \quad x = \sum_{i=1}^{m} c_i x_i \Rightarrow b = \sum_{i=1}^{m} c_i b_i. \quad (6) \)

Substituting \( x \) from (6) into \( Ax = b \) and using the linearity of \( A \) we obtain:

\[
Ax = \sum_{i=1}^{m} c_i Ax_i = \sum_{i=1}^{m} c_i b_i = b. \quad \text{Similarly for} \quad \Leftarrow \quad \Box
\]
Lemma 2

If the deflation matrix $\mathcal{Z}$ is constructed with a set of $m$ vectors such that $\mathbf{x} = \sum_{i=1}^{m} c_i \mathbf{x}_i$, with $\mathbf{x}_i$ l.i., then the solution of system (3) is obtained with one iteration of DCG.
Lemma 2 (proof)

Proof.
The relation between \( \hat{x} \) and \( x \) is given by [3]:

\[
x = Qb + P^T \hat{x}.
\]

For the first term \( Qb \), taking \( b = \sum_{i=1}^{m} c_i b_i \) we have:

\[
Qb = Z \mathcal{E}^{-1} Z^T \left( \sum_{i=1}^{m} c_i b_i \right) = Z (Z^T A Z)^{-1} Z^T \left( \sum_{i=1}^{m} c_i A x_i \right)
\]

\[
= Z (Z^T A Z)^{-1} Z^T (A x_1 c_1 + ... + A x_m c_m) = Z (Z^T A Z)^{-1} (Z^T A Z) c
\]

\[
= Z c = c_1 x_1 + c_2 x_2 + ... + c_m x_m = \sum_{i=1}^{m} c_i x_i = x.
\]
Lemma 2 (proof)

Therefore,

\[ x = Qb, \]  
(8)

is the solution to the original system.

For the second term of the equation, \( P^T\hat{x} \), we compute the deflated solution \( \hat{x} \).

\[
\begin{align*}
PA\hat{x} &= Pb \\
AP^T\hat{x} &= (I - AQ)b \\
AP^T\hat{x} &= b - AQb \\
AP^T\hat{x} &= b - Ax = 0 \\
P^T\hat{x} &= 0
\end{align*}
\]

using \( AP^T = PA \) [3] and definition of \( P \),

taking \( Qb = x \) from above, as \( A \) is invertible.

Then we have obtained the solution

\[ x = Qb + P^T\hat{x} = Qb, \]

in one step of DCG.
Deflation vectors

**Single-phase,** \( Tp^n = q^n \)

**Recycling linearly independent (l.i.) solutions**

- **Compute l.i. solutions with ICCG**
  \[ Tp_i = q_i, \]

- **Construct** \( Z \)
  \[ Z = \begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix}, \]

- **Use** \( Z \) **to solve with DICCG**
  \[ Tp = q. \]
Deflation vectors

Two-phases, $T^n p^n = q^n$

Training phase approach

Compute snapshots with ICCG

Construct correlation matrix

Obtain POD basis and use it to construct $Z_m$

$T^i p^i = q^i, \quad C = XX^T, \quad Z = [\phi_1, \ldots, \phi_n]$

$X = [p^1 \cdots p^n]$

Use $Z$ to solve $Tp = q$ with DICCG, different problems.
Numerical experiments

**Heterogeneous permeability (Neumann and Dirichlet boundary conditions).**

The experiments were performed for single-phase flow, with the following characteristics:

\[ nx = ny = 64 \text{ grid cells.} \]

5 linearly independent snapshots.

<table>
<thead>
<tr>
<th>System configuration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Well pressures (bars)</td>
</tr>
<tr>
<td>W1</td>
</tr>
<tr>
<td>-5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Snapshots</th>
</tr>
</thead>
<tbody>
<tr>
<td>W1</td>
</tr>
<tr>
<td>(z_1)</td>
</tr>
<tr>
<td>(z_2)</td>
</tr>
<tr>
<td>(z_3)</td>
</tr>
<tr>
<td>(z_4)</td>
</tr>
<tr>
<td>(z_5)</td>
</tr>
</tbody>
</table>

Table: Table with the well configuration and boundary conditions of the system and the snapshots used for the Case 1.
Numerical experiments

**Heterogeneous permeability (Neumann and Dirichlet boundary conditions).**

**Figure**: Heterogeneous permeability, 4 wells.

<table>
<thead>
<tr>
<th>$\kappa_2$ (mD)</th>
<th>$10^{-1}$</th>
<th>$10^{-2}$</th>
<th>$10^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ICCG</td>
<td>75</td>
<td>103</td>
<td>110</td>
</tr>
<tr>
<td>DICCG</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table**: Number of iterations for different contrasts between the permeability of the layers for the ICCG and DICCG methods.
Numerical experiments

**Heterogeneous permeability (Neumann boundary conditions).**
The experiments were performed for single-phase flow, with the following characteristics:

- \( nx = ny = 64 \) grid cells.
- Neumann boundary conditions.
- 15 snapshots, 4 linearly independent.
- \( W_1 = W_2 = W_3 = W_4 = -1 \) bars,
- \( W_5 = +4 \) bars.

