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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
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1.

a The local truncation error is defined as

$$\tau_{n+1}(h) = \frac{y_{n+1} - z_{n+1}}{h}, \tag{1}$$

where z_{n+1} is given by

$$z_{n+1} = y_n + h\left(a_1 f(t_n, y_n) + a_2 f(t_n + h, y_n + hf(t_n, y_n))\right). \tag{2}$$

A Taylor expansion of f around (t_n, y_n) yields

$$f(t_n+h, y_n+hf(t_n, y_n)) = f(t_n, y_n)+h\frac{\partial f}{\partial t}(t_n, y_n)+hf(t_n, y_n)\frac{\partial f}{\partial y}(t_n, y_n)+O(h^2). \tag{3}$$

This is substituted into equation (2) to obtain

$$z_{n+1} = y_n+h\left(a_1 f(t_n, y_n) + a_2\left[f(t_n, y_n) + h\frac{\partial f}{\partial t}(t_n, y_n) + hf(t_n, y_n)\frac{\partial f}{\partial y}(t_n, y_n)\right]\right)+O(h^3). \tag{4}$$

A Taylor series for $y(x)$ around t_n gives for y_{n+1}

$$y_{n+1} = y(t_n + h) = y_n + hy'(t_n) + \frac{h^2}{2}y''(t_n) + O(h^3). \tag{5}$$

From the differential equation we know that:

$$y'(t_n) = f(t_n, y_n) \tag{6}$$

From the Chain Rule of Differentiation, we derive

$$y''(t_n) = \frac{df(t_n, y_n)}{dt} = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y}y'(t_n) \tag{7}$$

after substitution of the differential equation one obtains:

$$y''(t_n) = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y}f(t_n, y_n) \tag{8}$$

Equations (5) and (4) are substituted into relation (1) to obtain

$$\tau_{n+1}(h) = f(t_n, y_n)(1 - (a_1 + a_2)) + h\left(\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}\right)\left(\frac{1}{2} - a_2\right) + O(h^2) \tag{9}$$

Hence

(a) $a_1 + a_2 = 1$ implies $\tau_{n+1}(h) = O(h)$;

(b) $a_1 + a_2 = 1$ and $a_2 = 1/2$, that is, $a_1 = a_2 = 1/2$, gives $\tau_{n+1}(h) = O(h^2)$.

b The test equation is given by

$$y' = \lambda y. \tag{10}$$

Application of the predictor step to the test equation gives

$$w^*_{n+1} = w_n + h\lambda w_n = (1 + h\lambda)w_n. \tag{11}$$

The corrector step yields

$$w_{n+1} = w_n + h\left(a_1\lambda w_n + a_2\lambda(1 + h\lambda)w_n\right) = (1 + (a_1 + a_2)h\lambda + a_2h^2\lambda^2)w_n. \tag{12}$$

Hence the amplification factor is given by

$$Q(h\lambda) = 1 + (a_1 + a_2)h\lambda + a_2h^2\lambda^2. \tag{13}$$

c Let $\lambda < 0$ (so λ is real), then, for stability, the amplification factor must satisfy

$$-1 \leq Q(h\lambda) \leq 1, \tag{14}$$

from the previous assignment, we have

$$-1 \leq 1 + (a_1 + a_2)h\lambda + a_2(h\lambda)^2 \leq 1 \Leftrightarrow -2 \leq (a_1 + a_2)h\lambda + a_2(h\lambda)^2 \leq 0. \tag{15}$$

First, we consider the left inequality:

$$a_2(h\lambda)^2 + (a_1 + a_2)h\lambda + 2 \geq 0 \tag{16}$$

For $h\lambda = 0$, the above inequality is satisfied, further the discriminant is given by $(a_1 + a_2)^2 - 8a_2 < 0$. Here the last inequality follows from the given hypothesis. Hence the left inequality in relation (15) is always satisfied. Next we consider the right hand inequality of relation (15)

$$a_2(h\lambda)^2 + (a_1 + a_2)h\lambda \leq 0. \tag{17}$$

This relation is rearranged into

$$a_2(h\lambda)^2 \leq -(a_1 + a_2)h\lambda, \tag{18}$$

hence

$$a_2|h\lambda|^2 \leq (a_1 + a_2)|h\lambda| \Leftrightarrow |h\lambda| \leq \frac{a_1 + a_2}{a_2}, \qquad a_2 \neq 0. \tag{19}$$

