Preconditioners for the incompressible Navier Stokes equations

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Incompressible Flow Solvers in MATLAB/COMSOL

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1. Introduction

The incompressible Navier Stokes equation

\[-\nu \nabla^2 \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = f \quad \text{in} \quad \Omega\]

\[\nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega.\]

\(\mathbf{u}\): fluid velocity; \(p\): pressure

\(\nu > 0\) is the kinematic viscosity coefficient (1/\(Re\)).

\(\Omega \subset \mathbb{R}^2\) is a bounded domain with boundary conditions:

\[\mathbf{u} = \mathbf{w} \quad \text{on} \quad \partial \Omega_D, \quad \nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} - \mathbf{n}p = 0 \quad \text{on} \quad \partial \Omega_N.\]
Finite element discretization

Discrete weak formulation

\[ X_h \subset (H_0^1(\Omega))^d, \quad M_h \subset L^2(\Omega) \]

Find \( u_h \in X_h \) and \( p_h \in M_h \)

\[
\nu \int_{\Omega} \nabla u_h : \nabla v_h \, d\Omega + \int_{\Omega} (u_h \cdot \nabla u_h) \cdot v_h \, d\Omega - \int_{\Omega} p_h (\nabla \cdot v_h) \, d\Omega = \int_{\Omega} f \cdot v_h \, d\Omega, \quad \forall v_h \in X_h,
\]

\[
\int_{\Omega} q_h (\nabla \cdot u_h) \, d\Omega = 0 \quad \forall q_h \in M_h.
\]

Matrix notation

\[
Au + N(u) + B^T p = f
\]

\[
Bu = 0.
\]
Choice of elements

Brezzi-Babuska condition

\[ \inf_{q \in Q_h} \sup_{v \in V_h} \frac{(\nabla \cdot v_h, q_h)}{\|v_h\|_{V_h} \|q_h\|_{Q_h}} \geq \gamma \geq 0. \]

Taylor Hood elements \((Q2 - Q1), (P2 - P1)\) and \((Q2 - Q1)\)
Choice of elements

Brezzi-Babuska condition

\[ \inf_{q \in Q_h} \sup_{v \in V_h} \frac{(\nabla \cdot v_h, q_h)}{\|v_h\|_{V_h} \|q_h\|_{Q_h}} \geq \gamma \geq 0. \]

Crouzeix Raviart \((Q2 - P0), (P2^+ - P1)\) and \((P2^+ - P1)\)
Choice of elements

Brezzi-Babuska condition

\[
\inf_{q \in Q_h} \sup_{v \in V_h} \frac{(\nabla \cdot v_h, q_h)}{v_h \| v_h \| q_h \| Q_h} \geq \gamma \geq 0.
\]

Taylor Hood mini elements \( Q_1^+ - Q_1 \) and \( P_1^+ - P_1 \)
Software packages

IFISS

- Incompressible Flow Iterative Solution Software
- Silvester, Elman, Ramage, Wathen
- Matlab, nice for experiments
- academic, 2D problems only
- modern block triangular preconditioners

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Preconditioners for the Navier Stokes problem
Software packages

Sepron

- Sepron = Segal + Praagman
- FORTRAN package, industrial and academic problems
- 1, 2, 3 Dimensional problems
- Complex geometries
- Taylor Hood and Raviart Thomas elements are implemented
Linearization

Stokes problem

\[- \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \]
\[\nabla \cdot \mathbf{u} = 0\]

Picard's method

\[- \nu \Delta \mathbf{u}^{(k+1)} + (\mathbf{u}^{(k)} \cdot \nabla) \mathbf{u}^{(k+1)} + \nabla p^{(k+1)} = \mathbf{f}\]
\[\nabla \cdot \mathbf{u}^{(k+1)} = 0\]

Newton's method

\[\nu \Delta \mathbf{u}^{k+1} + \mathbf{u}^{k+1} \cdot \nabla \mathbf{u}^k + \mathbf{u}^k \cdot \nabla \mathbf{u}^{k+1} + \nabla p^{k+1} = \mathbf{f} + \mathbf{u}^k \cdot \nabla \mathbf{u}^k,\]
\[\nabla \cdot \mathbf{u}^{k+1} = 0.\]
2. Solution techniques

