DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

Examiner responsible: D. den Ouden-van der Horst
Examination reviewer: C. Vuik

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS
(WI3097TU WI3097Minor WI3197Minor AESB2210 AESB2210-18 CTB2400)
Thursday January 31st 2019, 13:30-16:30

Number of questions: This is an exam with 12 open questions, subdivided in 3 main questions.

Answers All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will not give points.

Tools Only a non-graphical, non-programmable calculator is permitted. All other tools are not permitted.

Assessment In total 20 points can be earned. The final not-rounded grade is given by $^P/_2$, where P is the number of points earned.

1. We consider the following method

$$w_{n+1} = w_n + \frac{1}{2}\Delta t \left(f(t_n, w_n) + f(t_{n+1}, w_{n+1})\right) \tag{1}$$

for the integration of the **initial value problem** $y' = f(t, y)$, $y(t_0) = y_0$.

(a) Demonstrate that the *amplification factor* is given by

$$Q(\lambda\Delta t) = \frac{1 + \frac{1}{2}\lambda\Delta t}{1 - \frac{1}{2}\lambda\Delta t}.$$

($1\frac{1}{2}$ pt.)

(b) Show that the *local truncation error* of (1) for the test equation $y' = \lambda y$ takes on the form

$$\tau_{n+1} = T\Delta t^2 + \mathcal{O}(\Delta t^3),$$

and *give* a formula for T. ($3\frac{1}{2}$ pt.)

Hint: $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \mathcal{O}(x^4)$.
Hint: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \mathcal{O}(x^4)$.

(c) We consider the following *system of linear differential equations*:

$$\begin{cases} x_1' &= -x_1 + 2x_2 - 2x_3 + 9, \\ x_2' &= -2x_2 - 2x_3 + 4, \\ x_3' &= 2x_2 - 2x_3 + 8, \\ x_1(0) &= 1, x_2(0) = -1, x_3(0) = 3. \end{cases} \tag{2}$$

Write the above system as $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ and *show* that the application of (1) to (2) is stable for $\Delta t = 1$. ($3\frac{1}{2}$ pt.)

(d) The approximation $\mathbf{w}_1$ of the solution of the system (2) at time $t = 1$ obtained by applying (1) to (2) with $\Delta t = 1$ is calculated by us as

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}.$$

Show that the given value for $\mathbf{w}_1$ is correct. ($1\frac{1}{2}$ pt.)

please turn over

2. We consider the following boundary-value problem:

$$\begin{cases} -y''(x) + xy(x) &= \sin(2\pi x), \quad x \in (0,1), \\ y(0) &= 0, \\ y(1) &= 1. \end{cases} \tag{3}$$

In this exercise we approximate the exact solution with a numerical method.

(a) *Show* that

$$Q(\Delta x) = \frac{y(x+\Delta x) - 2y(x) + y(x-\Delta x)}{\Delta x^2}, \tag{4}$$

is a $\mathcal{O}(\Delta x^2)$ approximation of $y''(x)$. (1 pt.)

(b) We solve the boundary value problem (3) using the finite difference (4), upon setting $x_j = j\Delta x$, $(n+1)\Delta x = 1$, with Δx as the uniform stepsize. We then obtain the following formulas:

$$\begin{aligned} -\frac{w_2 - 2w_1}{(\Delta x)^2} + \Delta x w_1 &= \sin(2\pi\Delta x), \\ -\frac{w_{j+1} - 2w_j + w_{j-1}}{(\Delta x)^2} + j\Delta x w_j &= \sin(2\pi j\Delta x), \qquad \text{for } j \in \{2,\dots,n-1\}, \\ -\frac{-2w_n + w_{n-1}}{(\Delta x)^2} + n\Delta x w_n &= \sin(2\pi n\Delta x) + \frac{1}{\Delta x^2}. \end{aligned}$$

Give a *derivation* (with arguments) of this scheme. (2 pt.)

(c) The Gershgorin circle theorem states:

The eigenvalues of a general $n \times n$ matrix A are located in the complex plane in the union of circles

$$|z - a_{ii}| \le \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \quad \textit{where} \quad z \in \mathbb{C}.$$

Use this theorem to *show* that any eigenvalue λ of the given scheme (in the form $A\mathbf{w} = \mathbf{f}$) satisfies

$$\Delta x \le \lambda \le \frac{4}{(\Delta x)^2} + n\Delta x.$$

(2 pt.)

3. We have approximated a function f satisfying $f(-1) = 0$, $f(0) = 2$ and $f(1) = 1$ with a natural cubic spline s given by

$$s(x) = \begin{cases} -\frac{3}{4}x^3 - \frac{9}{4}x^2 + \frac{1}{2}x + 2 & \text{if} \quad x \in [-1,0), \\ \frac{3}{4}x^3 - \frac{9}{4}x^2 + \frac{1}{2}x + 2 & \text{if} \quad x \in [0,1]. \end{cases} \tag{5}$$

In the next exercises you will prove that s is indeed the natural cubic spline based on f. Then you will use s to approximate $f\left(-\frac{1}{2}\right)$.

(a) *Show* that s is a piecewise function consisting of polynomials of degree 3 or lower. ($\frac{1}{2}$ pt.)

(b) *Show* that $s(x)$ equals $f(x)$ in the nodes. (1 pt.)

(c) *Show* that s, s' and s'' are continuous on the interval $[-1,1]$. (2 pt.)

(d) *Show* that $s''(x)$ equals zero in the end points. ($\frac{1}{2}$ pt.)

(e) *Approximate* $f\left(-\frac{1}{2}\right)$ with the use of (5). (1 pt.)

For the answers of this test we refer to:
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html