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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (CTB2400 WI3097TU)
Thursday August 11th 2016, 18:30-21:30

1. (a) The local truncation error is defined as

$$\tau_{n+1}(\Delta t) = \frac{y_{n+1} - z_{n+1}}{\Delta t}, \tag{1}$$

where z_{n+1} is given by

$$z_{n+1} = y_n + \Delta t \left(a_1 f(t_n, y_n) + a_2 f(t_n + \Delta t, y_n + \Delta t f(t_n, y_n) \right). \tag{2}$$

A Taylor expansion of f around (t_n, y_n) yields

$$f(t_n+\Delta t, y_n+\Delta t f(t_n, y_n)) = f(t_n, y_n)+\Delta t\frac{\partial f}{\partial t}(t_n, y_n)+\Delta t f(t_n, y_n)\frac{\partial f}{\partial y}(t_n, y_n)+O((\Delta t)^2). \tag{3}$$

This is substituted into equation (2) to obtain

$$z_{n+1} = y_n+\Delta t \left(a_1 f(t_n, y_n) + a_2 \left[f(t_n, y_n) + \Delta t\frac{\partial f}{\partial t}(t_n, y_n) + \Delta t f(t_n, y_n)\frac{\partial f}{\partial y}(t_n, y_n) \right] \right)+O((\Delta t)^3) \tag{4}$$

A Taylor series for $y(t)$ around t_n gives for y_{n+1}

$$y_{n+1} = y(t_n + \Delta t) = y_n + \Delta t y'(t_n) + \frac{(\Delta t)^2}{2} y''(t_n) + O((\Delta t)^3). \tag{5}$$

From the differential equation we know that:

$$y'(t_n) = f(t_n, y_n) \tag{6}$$

From the Chain Rule of Differentiation, we derive

$$y''(t_n) = \frac{df(t_n, y_n)}{dt} = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y} y'(t_n) \tag{7}$$

after substitution of the differential equation one obtains:

$$y''(t_n) = \frac{\partial f(t_n, y_n)}{\partial t} + \frac{\partial f(t_n, y_n)}{\partial y} f(t_n, y_n) \tag{8}$$

Equations (5) and (4) are substituted into relation (1) to obtain

$$\tau_{n+1}(\Delta t) = f(t_n, y_n)(1-(a_1+a_2))+\Delta t \left(\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y} \right) \left(\frac{1}{2} - a_2 \right)+O((\Delta t)^2) \tag{9}$$

Hence

i. $a_1 + a_2 = 1$ implies $\tau_{n+1}(\Delta t) = O(\Delta t)$;

ii. $a_1+a_2 = 1$ and $a_2 = 1/2$, that is, $a_1 = a_2 = 1/2$, gives $\tau_{n+1}(\Delta t) = O((\Delta t)^2)$.

(b) The test equation is given by

$$y' = \lambda y. \tag{10}$$

Application of the predictor step to the test equation gives

$$w^*_{n+1} = w_n + \lambda \Delta t w_n = (1 + \lambda \Delta t) w_n. \tag{11}$$

The corrector step yields

$$w_{n+1} = w_n + \Delta t\,(a_1 \lambda w_n + a_2 \lambda (1 + \lambda \Delta t) w_n) = (1 + (a_1 + a_2)\lambda \Delta t + a_2 (\lambda \Delta t)^2) w_n. \tag{12}$$

Hence the amplification factor is given by

$$Q(\lambda \Delta t) = 1 + (a_1 + a_2)\lambda \Delta t + a_2 (\lambda \Delta t)^2. \tag{13}$$

(c) Let $\lambda < 0$ (so λ is real), then, for stability, the amplification factor must satisfy

$$-1 \leq Q(\lambda \Delta t) \leq 1, \tag{14}$$

from the previous assignment, we have

$$-1 \leq 1 + (a_1 + a_2)\lambda \Delta t + a_2 (\lambda \Delta t)^2 \leq 1 \Leftrightarrow -2 \leq (a_1 + a_2)\lambda \Delta t + a_2 (\lambda \Delta t)^2 \leq 0. \tag{15}$$

