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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS
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1. (a) The local truncation error is defined as

$$\tau_{n+1}(\Delta t) = \frac{y_{n+1} - z_{n+1}}{\Delta t}, \tag{1}$$

where z_{n+1} is given by

$$z_{n+1} = y_n + \Delta t \left(a_1 f(t_n, y_n) + a_2 f(t_n + \Delta t, y_n + \Delta t f(t_n, y_n)) \right). \tag{2}$$

A Taylor expansion of f around (t_n, y_n) of the last term yields

$$\begin{aligned} f(t_n + \Delta t, y_n + \Delta t f(t_n, y_n)) &= f(t_n, y_n) + \Delta t \frac{\partial f}{\partial t}(t_n, y_n) \\ &\quad + \Delta t f(t_n, y_n) \frac{\partial f}{\partial y}(t_n, y_n) + \mathcal{O}(\Delta t^2) \\ &= y'(t_n) + \Delta t y''(t_n) + \mathcal{O}(\Delta t^2). \end{aligned}$$

This is substituted into Equation (2) to obtain

$$\begin{aligned} z_{n+1} &= y_n + \Delta t \left(a_1 f(t_n, y_n) + a_2 \left[y'(t_n) + \Delta t y''(t_n) + \mathcal{O}(\Delta t)^2 \right] \right) \\ &= y_n + \Delta t y'(t_n) + a_2 \Delta t^2 y''(t_n) + \mathcal{O}(\Delta t^3). \end{aligned} \tag{3}$$

A Taylor series for $y(t)$ around t_n gives for y_{n+1}

$$y_{n+1} = y_n + \Delta t y'(t_n) + \frac{1}{2}(\Delta t)^2 y''(t_n) + \mathcal{O}(\Delta t^3). \tag{4}$$

Equations (4) and (3) are substituted into relation (1) to obtain

$$\tau_{n+1} = \left(\frac{1}{2} - a_2 \right) \Delta t y''(t_n) + \mathcal{O}(\Delta t^2).$$

Hence $\tau_{n+1} = \mathcal{O}(\Delta t)$ in general.
If $a_2 = \frac{1}{2}$, we have $\tau_{n+1} = \mathcal{O}(\Delta t^2)$.

(b) The test equation is given by

$$y' = \lambda y.$$

Application of the predictor step to the test equation gives

$$w^*_{n+1} = w_n + \lambda \Delta t w_n = (1 + \lambda \Delta t) w_n.$$

The corrector step yields

$$w_{n+1} = w_n + \Delta t \left(a_1 \lambda w_n + a_2 \lambda (1 + \lambda \Delta t) w_n \right) = (1 + \lambda \Delta t + a_2 (\lambda \Delta t)^2) w_n.$$

Hence the amplification factor is given by

$$Q(\lambda \Delta t) = 1 + \lambda \Delta t + a_2 (\lambda \Delta t)^2.$$

(c) Let $\lambda < 0$ (so λ is real), then, for stability, the amplification factor must satisfy

$$-1 \leq Q(\lambda\Delta t) \leq 1.$$

From the previous assignment, we have

$$-1 \leq 1 + \lambda\Delta t + a_2(\lambda\Delta t)^2 \leq 1,$$

which is equivalent to

$$-2 \leq \lambda\Delta t + a_2(\lambda\Delta t)^2 \leq 0.$$

First, we consider the left inequality:

$$a_2(\lambda\Delta t)^2 + \lambda\Delta t + 2 \geq 0$$

For $\Delta t = 0$, the above inequality is satisfied ($2 \geq 0$).
The discriminant of the quadratic inequality in Δt is given by $D = \lambda^2(1 - 8a_2)$. From the given assumption it follows that $D < 0$ so the quadratic equation does not have real roots. Hence the left inequality in relation (1) is always satisfied.
Next we consider the right hand inequality of relation (1)

$$a_2(\lambda\Delta t)^2 + \lambda\Delta t \leq 0.$$

This relation is rearranged into

$$a_2(\lambda\Delta t)^2 \leq -\lambda\Delta t,$$

hence

$$\Delta t \leq \frac{-1}{a_2\lambda}.$$

is the only stability condition.

(d) In order to compute the bound for Δt we need the eigenvalues of the Jacobian matrix in the initial point and we note that the right-hand side of the non linear system is given by:

$$f_1(t, x_1, x_2) = \cos x_1 - 2x_2 + t$$
$$f_2(t, x_1, x_2) = \frac{1}{2}x_1 - x_2^2$$

From the definition of the Jacobian it follows that:

$$J(t, x_1, x_2) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -\sin x_1 & -2 \\ \frac{1}{2} & -2x_2 \end{bmatrix}.$$

Substitution of $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} \pi \\ 1 \end{bmatrix}$ and $t = 0$ shows that

$$J(0, \pi, 1) = \begin{bmatrix} 0 & -2 \\ \frac{1}{2} & -2 \end{bmatrix}.$$

The eigenvalues of $J(0, \pi, 1)$ are calculated as $\lambda_1 = \lambda_2 = -1$.
Because λ_1 and λ_2 are real and negative, the stability bound from question (c) can be applied. Therefore the given method applied to the given system is stable for $\Delta t > 0$ satisfying

$$\Delta t \leq \frac{-1}{\frac{1}{2}(-1)} = 2,$$

at $t = 0$.