![Heterogeneous permeability layers.](image)

**Table**: Number of iterations.

<table>
<thead>
<tr>
<th></th>
<th>( 10^{-1} )</th>
<th>( 10^{-2} )</th>
<th>( 10^{-3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ICCG</td>
<td>90</td>
<td>115</td>
<td>131</td>
</tr>
<tr>
<td>DICCG(_4)</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>DICCG(_{15})</td>
<td>200*</td>
<td>200*</td>
<td>200*</td>
</tr>
<tr>
<td>DICCG(_{POD4})</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

![Eigenvalues of the data snapshot correlation matrix.](image)
Numerical experiments

**SPE 10 model**

60x220x85 grid cells.
Neumann boundary conditions.
15 snapshots, 4 linearly independent.
W1 = W2 = W3 = W4 = -1 bars, W5 = +4 bars.

![SPE 10 benchmark, permeability field.](image)

<table>
<thead>
<tr>
<th>Method</th>
<th>Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>ICCG</td>
<td>1011</td>
</tr>
<tr>
<td>DICCG_{15}</td>
<td>2000*</td>
</tr>
<tr>
<td>DICCG_{4}</td>
<td>1</td>
</tr>
<tr>
<td>DICCG_{POD4}</td>
<td>1</td>
</tr>
</tbody>
</table>

**Table**: Number of iterations for ICCG and DICCG methods.

Figure: SPE 10 benchmark, permeability field.
Numerical experiments

**Compressible heterogeneous layered problem**

35×35 grid cells.

Neumann boundary conditions.

W₁ = W₂ = W₃ = W₄ = 100 bars, W₅ = 600 bars.

Initial pressure 200 bars.

Contrast between permeability layers of \(10^1\), \(10^2\) and \(10^3\).

Figure : Solution, contrast between permeability layers of \(10^1\).

Figure : Eigenvalues of the data snapshot correlation matrix, contrast between permeability layers of \(10^1\).
Numerical experiments

<table>
<thead>
<tr>
<th>$\sigma_2/\sigma_1$</th>
<th>Total ICCG(only)</th>
<th>Method</th>
<th>ICCG Snapshots</th>
<th>DICCG</th>
<th>Total ICCG+DICCG</th>
<th>% of total ICCG(only)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^1$</td>
<td>780</td>
<td>ICCG$_{10}$</td>
<td>140</td>
<td>42</td>
<td>182</td>
<td>23</td>
</tr>
<tr>
<td></td>
<td>780</td>
<td>DICCG$_{POD_6}$</td>
<td>140</td>
<td>84</td>
<td>224</td>
<td>29</td>
</tr>
<tr>
<td>$10^2$</td>
<td>624</td>
<td>ICCG$_{10}$</td>
<td>100</td>
<td>42</td>
<td>142</td>
<td>23</td>
</tr>
<tr>
<td></td>
<td>624</td>
<td>DICCG$_{POD_7}$</td>
<td>100</td>
<td>42</td>
<td>142</td>
<td>23</td>
</tr>
<tr>
<td>$10^3$</td>
<td>364</td>
<td>ICCG$_{10}$</td>
<td>20</td>
<td>42</td>
<td>62</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>364</td>
<td>DICCG$_{POD_7}$</td>
<td>20</td>
<td>42</td>
<td>62</td>
<td>17</td>
</tr>
</tbody>
</table>

Table: Comparison between the ICCCC and DICCG methods of the average number of linear iterations for the first NR iteration for various contrast between permeability layers.

<table>
<thead>
<tr>
<th>$\sigma_2/\sigma_1$</th>
<th>Total ICCG(only)</th>
<th>Method</th>
<th>ICCG Snapshots</th>
<th>DICCG</th>
<th>Total ICCG+DICCG</th>
<th>% of total ICCG(only)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^1$</td>
<td>988</td>
<td>ICCG$_{10}$</td>
<td>180</td>
<td>78</td>
<td>258</td>
<td>26</td>
</tr>
<tr>
<td></td>
<td>988</td>
<td>DICCG$_{POD_6}$</td>
<td>180</td>
<td>198</td>
<td>378</td>
<td>38</td>
</tr>
<tr>
<td>$10^2$</td>
<td>832</td>
<td>ICCG$_{10}$</td>
<td>140</td>
<td>90</td>
<td>230</td>
<td>28</td>
</tr>
<tr>
<td></td>
<td>832</td>
<td>DICCG$_{POD_7}$</td>
<td>140</td>
<td>154</td>
<td>294</td>
<td>33</td>
</tr>
<tr>
<td>$10^3$</td>
<td>884</td>
<td>ICCG$_{10}$</td>
<td>110</td>
<td>90</td>
<td>200</td>
<td>23</td>
</tr>
<tr>
<td></td>
<td>884</td>
<td>DICCG$_{POD_7}$</td>
<td>110</td>
<td>150</td>
<td>260</td>
<td>29</td>
</tr>
</tbody>
</table>

