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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS
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1. (a) The test equation is given by

$$y' = \lambda y.$$

Application of the method to the test equation gives

$$w_{n+1} = w_n + \frac{1}{2}\lambda\Delta t w_n + \frac{1}{2}\lambda\Delta t w_{n+1}.$$

This is equivalent to

$$\left(1 - \frac{1}{2}\lambda\Delta t\right) w_{n+1} = \left(1 + \frac{1}{2}\lambda\Delta t\right) w_n.$$

Hence the amplification factor is given by

$$Q(\lambda\Delta t) = \frac{1 + \frac{1}{2}\lambda\Delta t}{1 - \frac{1}{2}\lambda\Delta t}.$$

(b) The local truncation error for the test equation is given as

$$\tau_{n+1}(\Delta t) = \frac{e^{\lambda\Delta t} - Q(\lambda\Delta t)}{\Delta t} y_n. \tag{1}$$

A Taylor expansion of $e^{\lambda\Delta t}$ around $\lambda\Delta t = 0$ yields

$$e^{\lambda\Delta t} = 1 + \lambda\Delta t + \frac{1}{2}(\lambda\Delta t)^2 + \frac{1}{6}(\lambda\Delta t)^3 + \mathcal{O}(\Delta t^4). \tag{2}$$

A Taylor expansion of $Q(\lambda\Delta t)$ around $\frac{1}{2}\lambda\Delta t = 0$ yields

$$\begin{aligned} Q(\lambda\Delta t) &= \frac{1 + \frac{1}{2}\lambda\Delta t}{1 - \frac{1}{2}\lambda\Delta t} \\ &= \left(1 + \frac{1}{2}\lambda\Delta t\right)\left(1 + \frac{1}{2}\lambda\Delta t + \left(\frac{1}{2}\lambda\Delta t\right)^2 + \left(\frac{1}{2}\lambda\Delta t\right)^3 + \mathcal{O}(\Delta t^4)\right) \\ &= 1 + \lambda\Delta t + \frac{1}{2}(\lambda\Delta t)^2 + \frac{1}{4}(\lambda\Delta t)^3 + \mathcal{O}(\Delta t^4). \end{aligned} \tag{3}$$

Equations (2) and (3) are substituted into relation (1) to obtain

$$\tau_{n+1} = -\frac{1}{12} y_n \lambda^3 \Delta t^2 + \mathcal{O}(\Delta t^3),$$

hence

$$T = -\frac{1}{12} y_n \lambda^3.$$

(c) With $\mathbf{x} = [x_1, x_2, x_3]^T$, the problem can be written as $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$, with

$$A = \begin{bmatrix} -1 & 2 & -2 \\ 0 & -2 & -2 \\ 0 & 2 & -2 \end{bmatrix},$$

and

$$\mathbf{b} = \begin{bmatrix} 9 \\ 4 \\ 8 \end{bmatrix}.$$

The characteristic equation of A is given by

$$\begin{aligned} & \det(A - \lambda I) = 0 \\ \Rightarrow \quad & \begin{vmatrix} -1-\lambda & 2 & -2 \\ 0 & -2-\lambda & -2 \\ 0 & 2 & -2-\lambda \end{vmatrix} = 0 \\ \Rightarrow \quad & (-1-\lambda) \begin{vmatrix} -2-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = 0 \\ \Rightarrow \quad & (-1-\lambda)\left((-2-\lambda)^2 + 4\right) = 0. \end{aligned}$$

The eigenvalues of A are calculated from this as $\lambda_1 = -1$ and $\lambda_2 = \overline{\lambda_3} = -2 + 2i$. Because λ_2 and λ_3 are each other complex conjugates, stability is governed by λ_1 and λ_2.

