Deflation type methods combined with shifted Laplace preconditioners for the Helmholtz equation

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May 2, 2013
Application: geophysical survey

hard Marmousi Model
Application: geophysical survey

hard Marmousi Model

- old method
- new method

<table>
<thead>
<tr>
<th>Frequency f</th>
<th>Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>100</td>
</tr>
<tr>
<td>10</td>
<td>500</td>
</tr>
<tr>
<td>15</td>
<td>1000</td>
</tr>
<tr>
<td>20</td>
<td>2000</td>
</tr>
<tr>
<td>25</td>
<td>3000</td>
</tr>
<tr>
<td>30</td>
<td>4000</td>
</tr>
</tbody>
</table>
1. Problem: The Helmholtz equation

The Helmholtz equation without damping

\[-\Delta u(x, y) - k^2(x, y)u(x, y) = g(x, y) \text{ in } \Omega\]

\(u(x, y)\) is the pressure field,
\(k(x, y)\) is the wave number,
\(g(x, y)\) is the point source function and
\(\Omega\) is the domain. Absorbing boundary conditions are used on \(\Gamma\).

\[
\frac{\partial u}{\partial n} - \nu u = 0
\]

\(n\) is the unit normal vector pointing outwards on the boundary.

Perfectly Matched Layer (PML) and Absorbing Boundary Layer (ABL)
Problem description

• Second order Finite Difference stencil:

\[
\begin{bmatrix}
-1 & & & \\
-1 & 4 - k^2 h^2 & -1 & \\
& -1 & & \\
& & -1 & \\
\end{bmatrix}
\]

• Linear system $A_h u_h = g_h$: properties
  
  Sparse & complex valued
  
  Symmetric & Indefinite for large $k$

• For high resolution a very fine grid is required: $30 - 60$ gridpoints per wavelength (or $\approx 5 - 10 \times k$) $\rightarrow A_h$ is extremely large!

• Traditionally solved by a Krylov subspace method, which exploits the sparsity.
2. Shifted Laplace Preconditioner

Laplace operator \( \text{Bayliss and Turkel, 1983} \)
Definite Helmholtz \( \text{Laird, 2000} \)
Shifted Laplace \( \text{Y.A. Erlangga, C. Vuik and C.W.Oosterlee, 2003} \)

Shifted Laplace preconditioner

\[
M \equiv -\Delta - (\beta_1 - i\beta_2)k^2, \quad \beta_1, \beta_2 \in \mathbb{R}.
\]

If \( \beta_1 \leq 0 \) holds than \( M \) is a (semi) definite operator.

\( \rightarrow \beta_1, \beta_2 = 0 \) : Bayliss and Turkel
\( \rightarrow \beta_1 = 1, \beta_2 = 0 \) : Laird
\( \rightarrow \beta_1 = -1, \beta_2 = 0.5 \) : Y.A. Erlangga, C. Vuik and C.W. Oosterlee
Numerical results for a wedge problem

<table>
<thead>
<tr>
<th>$k_2$</th>
<th>10</th>
<th>20</th>
<th>40</th>
<th>50</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>grid</td>
<td>32$^2$</td>
<td>64$^2$</td>
<td>128$^2$</td>
<td>192$^2$</td>
<td>384$^2$</td>
</tr>
<tr>
<td>No-Prec</td>
<td>201(0.56)</td>
<td>1028(12)</td>
<td>5170(316)</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>ILU($A$,0)</td>
<td>55(0.36)</td>
<td>348(9)</td>
<td>1484(131)</td>
<td>2344(498)</td>
<td>–</td>
</tr>
<tr>
<td>ILU($A$,1)</td>
<td>26(0.14)</td>
<td>126(4)</td>
<td>577(62)</td>
<td>894(207)</td>
<td>–</td>
</tr>
<tr>
<td>ILU($M$,0)</td>
<td>57(0.29)</td>
<td>213(8)</td>
<td>1289(122)</td>
<td>2072(451)</td>
<td>–</td>
</tr>
<tr>
<td>ILU($M$,1)</td>
<td>28(0.28)</td>
<td>116(4)</td>
<td>443(48)</td>
<td>763(191)</td>
<td>2021(1875)</td>
</tr>
<tr>
<td>MG(V(1,1))</td>
<td>13(0.21)</td>
<td>38(3)</td>
<td>94(28)</td>
<td>115(82)</td>
<td>252(850)</td>
</tr>
</tbody>
</table>
Shifted Laplace Preconditioner Spectrum

- Eigenvalues of the preconditioned operator are bounded by 1
- Small eigenvalues move to zero, as $k$ increases.