This results into the following condition for stability

$$h \leq \frac{a_1 + a_2}{a_2|\lambda|}, \qquad a_2 \neq 0. \tag{20}$$

d The Jacobian, J, is given by

$$J = \begin{pmatrix} \dfrac{\partial f_1}{\partial y_1} & \dfrac{\partial f_1}{\partial y_2} \\ \\ \dfrac{\partial f_2}{\partial y_1} & \dfrac{\partial f_2}{\partial y_2} \end{pmatrix}. \tag{21}$$

Since $f_1(y_1, y_2) = -y_1 y_2$ and $f_2(y_1, y_2) = y_1 y_2 - y_2$, we obtain

$$J = \begin{pmatrix} -y_2 & -y_1 \\ y_2 & y_1 - 1 \end{pmatrix}. \tag{22}$$

Substitution of the initial values $y_1(0) = 1$ and $y_2(0) = 2$, gives

$$J = \begin{pmatrix} -2 & -1 \\ 2 & 0 \end{pmatrix}. \tag{23}$$

e The eigenvalues of the Jacobian at $y_1(0) = y_2(0) = 1$ are given by $\lambda_{1,2} = -1 \pm i$. For our case, we have

$$Q(h\lambda) = 1 + h\lambda + 1/2(h\lambda)^2. \tag{24}$$

Since our eigenvalues are not real valued, it is required for stability that

$$|Q(h\lambda)| \leq 1. \tag{25}$$

Since the eigenvalues are complex conjugates, we can proceed with one of the eigenvalues, say $\lambda = -1 + i$ with $\lambda^2 = -2i$ to obtain

$$Q(h\lambda) \quad = 1 + h(-1 + i) + 1/2h^2(-2i) \tag{26}$$

Substitution of $h = 1$ shows that $Q(h\lambda) = 0$. This implies that $|Q(h\lambda)| = 0 \leq 1$ so the method is stable.

2. a First, we check the boundary conditions:

$$u(0) = \frac{e^0 - 1}{e - 1} = \frac{1 - 1}{e - 1} = 0, \quad u(1) = \frac{e^1 - 1}{e - 1} = 1. \tag{27}$$

Further, we have

$$u'(x) = \frac{e^x}{e - 1} = u''(x). \tag{28}$$

Hence, we immediately see

$$-u''(x) + u'(x) = -\frac{e^x}{e - 1} + \frac{e^x}{e - 1} = 0. \tag{29}$$

Hence, the solution $u(x) = \frac{e^x - 1}{e - 1}$ satisfies the differential and the boundary conditions, and therewith $u(x)$ is the solution to the boundary value problem (uniqueness can be demonstrated in a straightforward way, but this was not asked for).

b The domain of computation, being $(0,1)$, is divided into subintervals with mesh-points, we set $x_j = jh$, where we use n unknowns, such that $x_{n+1} = (n+1)h = 1$. We are looking for a discretization with an error of second order, $O(h^2)$. To this extent, we use the following central differences approximation at x_j:

$$u'(x_j) \approx \frac{u(x_{j+1}) - u(x_{j-1})}{2h}, \text{ for } j \in \{1, \ldots, n\}. \tag{30}$$

We note that the above formula can be derived formally by writing the derivative as

$$u'(x_j) = \frac{\alpha_0 u(x_{j-1}) + \alpha_1 u(x_j) + \alpha_2 u(x_{j+1})}{h}, \tag{31}$$

and solve α_0, α_1 and α_2 from checking the zeroth, first and second order derivatives of $u(x)$. Further, the second order derivative is approximated by

$$u''(x_j) \approx \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2}. \tag{32}$$

Since we approximate the derivatives at the point x_j, we use Taylor Series around x_j, to obtain:

$$\begin{aligned} u(x_{j+1}) &= u(x_j + h) = u(x_j) + hu'(x_j) + \frac{h^2}{2}u''(x_j) + \frac{h^3}{6}u'''(x_j) + \frac{h^4}{24}u''''(x_j) + O(h^5), \\ u(x_{j-1}) &= u(x_j - h) = u(x_j) - hu'(x_j) + \frac{h^2}{2}u''(x_j) - \frac{h^3}{6}u'''(x_j) + \frac{h^4}{24}u''''(x_j) + O(h^5), \end{aligned} \tag{33}$$