Matrix form after linearization

\[
\begin{bmatrix}
F & B^T \\
B & 0 \\
\end{bmatrix}
\begin{bmatrix}
u \\
p \\
\end{bmatrix}
= 
\begin{bmatrix}
f \\
0 \\
\end{bmatrix}
\text{ or } Ax = b
\]

- \( F \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times n}, f \in \mathbb{R}^{n} \text{ and } m \leq n \)
- Sparse linear system, symmetric (Stokes problem), nonsymmetric (Navier Stokes) and always indefinite.
- For unique solution \( u \) and \( p \), finite elements must satisfy BB condition.
- Saddle point problem having large number of zeros on the main diagonal
Preconditioner for the Navier Stokes equations

**Definition**

A linear system $Ax = b$ is transformed into $P^{-1}Ax = P^{-1}b$.

- Eigenvalues of $P^{-1}A$ are more clustered than $A$
- $P \approx A$
- $Pz = r$ is cheap to solve for $z$

Block triangular preconditioners

\[
\begin{bmatrix}
F & B^T \\
B & 0
\end{bmatrix} =
\begin{bmatrix}
I & 0 \\
BF^{-1} & I
\end{bmatrix}
\begin{bmatrix}
F & 0 \\
0 & S
\end{bmatrix}
\begin{bmatrix}
I & F^{-1}B^T \\
0 & I
\end{bmatrix} =
\begin{bmatrix}
I & 0 \\
BF^{-1} & I
\end{bmatrix}
\begin{bmatrix}
F & B^T \\
0 & S
\end{bmatrix}
\]

\[
P^{-1} = \begin{bmatrix}
F & B^T \\
0 & S
\end{bmatrix}^{-1}, \quad S = -BF^{-1}B^T \text{(Schur complement matrix)}
\]

\[
Sz_2 = r_2, \quad Fz_1 = r_1 - B^Tz_2
\]

- GMRES converges in two iterations if exact arithmetic is used [Murphy, Golub, Wathen -2000]
- In practice $F^{-1}$ and $S^{-1}$ are expensive, so they are approximated
Preconditioners for the Navier Stokes equations

Block triangular preconditioners

- **Pressure convection diffusion (PCD)** [Kay, Login and Wathen, 2002]
  
  \[ S \approx -A_p F_p^{-1} Q_p \]

- **Least squares commutator (LSC)** [Elman, Howle, Shadid, Silvester and Tuminaro, 2002]
  
  \[ S \approx -(BQ^{-1} B^T)(BQ^{-1} FQ^{-1} B^T)^{-1}(BQ^{-1} B^T) \]

- one of the best approximations available in the literature
- Convergence independent of the mesh size and mildly dependent on Reynolds number
- Require extra operators
- Require iterative solvers (Geometric multigrid, algebraic multigrid) for the (1,1) and (2,2) blocks
Augmented Lagrangian Approach (AL) [Benzi, Olshanski, 2007]

Adapted system

\[
\begin{bmatrix}
F + \gamma B^T W^{-1} B & B^T \\
B & 0
\end{bmatrix}
\begin{bmatrix}
u \\
p
\end{bmatrix}
= 
\begin{bmatrix}
f \\
0
\end{bmatrix}
\]

\(\hat{S}^{-1} = - (\nu \hat{Q}_p^{-1} + \gamma W^{-1})\)

- \(\hat{Q}_p\) approximation of the pressure mass matrix
- \(W = \hat{Q}_p\), \(\gamma\) Lagrange multiplier, \(\nu\) viscosity
- Convergence independent of the mesh size and mildly dependent on Reynolds number
- Require iterative solvers (Geometric multigrid, algebraic multigrid) for the (1,1) block
Incomplete LU preconditioners

\[ A = LU - R, \] where \( R \) consist of dropped entries that are absent in the index set \( S(i, j) \).

\[ S = \{(i, j) | a_{ij} \neq 0\} \] [Classical ILU by Meijerink and van der Vorst, 1977].

In our case, \( S(i, j) = \{(i, j) | i, j \text{ are connected in the finite element grid}\} \). So zeros in the matrix, due to the coefficients are considered to be non-zero in the structure.