First, we consider the left inequality:

$$a_2 (\lambda \Delta t)^2 + (a_1 + a_2)\lambda \Delta t + 2 \geq 0 \tag{16}$$

For $\lambda \Delta t = 0$, the above inequality is satisfied. The discriminant of the quadratic equation is given by $(a_1 + a_2)^2 - 8a_2$. From the given assumption it follows that $(a_1 + a_2)^2 - 8a_2 < 0$ so the quadratic equation does not have real roots. Hence the left inequality in relation (15) is always satisfied. Next we consider the right hand inequality of relation (15)

$$a_2 (\lambda \Delta t)^2 + (a_1 + a_2)\lambda \Delta t \leq 0. \tag{17}$$

This relation is rearranged into

$$a_2 (\lambda \Delta t)^2 \leq -(a_1 + a_2)\lambda \Delta t, \tag{18}$$

hence

$$a_2 |\lambda \Delta t|^2 \leq (a_1 + a_2)|\lambda \Delta t| \Leftrightarrow |\lambda \Delta t| \leq \frac{a_1 + a_2}{a_2}, \qquad a_2 \neq 0. \tag{19}$$

This results into the following condition for stability

$$\Delta t \leq \frac{a_1 + a_2}{a_2 |\lambda|}, \qquad a_2 \neq 0. \tag{20}$$

(d) In order to compute the Jacobian, we note that the right-hand side of the non linear system is given by:

$$f_1(x_1, x_2) = -\sin x_1 + 2x_2 + t$$

$$f_2(x_1, x_2) = x_1 - x_2^2$$

From the definition of the Jacobian it follows that:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -\cos x_1 & 2 \\ 1 & -2x_2 \end{pmatrix}.$$

Substitution of $\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ shows that

$$J = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}.$$

(e) For the stability it is sufficient to check that $|Q(\lambda_i \Delta t)| \leq 1$ for all the eigenvalues of the Jacobian matrix. It is easy to see that the eigenvalues of the Jacobian matrix are $\lambda_1 = -3$ and $\lambda_2 = 0$.

For the choice $a_1 = a_2 = \frac{1}{2}$ we use the expression

$$Q(\lambda \Delta t) = 1 + \lambda \Delta t + \frac{1}{2}(\lambda \Delta t)^2$$

For $\lambda_2 = 0$ it appears that $Q(\lambda_2 \Delta t) = 1$ so the inequality is satisfied for all Δt. For $\lambda_1 = -3$ we have to check the following inequalities:

$$-1 \leq 1 - 3\Delta t + \frac{9}{2}(\Delta t)^2 \leq 1$$

For the left-hand inequality we arrive at

$$0 \leq \frac{9}{2}(\Delta t)^2 - 3\Delta t + 2$$

It appears that the discriminant $9 - 4 \cdot \frac{9}{2} \cdot 2$ is negative, so there are no real roots which implies that the inequality is satisfied for all Δt.

For the right-hand inequality we get

$$-3\Delta t + \frac{9}{2}(\Delta t)^2 \leq 0$$

$$\frac{9}{2}(\Delta t)^2 \leq 3\Delta t$$

so

$$\Delta t \leq \frac{2}{3}$$

(another option is to see that for $a_1 = a_2 = \frac{1}{2}$ the method is equal to the modified Euler method, and remember that this method is stable for real eigenvalues if $\Delta t \leq \frac{-2}{\lambda}$)

2. (a) The *Taylor polynomials* around 0 are given by:

$$\begin{aligned} f(0) &= f(0) , \\ f(-h) &= f(0) - hf'(0) + \frac{h^2}{2} f''(0) - \frac{h^3}{6} f'''(\xi_1) , \\ f(-2h) &= f(0) - 2hf'(0) + 2h^2 f''(0) - \frac{(2h)^3}{6} f'''(\xi_2) . \end{aligned}$$

Here $\xi_1 \in (-h, 0)$, $\xi_2 \in (-2h, 0)$. We know that $Q(h) = \frac{\alpha_0}{h^2} f(0) + \frac{\alpha_{-1}}{h^2} f(-h) + \frac{\alpha_{-2}}{h^2} f(-2h)$, which should be equal to $f''(0) + \mathcal{O}(h)$. This leads to the following conditions:

$$\begin{array}{rccccccc} f(0): & \frac{\alpha_0}{h^2} & + & \frac{\alpha_{-1}}{h^2} & + & \frac{\alpha_{-2}}{h^2} & = & 0 , \\ f'(0): & & & -\frac{h\alpha_{-1}}{h^2} & - & \frac{2h\alpha_{-2}}{h^2} & = & 0 , \\ f''(0): & & & \frac{h^2}{2h^2}\alpha_{-1} & + & \frac{2h^2\alpha_{-2}}{h^2} & = & 1 . \end{array}$$

This can also be written as

$$\begin{array}{rccccccc} f(0): & \alpha_0 & + & \alpha_{-1} & + & \alpha_{-2} & = & 0 , \\ f'(0): & & & -\alpha_{-1} & - & 2\alpha_{-2} & = & 0 , \\ f''(0): & & & \frac{\alpha_{-1}}{2} & + & 2\alpha_{-2} & = & 1 . \end{array}$$