This gives

$$\begin{aligned} &-\frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} + \frac{u(x_{j+1}) - u(x_{j-1})}{2h} = -u''(x_j) + u'(x_j) \\ &+\frac{\frac{h^3}{3}u'''(x_j) + O(h^4)}{2h} + \frac{\frac{h^4 u''''(x_j)}{12} + O(h^5)}{h^2} = -u''(x_j) + u'(x_j) + O(h^2). \end{aligned} \tag{34}$$

Hence the error is second order, that is $O(h^2)$. Next, we neglect the truncation error, and set $v_j := u(x_j)$ to get

$$-\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2} + \frac{v_{j+1} - v_{j-1}}{2h} = 0, \text{ for } j \in \{1, \ldots, n\}. \tag{35}$$

At the boundaries, we see for $j = 1$ and $j = n$, upon substituting $v_0 = 0$ and $v_{n+1} = 1$:

$$\begin{aligned} -\frac{v_2 - 2v_1 + 0}{h^2} + \frac{v_2 - 0}{2h} &= 0, \\ -\frac{1 - 2v_n + v_{n-1}}{h^2} + \frac{1 - v_{n-1}}{2h} &= 0. \end{aligned} \tag{36}$$

This is rewritten more neatly by

$$\begin{aligned}\frac{-v_2+2v_1}{h^2}+\frac{v_2}{2h}&=0,\\ \frac{2v_n-v_{n-1}}{h^2}-\frac{v_{n-1}}{2h}&=\frac{1}{h^2}-\frac{1}{2h}.\end{aligned} \tag{37}$$

c Since the exact solution is given by

$$u(x)=\frac{e^x-1}{e-1}, \tag{38}$$

we immediately see that the exact solution is a real–valued exponential, which is monotonic by its nature. Since the numerical solution should have the same characteristics as the exact solution, non–monotonic (and hence oscillatory) solutions should be considered as not reflecting the analytic solution.

d i The magnitude of the local truncation error is given by

$$|\varepsilon|=|\int_a^b f(x)dx-\frac{b-a}{2}\left(f(a)+f(b)\right)|. \tag{39}$$

We use linear interpolation over the interval (a,b), which gives

$$f(x)-\left(f(a)\frac{x-b}{a-b}+f(b)\frac{x-a}{b-a}\right)=\frac{1}{2}(x-a)(x-b)f''(\xi),\ \text{for a } \xi\in(a,b)\backslash\{x\}. \tag{40}$$

Since

$$\int_a^b\left(f(a)\frac{x-b}{a-b}+f(b)\frac{x-a}{b-a}\right)dx=\frac{b-a}{2}\left(f(a)+f(b)\right), \tag{41}$$

this implies that

$$\begin{aligned}&|\int_a^b f(x)dx-\frac{b-a}{2}\left(f(a)+f(b)\right)|=|\int_a^b\frac{1}{2}(x-a)(x-b)f''(\xi(x))dx|\\ &\le\frac{1}{2}\int_a^b(x-a)(b-x)|f''(\xi(x))|dx\le\frac{M}{2}\int_a^b(x-a)(b-x)dx.\end{aligned} \tag{42}$$

Continuity, and hence boundedness, of the second–order derivatives on the interval $(0,1)$ implies the existence of an $M>0$ such that $|f''(x)|\le M$ on the interval (a,b), since the derivatives of f up to second order are bounded. Furthermore, we have

$$\int_a^b(x-a)(x-b)dx=\frac{(b-a)^3}{6}, \tag{43}$$

by which we finally conclude

$$|\int_a^b f(x)dx - \frac{b-a}{2}\left(f(a)+f(b)\right)| \leq \frac{M}{12}(b-a)^3, \tag{44}$$

where, as we saw earlier, M is an upper bound for the second order derivative on the interval (a,b).

ii Application over the gridnodes $\{x_j\}$, with gridspacing h, gives

$$\begin{aligned} \int_0^{0.3} f(x)dx &\approx \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(x)dx = \sum_{j=1}^{n} \frac{h}{2}(f(x_{j-1})+f(x_j)) \\ &= h\left(\frac{f(x_0)}{2} + f(x_1) + \ldots + f(x_{n-1}) + \frac{f(x_n)}{2}\right). \end{aligned} \tag{45}$$

Application to the current situation, with $h = 0.1$, $n = 3$, and the tabular values, gives

$$\int_0^{0.3} f(x)dx \approx 0.1\left(\frac{0}{2} + 0.01 + 0.04 + \frac{0.09}{2}\right) = 0.0095. \tag{46}$$