DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU AESB2210)
Thursday January 29 2015, 18:30-21:30

1. For the initial value problem $y' = f(t,y)$, $y(t_0) = y_0$, we use the following integration method:

$$\begin{cases} w^*_{n+1} = w_n + hf(t_n, w_n) \\ w_{n+1} = w_n + \frac{h}{2}\left(f(t_n, w_n) + f(t_{n+1}, w^*_{n+1})\right). \end{cases} \tag{1}$$

Here h denotes the timestep and w_n represents the numerical approximation at time t_n.

(a) Show that the local truncation error of the integration method is of the order $O(h^2)$. (You are not allowed to use the test equation here.) (3pt.)

Consider the following initial value problem

$$\begin{cases} \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = \cos t, \\ y(0) = 1, \quad \frac{dy}{dt}(0) = 2. \end{cases} \tag{2}$$

(b) Show that the above initial value problem can be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos t \end{pmatrix}. \tag{3}$$

Give the initial conditions for $x_1(0)$ and $x_2(0)$ as well. (1pt.)

(c) Calculate one step with the integration method, in which $h = 0.1$ and $t_0 = 0$, and use the given initial conditions. (2pt.)

(d) Derive the amplification factor for the integration method. (2pt.)

(e) Examine for which stepsizes $h > 0$, the integration method, applied to the initial value problem (2), is stable. (2pt.)

[0]**please turn over, For the answers of this test we refer to:**
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html

2. In this exercise an estimate is determined for the velocity of a rowing boat. The measured distance of the boat from the starting line are given in the table below.

t (s)	0	10	20
$d(t)$ (m)	0	40	100

(a) Give the first order backward difference formula and use this to determine an estimate of the velocity for $t = 20$ ($d'(20)$). (1 pt.)

(b) We are looking for a difference formula of the first derivative of d in $2h$ of the form:

$$Q(h) = \frac{\alpha_0}{h}d(0) + \frac{\alpha_1}{h}d(h) + \frac{\alpha_2}{h}d(2h),$$

such that

$$d'(2h) - Q(h) = O(h^2).$$

In the remainder of this exercise we use this formula. Show that the coefficients α_0, α_1 and α_2 should satisfy the next system:

$$\begin{array}{rcrcrcl} \frac{\alpha_0}{h} & + & \frac{\alpha_1}{h} & + & \frac{\alpha_2}{h} & = & 0\,, \\ -2\alpha_0 & - & \alpha_1 & & & = & 1\,, \\ 2\alpha_0 h & + & \frac{1}{2}\alpha_1 h & & & = & 0\,. \end{array}$$

(2 pt.)

(c) The solution of this system is given by $\alpha_0 = \frac{1}{2}$, $\alpha_1 = -2$ and $\alpha_2 = \frac{3}{2}$. Give for these values an expression for the truncation error $d'(2h) - Q(h)$. Use this formula to give an estimate of the velocity at $t = 20$. (2 pt.)

(d) The maximal measuring error in the measured position is bounded by ϵ: $|d(t) - \hat{d}(t)| \leq \epsilon$. Show that this implies a measuring error in the estimate of

$$|Q(h) - \hat{Q}(h)| \leq \frac{C_1 \epsilon}{h}$$

and give C_1. (1.5 pt.)

The Newton-Raphson method is based on the following formula:

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}.$$

(e) Derive the above formula for the Newton-Raphson method. (1 pt.)

(f) We are searching the positive zero of $f(x) = e^{\sin(x)} - \frac{1}{e}$.

- Use $p_0 = \pi$ as the initial guess and determine p_1 by the use of the Newton-Raphson method.
- Use $p_0 = \frac{3}{2}\pi$ as the initial guess and discuss if this is a sensible choice for the Newton-Raphson method.

(2.5 pt.)