(b) The *truncation error* follows from the Taylor polynomials:

$$f''(0) - Q(h) = f''(0) - \frac{f(0) - 2f(-h) + f(-2h)}{h^2} = -\left(\frac{\frac{2h^3}{6} f'''(\xi_1) - \frac{8h^3}{6} f'''(\xi_2)}{h^2} \right)$$

$$= hf'''(\xi).$$

(c) Note that

$$f''(0) - Q(2h) = K2h \tag{21}$$

$$f''(0) - Q(h) = Kh \tag{22}$$

Subtraction gives:

$$Q(h) - Q(2h) = K2h - Kh = Kh. \tag{23}$$

We choose $h = \frac{1}{4}$. Then $Q(2h) = Q(\frac{1}{2}) = \frac{0-2\times 0.1250+1}{0.25} = 3$ and $Q(h) = Q(\frac{1}{4}) = \frac{0-2\times 0.0156+0.1250}{(\frac{1}{4})^2} = 1.5008$. Combining (22) and (23) shows that

$$f''(0) - Q(\frac{1}{4}) = Q(\frac{1}{4}) - Q(\frac{1}{2}) = -1.4992$$

3. (a) First, we check the boundary conditions:

$$u(0) = 0 - \frac{1-e^{0/\epsilon}}{1-e^{1/\epsilon}} = \frac{1-1}{1-e^{1/\epsilon}} = 0, \quad u(1) = 1 - \frac{1-e^{1/\epsilon}}{1-e^{1/\epsilon}} = 0. \tag{24}$$

Further, we have

$$u'(x) = 1 + \frac{e^{x/\epsilon}}{\epsilon(1-e^{1/\epsilon})}, \tag{25}$$

$$u''(x) = \frac{e^{x/\epsilon}}{\epsilon^2(1-e^{1/\epsilon})}. \tag{26}$$

Hence, we immediately see

$$-\epsilon u''(x) + u'(x) = -\frac{\epsilon e^{x/\epsilon}}{\epsilon^2(1-e^{1/\epsilon})} + 1 + \frac{e^{x/\epsilon}}{\epsilon(1-e^{1/\epsilon})} = 1. \tag{27}$$

Hence, the solution $u(x) = 1 - \frac{1-e^{x/\epsilon}}{1-e^{1/\epsilon}}$ satisfies the differential equation and the Dirichlet boundary conditions, and therewith $u(x)$ is the solution to the boundary value problem (uniqueness can be demonstrated in a straightforward way, but this was not asked for).

(b) The domain of computation, being $(0,1)$, is divided into subintervals with mesh points, we set $x_j = j\Delta x$, where we use n unknowns, such that $x_{n+1} = (n+1)\Delta x = 1$. The discretization method for an interior node is given by

$$-\epsilon\frac{w_{j+1} - 2w_j + w_{j-1}}{(\Delta x)^2} + \frac{w_j - w_{j-1}}{\Delta x} = 1, \text{ for } j \in \{1,\dots,n\}. \tag{28}$$

At the boundaries, we see for $j = 1$ and $j = n$, upon substituting $w_0 = 0$ and $w_{n+1} = 0$, respectively:

$$\begin{aligned} -\epsilon\frac{w_2 - 2w_1 + 0}{(\Delta x)^2} + \frac{w_1 - 0}{\Delta x} &= 1, \\ -\epsilon\frac{0 - 2w_n + w_{n-1}}{(\Delta x)^2} + \frac{w_n - w_{n-1}}{\Delta x} &= 1. \end{aligned} \tag{29}$$

This can be rewritten more neatly as follows:

$$\begin{aligned} \epsilon\frac{-w_2 + 2w_1}{(\Delta x)^2} + \frac{w_1}{\Delta x} &= 1, \\ \epsilon\frac{2w_n - w_{n-1}}{(\Delta x)^2} + \frac{w_n - w_{n-1}}{\Delta x} &= 1. \end{aligned} \tag{30}$$

(c) Next, we use $\Delta x = 1/4$ and thus $n = 3$. Then, from equations (28) and (30), one obtains the following 3×3 linear system of equations

$$\begin{bmatrix} 32\epsilon + 4 & -16\epsilon & 0 \\ -16\epsilon - 4 & 32\epsilon + 4 & -16\epsilon \\ 0 & -16\epsilon - 4 & 16\epsilon + 4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{31}$$

(d) No, the upwind difference method for the convective terms is designed to not produce oscillatory solutions independent of the size of the diffusion coefficient ϵ and the velocity v which is equal to one in this case.