DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU/Minor AESB2210)
Thursday April 14th 2016, 18:30-21:30

1. We consider the general initial value problem

$$y' = f(t,y), \qquad y(0) = y_0, \tag{1}$$

which we solve using the **backward Euler time integration method**.

$$w_{n+1} = w_n + \Delta t f(t_{n+1}, w_{n+1}). \tag{2}$$

(a) Use the **test equation**, to demonstrate that the local truncation error of the **backward Euler method** is order $\mathcal{O}(\Delta t)$.
Hint:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, \qquad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots. \tag{3}$$

(3 pt.)

(b) Use the **test equation**, to show that for general complex $\lambda = \mu + i\nu$, the numerical solution is stable if

$$(1 - \Delta t\mu)^2 + (\Delta t\nu)^2 \geq 1. \tag{4}$$

Sketch the stability region in the complex plane. (2 pt.)

We apply the **backward Euler method** to the following equations

$$\begin{aligned} y_1' &= -y_1^2 + 2y_1y_2, \\ y_2' &= -y_1y_2 - \tfrac{1}{2}y_2^2, \end{aligned} \tag{5}$$

subject to initial conditions, which will be specified later.

(c) Derive the Jacobian matrix from linearization of system (5) around $(y_1, y_2) = (1, 0)$, and give its eigenvalues. (1 pt.)

(d) Determine the maximum allowable time step size around $(y_1, y_2) = (1, 0)$ that warrants linear stability for the **backward Euler time integration method**. (1.5 pt.)
Hint: If you cannot find an answer to part (c) you can use $\lambda_1 = -4$ and $\lambda_2 = -3$ (note that these are **not** the correct eigenvalues).
Do the same for the the **forward Euler time integration method**. (1.5 pt.)

We use the **forward Euler method** to approximate the solution.

(e) Use the initial condition $(y_1(0), y_2(0)) = (1, 0)$ and time-step $\Delta t = 1$ to compute the numerical solution after one time-step. (1 pt.)

2. In this exercise an estimate is determined for the velocity of a drilling rig as it is used in geoscience applications to create holes in the earth sub-surface. The measured depth d of the drill bit from the surface of the earth are given in the table below:

t (min)	0	5	10
$d(t)$ (cm)	0	250	550

   (a) Give the **first order backward difference formula** and use this to determine an estimate of the velocity for $t = 10$, that is, $d'(10)$. (1.5 pt.)

   (b) We are looking for a difference formula of the first derivative of d in $2h$ of the form:
$$Q(h) = \frac{\alpha_0}{h}d(0) + \frac{\alpha_1}{h}d(h) + \frac{\alpha_2}{h}d(2h),$$
   such that
$$d'(2h) - Q(h) = O(h^2).$$
   In the remainder of this exercise we use this formula. Show that the coefficients $\alpha_0,\ \alpha_1$ and α_2 should satisfy the next system:
$$\begin{array}{rcrcrcl} \frac{\alpha_0}{h} & + & \frac{\alpha_1}{h} & + & \frac{\alpha_2}{h} & = & 0\,, \\ -2\alpha_0 & - & \alpha_1 & & & = & 1\,, \\ 2\alpha_0 h & + & \frac{1}{2}\alpha_1 h & & & = & 0\,. \end{array}$$
   (2 pt.)

   (c) The solution of this system is given by $\alpha_0 = \frac{1}{2}$, $\alpha_1 = -2$ and $\alpha_2 = \frac{3}{2}$. Give for these values an expression for the truncation error $d'(2h) - Q(h)$. Use this formula to give an estimate of the velocity at $t = 10$. (1.5 pt.)

3. We analyse **Lagrangian interpolation**. For given points $x_0, x_1, \ldots, x_n$, with their respective function values $f(x_0), f(x_1), \ldots, f(x_n)$, the interpolatory polynomial $L_n(x)$ is given by
$$L_n(x) = \sum_{k=0}^{n} f(x_k)L_{kn}(x), \text{ with}$$
$$L_{kn}(x) = \frac{(x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)}. \qquad (6)$$

   (a) Give the **linear Lagrangian interpolatory polynomial** $L_1(x)$ with nodes x_0 and x_1. (1 pt.)

   (b) Give the **quadratic Lagrangian interpolatory polynomial** $L_2(x)$ with nodes x_0, x_1 and x_2. (2 pt.)

   (c) Calculate $L_n(2)$ and $L_n(3)$ both by using linear and quadratic Lagrangian interpolation using the following measured values:

k	x_k	$f(x_k)$
0	1	3
1	3	6
2	4	5

   (2 pt.)

For the answers of this test we refer to:
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html