- if \( \|R\| \) is large, give poor convergence (**reordering**)
- Instability due to large \( \|L^{-1}\| \) and \( \|U^{-1}\| \)
Preconditioners for the Navier Stokes equations

ILUPACK
- Bollhöfer and Saad
- Static reordering schemes
- Inverse-based ILU with diagonal pivoting
- Multilevel framework
- Iterative solver
3. Advanced ILU preconditioner

Effect of reordering

- In direct solver, reordering improves the profile and bandwidth of the matrix.
- Improve the convergence of the ILU preconditioned Krylov subspace method
- Minimizes dropped entries in ILU ($\|A - LU\|_F$)
- May give stable factorization ($\|I - A(\bar{L}\bar{U})^{-1}\|_F$)


Well-known renumbering schemes

- Cuthill McKee renumbering (RCM) [Cuthill McKee - 1969]
- Sloan renumbering [Sloan - 1986]
- Minimum degree renumbering (MD) [Tinney and Walker - 1967]
New renumbering scheme

- Renumbering of grid points: Grid points are renumbered with Sloan or Cuthill McKee algorithms
- The unknowns are reordered by p-last or p-last per level methods

In **p-last reordering**, first all the velocity unknowns are ordered followed by pressure unknowns. Usually, this produces a large profile but avoids breakdown of the $LU$ decomposition.

**p-last per level reordering**, smaller profile

**p-last per element reordering**, smallest profile
p-last per level reordering

Q2–Q1 finite element subdivision

First level

Second level

Third level
Special features of Advanced ILU

- lumping of positive off-diagonal elements
- extra fill in (global, or pressure only)
- $\epsilon$ stabilization parameter
4. Numerical Experiments

Flow domains

- **Channel flow**  The Poiseuille channel flow in a square domain \((-1, 1)^2\) with a parabolic inflow boundary condition and the natural outflow condition having the analytic solution: \(u_x = 1 - y^2; \; u_y = 0; \; p = 2\nu x\)

- **Backward facing step**

  ![Diagram of channel flow and backward facing step](image)

  - Q2-Q1 finite element discretization [Taylor, Hood - 1973]
  - Q2-P1 finite element discretization [Crouzeix, Raviart - 1973]
Comparison of p-last and p-last per level

Square channel, Stokes, Q2-P1

GMRES(20) costs more CPU time

GMRESR is comparable with Bi-CGSTAB, wrt CPU time
Dependence on the Reynolds number

Backward Facing Step, Navier Stokes, $16 \times 48$ with Q2-Q1 discretization

Sloan reordering is faster than RCM reordering
Comparison of ILU preconditioner with block triangular preconditioners using GMRES ($accuracy = 10^{-4}$) and Newton linearization for the backward facing step Navier Stokes problem, Q2-Q1 element

Direct solver for $(1,1),(2,2)$ blocks of the block triangular preconditioners

<table>
<thead>
<tr>
<th>Re=100</th>
<th>PCD</th>
<th>p-last-level(Sloan)</th>
<th>LSC</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Iter</td>
<td>sec</td>
<td>Iter</td>
</tr>
<tr>
<td>-</td>
<td>Iter</td>
<td>sec</td>
<td>Iter</td>
</tr>
<tr>
<td>8x24</td>
<td>32</td>
<td>0.50</td>
<td>16</td>
</tr>
<tr>
<td>16x24</td>
<td>33</td>
<td>1.21</td>
<td>21</td>
</tr>
<tr>
<td>32x24</td>
<td>37</td>
<td>3.16</td>
<td>68</td>
</tr>
<tr>
<td>64x24</td>
<td>45</td>
<td>8.30</td>
<td>61</td>
</tr>
<tr>
<td>Re = 200</td>
<td>8x24</td>
<td>45</td>
<td>7.26</td>
</tr>
<tr>
<td></td>
<td>16x24</td>
<td>50</td>
<td>1.90</td>
</tr>
<tr>
<td></td>
<td>32x24</td>
<td>52</td>
<td>4.30</td>
</tr>
<tr>
<td></td>
<td>64x24</td>
<td>60</td>
<td>11.0</td>
</tr>
</tbody>
</table>

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Preconditioners for the Navier Stokes problem
Comparison of ILU preconditioner with block triangular preconditioners using GCR(30) and Newton linearization for the backward facing step Navier Stokes problem