For $\lambda_1 = -1$ and $\Delta t = 1$ we obtain

$$\begin{aligned} Q(\lambda_1 \Delta t) &= Q(-1) \\ &= \frac{1 + \frac{1}{2}(-1)}{1 - \frac{1}{2}(-1)} \\ &= \frac{\frac{1}{2}}{\frac{3}{2}} \\ &= \frac{1}{3}, \end{aligned}$$

and therefore

$$|Q(\lambda_1 \Delta t)| = \frac{1}{3} \leq 1. \tag{4}$$

For $\lambda_2 = -2 + 2i$ and $\Delta t = 1$ we obtain

$$\begin{aligned} Q(\lambda_2 \Delta t) &= Q(-2 + 2i) \\ &= \frac{1 + \frac{1}{2}(-2 + 2i)}{1 - \frac{1}{2}(-2 + 2i)} \\ &= \frac{i}{2 - i)} \\ &= -\frac{1}{5} + \frac{2}{5}i, \end{aligned}$$

and therefore

$$|Q(\lambda_2 \Delta t)| = \sqrt{\frac{1}{25} + \frac{4}{25}} = \sqrt{\frac{1}{5}} \leq 1. \tag{5}$$

From (4) and (5) it follows that the method applied to the given IVP is stable for $\Delta t = 1$.

(d) First note that holds

$$\mathbf{w}_0 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}.$$

We can show that

$$A\mathbf{w}_0 + \mathbf{b} = \mathbf{0}. \tag{6}$$

The given value for $\mathbf{w}_1$ is exactly equal to $\mathbf{w}_0$, so we also have as a direct consequence:

$$A\mathbf{w}_1 + \mathbf{b} = \mathbf{0}. \tag{7}$$

(6), (7) and the values for $\mathbf{w}_0$ and $\mathbf{w}_1$ can be substituted in the method, which leads to

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix},$$

which is mathematically correct. Therefor $\mathbf{w}_1$ as given is indeed the approximation of the exact solution at time $t = 1$.

Alternative solution: $\mathbf{w}_1$ can also be calculated explicitly be direct application of the method, which has the following calculations:

$$\mathbf{w}_0 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix},$$

$$\text{Method:} \qquad \mathbf{w}_1 = \mathbf{w}_0 + \frac{1}{2}\left(A\mathbf{w}_0 + \mathbf{b} + A\mathbf{w}_1 + \mathbf{b}\right),$$

$$\Rightarrow \qquad \left(I - \frac{1}{2}A\right)\mathbf{w}_1 = \left(I + \frac{1}{2}A\right)\mathbf{w}_0 + \mathbf{b},$$

$$\Rightarrow \qquad \begin{bmatrix} 3/2 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}\mathbf{w}_1 = \begin{bmatrix} 11/2 \\ 1 \\ 7 \end{bmatrix},$$

$$\Rightarrow \qquad \mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}.$$

No points will be given if a different method is used or a different system of differential equations is solved.

2. (a) The order can be determined with the use of Taylor expansions. We need the following:

$$\begin{aligned} y(x+\Delta x) &= y(x) + y'(x)\Delta x + \frac{1}{2}y''(x)\Delta x^2 + \frac{1}{6}y'''(x)\Delta x^3 + \mathcal{O}(\Delta x^4), \\ y(x) &= y(x), \\ y(x-\Delta x) &= y(x) - y'(x)\Delta x + \frac{1}{2}y''(x)\Delta x^2 - \frac{1}{6}y'''(x)\Delta x^3 + \mathcal{O}(\Delta x^4). \end{aligned}$$

Substitution of the above in the given finite difference $Q(\Delta x)$ gives

$$\begin{aligned} Q(\Delta x) &= \frac{y''(x)\Delta x^2 + \mathcal{O}(\Delta x^4)}{\Delta x^2} \\ &= y''(x) + \mathcal{O}(\Delta x^2). \end{aligned}$$

This shows that $Q(\Delta x)$ is indeed a $\mathcal{O}(\Delta x^2)$ approximation of $y''(x)$.

(b) Evaulation of the ode in $x = x_j$ and replacing $y''(x_j)$ with the given finite difference $Q(\Delta x)$ gives

$$-\frac{y(x_{j+1}) - 2y(x_j) + y(x_{j-1})}{\Delta x^2} + \mathcal{O}\left(\Delta x^2\right) + x_j y(x_j) = \sin(2\pi x_j).$$

Next, we neglect the truncation error, and set $w_j \approx y(x_j)$ to obtain

$$-\frac{w_{j+1} - 2w_j + w_{j-1}}{\Delta x^2} + j\Delta x w_j = \sin(2\pi j\Delta x), \tag{8}$$

which is the second of the given equations.