Spectrum of $M^{-1}A$, where $(\beta_1, \beta_2) = (1, 0.5)$

$k = 30$

$k = 120$
3. Second Level Preconditioners

Number of GMRES iterations. Shifts in the preconditioner are $(1, 0.5)$

<table>
<thead>
<tr>
<th>Grid</th>
<th>$k = 10$</th>
<th>$k = 20$</th>
<th>$k = 30$</th>
<th>$k = 40$</th>
<th>$k = 50$</th>
<th>$k = 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 32$</td>
<td>5/10</td>
<td>8/17</td>
<td>14/28</td>
<td>26/44</td>
<td>42/70</td>
<td>13/14</td>
</tr>
<tr>
<td>$n = 64$</td>
<td>4/10</td>
<td>6/17</td>
<td>8/28</td>
<td>12/36</td>
<td>18/45</td>
<td>173/163</td>
</tr>
<tr>
<td>$n = 96$</td>
<td>3/10</td>
<td>5/17</td>
<td>7/27</td>
<td>9/35</td>
<td>12/43</td>
<td>36/97</td>
</tr>
<tr>
<td>$n = 128$</td>
<td>3/10</td>
<td>4/17</td>
<td>6/27</td>
<td>7/35</td>
<td>9/43</td>
<td>36/85</td>
</tr>
</tbody>
</table>

Erlangga and Nabben, 2008, seems to be independent of $k$. with / without deflation.
Deflation: or two-grid method

Deflation, a projection preconditioner

\[ P = I - AQ, \quad \text{with} \quad Q = ZE^{-1}Z^T \quad \text{and} \quad E = Z^TAZ \]

where,

\[ Z \in \mathbb{R}^{n \times r}, \quad \text{with deflation vectors} \quad Z = [z_1, ..., z_r], \quad rank(Z) = r \leq n \]

Along with a traditional preconditioner \( M \), deflated preconditioned system reads

\[ PM^{-1}Au = PM^{-1}g. \]

Deflation vectors shifted the eigenvalues to zero.
Spectrum as function of $k$
Deflation for Helmholtz

With choice of multigrid inter-grid transfer operator (Prolongation) as deflation matrix, i.e. $Z = I^h_{2h}$ and $Z^T = I^h_{2h}$ then

$$P_h = I_h - A_h Q_h, \quad \text{with} \quad Q_h = I^h_{2h} A_{2h}^{-1} I^h_{2h} \quad \text{and} \quad A_{2h} = I^h_{2h} A_h I^h_{2h}$$

where

- $P_h$ can be interpreted as a coarse grid correction and
- $Q_h$ as the coarse grid operator
Deflation: ADEF1

Deflation can be implemented combined with SLP $M_h$,

$$M_h^{-1} P_h A_h u_h = M_h^{-1} P_h g_h$$

$A_h u_h = g_h$ is preconditioned by two-level preconditoner $M_h^{-1} P_h$.

For large problems, $A_{2h}$ is too large to invert exactly. Inversion of $A_{2h}$ is sensitive, since $P_h$ deflates the spectrum to zero.

To do is: Solve $A_{2h}$ iteratively to required accuracy on certain levels, and shift the deflated spectrum to $\lambda_{h}^{max}$ by adding a shift in two level preconditioner. This leads to the ADEF1 preconditioner

$$P_{(h,ADEF1)} = M_h^{-1} P_h + \lambda_{h}^{max} Q_h$$
Deflation: MLKM

Multi Level Krylov Method, take $\hat{A}_h = M_h^{-1} A_h$, and define $\hat{P}_h$ by using $\hat{A}_h$ (instead of $A_h$) will be

$$\hat{P}_h = I_h - \hat{A}_h \hat{Q}_h,$$

where

$$\hat{Q}_h = I_{2h}^2 \hat{A}_{2h}^{-1} I_{2h}^h$$ and $$\hat{A}_{2h} = I_{2h}^h \hat{A}_h I_{2h}^2 = I_{2h}^h (M_h^{-1} A_h) I_{2h}^2$$

Construction of coarse matrix $A_{2h}$ at level $2h$ costs inversion of preconditioner at level $h$.

Approximate $A_{2h}$

<table>
<thead>
<tr>
<th>Ideal</th>
<th>Practical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{2h} = I_{2h}^h (M_h^{-1} A_h) I_{2h}^2$</td>
<td>$A_{2h} = I_{2h}^h (M_h^{-1} A_h) I_{2h}^2$</td>
</tr>
<tr>
<td></td>
<td>$A_{2h} \approx I_{2h}^h I_{2h}^2 M_{2h}^{-1} A_{2h}$</td>
</tr>
</tbody>
</table>

Erlangga, Y.A and Nabben R., ETNA 2008
4. Fourier Analysis

Dirichlet boundary conditions for analysis.
With above deflation,

$$\text{spec}(PM^{-1}A) = f(\beta_1, \beta_2, k, h)$$

is a complex valued function.