Iterative solver for (1,1),(2,2) blocks of the block triangular preconditioners

<table>
<thead>
<tr>
<th>Q2-Q1</th>
<th>AL</th>
<th>LSC</th>
<th>p-last-level(Sloan)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Iter</td>
<td>sec</td>
<td>Iter</td>
</tr>
<tr>
<td>8x24</td>
<td>9</td>
<td>0.10</td>
<td>17</td>
</tr>
<tr>
<td>16x24</td>
<td>9</td>
<td>0.44</td>
<td>18</td>
</tr>
<tr>
<td>32x24</td>
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<td>2.72</td>
<td>23</td>
</tr>
<tr>
<td>64x24</td>
<td>9</td>
<td>9.30</td>
<td>27</td>
</tr>
<tr>
<td>128x24</td>
<td>9</td>
<td>44.5</td>
<td>42</td>
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</table>

<table>
<thead>
<tr>
<th>Q2-P1</th>
<th>AL</th>
<th>LSC</th>
<th>p-last-level(Sloan)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Iter</td>
<td>sec</td>
<td>Iter</td>
</tr>
<tr>
<td>8x24</td>
<td>8</td>
<td>0.12</td>
<td>14</td>
</tr>
<tr>
<td>16x24</td>
<td>8</td>
<td>0.28</td>
<td>11</td>
</tr>
<tr>
<td>32x24</td>
<td>8</td>
<td>0.64</td>
<td>20</td>
</tr>
<tr>
<td>64x24</td>
<td>8</td>
<td>1.50</td>
<td>NC</td>
</tr>
<tr>
<td>128x24</td>
<td>8</td>
<td>3.43</td>
<td>NC</td>
</tr>
</tbody>
</table>
### ILUPACK with GMRES(20)

<table>
<thead>
<tr>
<th>Grid</th>
<th>Iterations</th>
<th>nnz(A)</th>
<th>nnz(ILU)</th>
<th>Growth factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>8x24</td>
<td>4</td>
<td>7040</td>
<td>15020</td>
<td>2.13</td>
</tr>
<tr>
<td>16x48</td>
<td>4</td>
<td>33122</td>
<td>96227</td>
<td>2.90</td>
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<tr>
<td>32x96</td>
<td>4</td>
<td>143548</td>
<td>797598</td>
<td>5.56</td>
</tr>
<tr>
<td>64x192</td>
<td>5</td>
<td>598832</td>
<td>3951127</td>
<td>6.60</td>
</tr>
</tbody>
</table>

Backward facing step, Stokes problem, Q2-Q1
ILUPACK with GMRES(20)

<table>
<thead>
<tr>
<th>Grid</th>
<th>ILUPACK</th>
<th>ILU p-last-level(Sloan)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Iter. (time(s))</td>
<td>Total time(s)</td>
</tr>
<tr>
<td>8x24</td>
<td>4 (0.01)</td>
<td>0.04</td>
</tr>
<tr>
<td>16x48</td>
<td>4(0.04)</td>
<td>0.23</td>
</tr>
<tr>
<td>32x96</td>
<td>4 (0.19)</td>
<td>2.42</td>
</tr>
<tr>
<td>64x192</td>
<td>5 (1.0)</td>
<td>21.00</td>
</tr>
</tbody>
</table>

Backward facing step, Stokes problem, Q2-Q1
5. Conclusions

- IFISS is a nice tool to investigate the incompressible Navier Stokes equations
- Advanced ILU:
  - renumbering of grid points and reordering of unknowns
  - no break down and fast convergence
  - iterations increase with increase in Reynolds number and grid points
- Block preconditioners are better for large grid sizes and large Reynolds numbers
- ILUPACK needs small number of iterations, but memory and CPU time can be large
- Stretched grids?
References

- ta.twi.tudelft.nl/nw/users/vui/k/pub_it_navstok.html
- M. ur Rehman and C. Vuik and G. Segal
  Solution of the incompressible Navier Stokes equations with preconditioned Krylov subspace methods TUD Report 06-05
- M. ur Rehman and C. Vuik and G. Segal
  Study Report of IFISS Package (version 2.1) TUD Report 06-06
- C. Li and C. Vuik
  Eigenvalue analysis of the SIMPLE preconditioning for incompressible flow
  Numerical Linear Algebra with Applications, 11, pp.511-523, 2004
- C. Vuik and A. Saghir and G.P. Boerstoel
  The Krylov accelerated SIMPLE(R) method for flow problems in industrial furnaces