At the left boundary, $x = 0$, we have $w_0 = 0$, which after substitution in (8) for $j = 1$ gives

$$-\frac{w_2 - 2w_1}{\Delta x^2} + \Delta x w_1 = \sin(2\pi\Delta x),$$

which is the first of the given equations.

At the right boundary, $x = 1$, we have $w_{n+1} = 1$, which after substitution in (8) for $j = n$ gives

$$-\frac{-2w_n + w_{n-1}}{\Delta x^2} + n\Delta x w_n = \sin(2\pi n\Delta x) + \frac{1}{\Delta x^2},$$

which is the third of the given equations.

(c) The matrix A of the scheme is given by

$$A = \frac{1}{\Delta x^2}\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix} + \Delta x \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n \end{bmatrix}.$$

Because A is symmetric ($A^T = A$), all eigenvalues of A are real, so the Gershgorin circle theorem will give intervals in $\mathbb{R}$ instead of circles in $\mathbb{C}$.

The first row of A gives the inequalities

$$-\frac{1}{\Delta x^2} \leq \lambda - \left(\frac{2}{\Delta x^2} + \Delta x\right) \leq \frac{1}{\Delta x^2}.$$

These can be rewritten to

$$\frac{1}{\Delta x^2} + \Delta x \leq \lambda \leq \frac{3}{\Delta x^2} + \Delta x. \tag{9}$$

In a similar manner, row j of matrix A with $j \in \{2, \ldots, n-1\}$ gives

$$j\Delta x \leq \lambda \leq \frac{4}{\Delta x^2} + j\Delta x, \tag{10}$$

and row n finally gives

$$\frac{1}{\Delta x^2} + n\Delta x \leq \lambda \leq \frac{3}{\Delta x^2} + n\Delta x. \tag{11}$$

Combining Equations (9), (10) and (11) gives

$$\Delta x \leq \lambda \leq \frac{4}{\Delta x^2} + n\Delta x,$$

as was requested to show.

3. (a) The given formula for $s(x)$ consists of two polynomials of degree 3 on disjunct intervals dividing $[-1,1]$, so $s(x)$ is a piecewise function consisting of polynomials of degree 3 or lower.

(b) The nodes of the spline are $x=-1$, $x=0$ and $x=1$.

We will evaluate $s(x)$ in these three nodes and show that it is equal to $f(x)$ in these nodes:

$$\begin{aligned}
s(-1) &= -\frac{3}{4}x^3 - \frac{9}{4}x^2 + \frac{1}{2}x + 2\bigg|_{x=-1} \\
&= -\frac{3}{4}(-1)^3 - \frac{9}{4}(-1)^2 + \frac{1}{2}(-1) + 2 \\
&= \frac{3}{4} - \frac{9}{4} - \frac{1}{2} + 2 \\
&= 0 \\
&= f(-1), \\
s(0) &= \frac{3}{4}x^3 - \frac{9}{4}x^2 + \frac{1}{2}x + 2\bigg|_{x=0} \\
&= \frac{3}{4}(0)^3 - \frac{9}{4}(0)^2 + \frac{1}{2}(0) + 2 \\
&= 0 - 0 + 0 + 2 \\
&= 2 \\
&= f(0), \\
s(1) &= \frac{3}{4}x^3 - \frac{9}{4}x^2 + \frac{1}{2}x + 2\bigg|_{x=1} \\
&= \frac{3}{4}(1)^3 - \frac{9}{4}(1)^2 + \frac{1}{2}(1) + 2 \\
&= \frac{3}{4} - \frac{9}{4} + \frac{1}{2} + 2 \\
&= 1 \\
&= f(1).
\end{aligned}$$

(c) Because $s(x)$ consists of polynomials, the only possible point of discontinuity is the node $x=0$, so $s(x)$ is continuous if it is continuous in $x=0$.

Therefore we have to show

$$\lim_{x\to 0^-} s(x) = \lim_{x\to 0^+} s(x).$$

The left limit equals:

$$\begin{aligned}
\lim_{x\to 0^-} s(x) &= \lim_{x\to 0^-} -\frac{3}{4}x^3 - \frac{9}{4}x^2 + \frac{1}{2}x + 2 \\
&= 2.
\end{aligned}$$

The right limit equals:

$$\begin{aligned}
\lim_{x\to 0^+} s(x) &= \lim_{x\to 0^+} \frac{3}{4}x^3 - \frac{9}{4}x^2 + \frac{1}{2}x + 2 \\
&= 2.
\end{aligned}$$

So $s(x)$ is continuous.