Table: Comparison between the ICCCC and DICCG methods of the average number of linear iterations for the second NR iteration for various contrast between permeability layers.
Numerical experiments

**Compressible SPE 10 problem**

60x220x85 grid cells.

Neumann boundary conditions.

$W_1 = W_2 = W_3 = W_4 = 100$ bars, $W_5 = 600$ bars.

Initial pressure 200 bars.

Contrast in permeability of $3 \times 10^7$.

![Figure: Eigenvalues of the data snapshot correlation matrix.](image)
Numerical experiments

<table>
<thead>
<tr>
<th>Total ICCG(only)</th>
<th>Method</th>
<th>ICCG Snapshots</th>
<th>DICCG</th>
<th>Total ICCG+DICCG</th>
<th>% of total ICCG(only)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10173</td>
<td>DICCG$_{10}$</td>
<td>1770</td>
<td>1134</td>
<td>2904</td>
<td>28</td>
</tr>
<tr>
<td>10173</td>
<td>DICCG$_{POD_4}$</td>
<td>1770</td>
<td>1554</td>
<td>3324</td>
<td>32</td>
</tr>
</tbody>
</table>

Table: Total number of linear iterations for the first NR iteration, full SPE 10 benchmark.

<table>
<thead>
<tr>
<th>Total ICCG(only)</th>
<th>Method</th>
<th>ICCG Snapshots</th>
<th>DICCG</th>
<th>Total ICCG+DICCG</th>
<th>% of total ICCG(only)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10231</td>
<td>DICCG$_{10}$</td>
<td>1830</td>
<td>200</td>
<td>2030</td>
<td>20</td>
</tr>
<tr>
<td>10231</td>
<td>DICCG$_{POD_4}$</td>
<td>1830</td>
<td>200</td>
<td>2030</td>
<td>20</td>
</tr>
</tbody>
</table>

Table: Total number of linear iterations for the second NR iteration, full SPE 10 benchmark.
Numerical experiments, two-phase flow

Injection through wells, *training phase approach*

<table>
<thead>
<tr>
<th>Well pressures [bars], $P_0 = 500$ [bars]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Training phase</td>
</tr>
<tr>
<td>$P_1$</td>
</tr>
<tr>
<td>137-275</td>
</tr>
</tbody>
</table>

Same pressure in the producers

| 200 | 200 | 200 | 200 | 800 |

Different pressure in the producers

| 20  | 500 | 500 | 500 | $4 \times P$ |

![Graph showing pressure over time]
### Injection through wells, training phase approach

**Pressure Field**

#### Same pressure in production wells

\[ P_{bhp} = 200 \text{ [bars]} \]

<table>
<thead>
<tr>
<th>dv</th>
<th>Total ICCG</th>
<th>DICCG Method</th>
<th>% of ICCG</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>90130</td>
<td>13720</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>90130</td>
<td>21522</td>
<td>24</td>
</tr>
</tbody>
</table>

#### Different pressure in production wells

\[ P_{2,3,4} = 500 \text{ [bars]}, P_1 = 20 \text{ [bars]} \]

<table>
<thead>
<tr>
<th>dv</th>
<th>Total ICCG</th>
<th>DICCG Method</th>
<th>% of ICCG</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>90130</td>
<td>11740</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>90130</td>
<td>17855</td>
<td>20</td>
</tr>
</tbody>
</table>

**Table**: Number of iterations.
Numerical experiments, two-phase flow

**Injection through wells, training phase approach**

**Figure**: Relative residual.

**Figure**: Relative true error.
Conclusions

- Solution is reached in few (1 or 2) iterations for the DICCG method in the incompressible case.
- A good choice of snapshots takes into account the boundary conditions of the problem.
- The number of iterations of the DICCG method does not depend on the contrast between the coefficients (Heterogeneous permeability example).
- The number of iterations of the ICCG method is reduced up to 80% with the DICCG method in the compressible case.
- Only a limited number of POD basis vectors is necessary to obtain a good speed-up. (for more info see [10, 11])
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