Setting $kh = 0.625$,

- Spectrum of $PM^{-1}A$ with shifts $(1, 0.5)$ is clustered around 1 with a few outliers.
- Spectrum remains almost the same, when the imaginary shift for the preconditioner is varied from 0.5 to 1.
Fourier Analysis

**ADEF1**: Analysis shows spectrum clustered around 1 with few outliers.

\[ k = 30 \quad k = 120 \]
Fourier Analysis

Spectrum of Helmholtz preconditioned by MLKM, $k = 160$ and 20 gp/wl

Ideal

Practical

Two-level
5. Numerical results

Number of GMRES iterations for the 1D Helmholtz equation

\[ 10 \leq k \leq 800 \]
Numerical results

Number of GMRES iterations for the 1D Helmholtz equation

\[ 1000 \leq k \leq 20000 \]
Numerical results

Number of GMRES outer-iterations in multilevel algorithm.

\[(\beta_1, \beta_2) = (1, 0.5) \quad kh = .3125 \text{ or } 20 \text{ gp/wl}\]

and SLP approximated by multigrid Vcycle V(1,1)

<table>
<thead>
<tr>
<th>Grid</th>
<th>(k = 10)</th>
<th>(k = 20)</th>
<th>(k = 40)</th>
<th>(k = 80)</th>
<th>(k = 160)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ADEF1-V(4,2,1)</td>
<td>9</td>
<td>11</td>
<td>16</td>
<td>27</td>
<td>100+</td>
</tr>
<tr>
<td>ADEF1-V(6,2,1)</td>
<td>9</td>
<td>10</td>
<td>14</td>
<td>21</td>
<td>47</td>
</tr>
<tr>
<td>ADEF1-V(8,2,1)</td>
<td>9</td>
<td>10</td>
<td>13</td>
<td>20</td>
<td>38</td>
</tr>
<tr>
<td>ADEF1-V(8,3,2)</td>
<td>9</td>
<td>10</td>
<td>13</td>
<td>19</td>
<td>37</td>
</tr>
</tbody>
</table>

ADEF1-V(8,2,1), a multilevel solver where 8 and 2 Krylov iterations performed on \(2^{nd}\) and \(3^{rd}\) levels and 1 iteration on further levels and Vcycle approximates SLP.
## Results

Petsc solve-time in Seconds; a Two-level solver.

<table>
<thead>
<tr>
<th>Solver</th>
<th>20</th>
<th>40</th>
<th>80</th>
<th>120</th>
<th>160</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>SLP</td>
<td>0.01(23)</td>
<td>0.24(54)</td>
<td>2.62(113)</td>
<td>11.60(168)</td>
<td>33.59(222)</td>
<td>83.67(274)</td>
</tr>
<tr>
<td>ADEF1/SLP</td>
<td>0.03(10)</td>
<td>0.14(14)</td>
<td>0.82(23)</td>
<td>2.92(37)</td>
<td>8.98(61)</td>
<td>23.13(87)</td>
</tr>
</tbody>
</table>

SLP : GCR preconditioned with SLP \( M(1, 1) \).

Def/SLP: Deflated and preconditioned GCR.

Grid resolution is such that there are 10 grid points per wavelength.
Numerical results

Comparison of number of iterations by ADEF1 and MLKM.
Numerical results

3D Helmholtz on unit cube with Sommerfeld b.c. on all faces.

<table>
<thead>
<tr>
<th>Solver Type</th>
<th>Wavenumber $k$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5</td>
</tr>
<tr>
<td>SL Prec.</td>
<td>11</td>
</tr>
<tr>
<td>ADEF1-F(8,2,1)</td>
<td>9</td>
</tr>
<tr>
<td>ADEF1-V(8,2,1)</td>
<td>11</td>
</tr>
</tbody>
</table>

SL Prec. : Only shifted Laplace preconditioner

ADEF1-F : Multilevel solver , Fcycle for slp.

ADEF1-V : Multilevel solver , Vcycle for slp.
Numerical results

Multilevel solver is coded in Petsc.
3D Helmholtz on unit cube with constant wavenumber.

ADEF1 solve time and Setup time.
Conclusions

• Without deflation, when imaginary shift is increased in SLP, spectrum remains bounded above 1, but lower part moves to zero.
• With deflation the convergence is nearly independent of the imaginary shift.
• With deflation the convergence is initially weakly depending on $k$. For large $k$ is scales again linearly.
• With deflation the CPU time is less than without deflation.
• The convergence of ADEF1 and the practical variant of MLKM are similar.
References


• http://ta.twi.tudelft.nl/nw/users/vuik/pub_it_helmholtz.html