The derivative $s'(x)$ is given by

$$s'(x) = \begin{cases} -\frac{9}{4}x^2 - \frac{9}{2}x + \frac{1}{2} & \text{if} \quad x \in [-1, 0), \\ \\ \frac{9}{4}x^2 - \frac{9}{2}x + \frac{1}{2} & \text{if} \quad x \in [0, 1]. \end{cases}$$

$s'(x)$ is continuous if it is continuous in $x = 0$, so we have to show

$$\lim_{x \to 0^-} s'(x) = \lim_{x \to 0^+} s'(x).$$

The left limit equals:

$$\begin{aligned} \lim_{x \to 0^-} s'(x) &= \lim_{x \to 0^-} -\frac{9}{4}x^2 - \frac{9}{2}x + \frac{1}{2} \\ &= \frac{1}{2}. \end{aligned}$$

The right limit equals:

$$\begin{aligned} \lim_{x \to 0^+} s'(x) &= \lim_{x \to 0^+} \frac{9}{4}x^2 - \frac{9}{2}x + \frac{1}{2} \\ &= \frac{1}{2}. \end{aligned}$$

So $s'(x)$ is continuous.
The second derivative $s''(x)$ is given by

$$s''(x) = \begin{cases} -\frac{9}{2}x - \frac{9}{2} & \text{if} \quad x \in [-1, 0), \\ \\ \frac{9}{2}x - \frac{9}{2} & \text{if} \quad x \in [0, 1]. \end{cases}$$

$s''(x)$ is continuous if it is continuous in $x = 0$, so we have to show

$$\lim_{x \to 0^-} s''(x) = \lim_{x \to 0^+} s''(x).$$

The left limit equals:

$$\begin{aligned} \lim_{x \to 0^-} s''(x) &= \lim_{x \to 0^-} -\frac{9}{2}x - \frac{9}{2} \\ &= -\frac{9}{2}. \end{aligned}$$

The right limit equals:

$$\begin{aligned} \lim_{x \to 0^+} s''(x) &= \lim_{x \to 0^+} \frac{9}{2}x - \frac{9}{2} \\ &= -\frac{9}{2}. \end{aligned}$$

So $s''(x)$ is continuous.

(d) Evualuating $s''(x)$ in $x = -1$ gives:

$$s''(-1) = -\frac{9}{2}x - \frac{9}{2}\bigg|_{x=-1} = \frac{9}{2} - \frac{9}{2} = 0,$$

and evaluation at $x = 1$ gives

$$s''(1) = \frac{9}{2}x - \frac{9}{2}\bigg|_{x=1} = \frac{9}{2} - \frac{9}{2} = 0,$$

so indeed $s''(x) = 0$ in the end points.

(e) $x = -\frac{1}{2}$ lies in the left interval, so we need to perform the next calculation:

$$\begin{aligned}
f\left(-\frac{1}{2}\right) &\approx s\left(-\frac{1}{2}\right) \\
&= -\frac{3}{4}x^3 - \frac{9}{4}x^2 + \frac{1}{2}x + 2\bigg|_{x=-\frac{1}{2}} \\
&= -\frac{3}{4}\left(-\frac{1}{2}\right)^3 - \frac{9}{4}\left(-\frac{1}{2}\right)^2 + \frac{1}{2}\left(-\frac{1}{2}\right) + 2 \\
&= -\frac{3}{4}\left(-\frac{1}{8}\right) - \frac{9}{4}\left(\frac{1}{4}\right) + \frac{1}{2}\left(-\frac{1}{2}\right) + 2 \\
&= \frac{3}{32} - \frac{9}{16} - \frac{1}{4} + 2 \\
&= \frac{3}{32} - \frac{18}{32} - \frac{8}{32} + \frac{64}{32} \\
&= \frac{3 - 18 - 8 + 64}{32} \\
&= \frac{41}{32}.
\end{aligned}$$