(e) With $\mathbf{f}(t,\mathbf{x}) = [f_1(t,x_1,x_2), f_2(t,x_1,x_2)]^T$ and $\mathbf{x} = [x_1,x_2]^T$, applying the method as stated gives

$$\begin{aligned}
\mathbf{w}_0 &= \begin{bmatrix} \pi \\ 1 \end{bmatrix}, \\
\mathbf{f}(0,\mathbf{w}_0) &= \begin{bmatrix} -3 \\ \frac{1}{2}\pi - 1 \end{bmatrix}, \\
\mathbf{w}_1^* &= \begin{bmatrix} \pi - 3 \\ \frac{1}{2}\pi \end{bmatrix} = \begin{bmatrix} 0.1415 \\ 1.5707 \end{bmatrix}, \\
\mathbf{f}(1,\mathbf{w}_1^*) &= \begin{bmatrix} \cos(\pi-3) - \pi + 1 \\ \frac{1}{2}\pi - \frac{3}{2} - \frac{1}{4}\pi^2 \end{bmatrix}, \\
\mathbf{w}_1 &= \begin{bmatrix} \frac{1}{2}\pi + \frac{1}{2}\cos(\pi-3) - 1 \\ \frac{1}{2}\pi - \frac{1}{4} - \frac{1}{8}\pi^2 \end{bmatrix} = \begin{bmatrix} 1.0658 \\ 0.0871 \end{bmatrix}.
\end{aligned}$$

2. (a) The solution and its first and second derivative are given by

$$\begin{aligned} y(x) &= e^x(2-x), \\ y'(x) &= e^x(1-x), \\ y''(x) &= -xe^x. \end{aligned}$$

Substitution of the above in the left hand side of the ordinary differential equation gives

$$\begin{aligned} -y''(x)+y(x) &= -xe^x + e^x(2-x) \\ &= -xe^x + 2e^2 - xe^x \\ &= 2e^x, \end{aligned}$$

which shows that it is indeed a solution to the given ode.
The left boundary condition is also satisfied:

$$y(0) = e^0(2-0) = 2,$$

and so is the right boundary condition:

$$y'(1) = e^1(1-1) = 0.$$

(b) The given formulas show that central difference approximations are used so we need formulae with a local truncation error of $\mathcal{O}\left(\Delta x^2\right)$.
Evaulation of the ode in $x = x_j$ and replacing $y''(x_j)$ with a finite difference of $\mathcal{O}\left(\Delta x^2\right)$ gives

$$-\frac{y(x_{j+1}) - 2y(x_j) + y(x_{j-1})}{\Delta x^2} + \mathcal{O}\left(\Delta x^2\right) + y(x_j) = 2e^{j\Delta x}.$$

Next, we neglect the truncation error, and set $w_j \approx y(x_j)$ to obtain

$$-\frac{w_{j+1} - 2w_j + w_{j-1}}{\Delta x^2} + w_j = 2e^{j\Delta x}, \tag{5}$$

which is the second of the given equations.
At the left boundary, $x = 0$, we have $w_0 = 2$, which after substitution in (5) for $j = 1$ gives

$$-\frac{w_2 - 2w_1}{\Delta x^2} + w_1 = 2e^{\Delta x} + \frac{2}{\Delta x^2},$$

which is the first of the given equations.
At the right boundary, $x = 1$, we approximate $y'(1)$ with a second-order central finite-difference:

$$\frac{y(x_{n+1}) - y(x_{n-1})}{2\Delta x} + \mathcal{O}(\Delta x^2) = 0,$$

which after neglecting the errors results in

$$w_{n+1} = w_{n-1}.$$

Substitution of the above in (5) with $j = n$ gives

$$-\frac{-2w_n + 2w_{n-1}}{\Delta x^2} + w_n = 2e,$$

which is the third of the given equations.

(c) Next, we use $\Delta x = 1/3$, so $n = 3$ and then, from the given equations, one obtains the following system:

$$\begin{aligned} 19w_1 - 9w_2 &= 2e^{1/3} + 18 \\ -9w_1 + 19w_2 - 9w_3 &= 2e^{2/3} \\ -18w_2 + 19w_3 &= 2e \end{aligned}$$

This means with $\mathbf{w} = [w_1, w_2, w_3]^T$ that

$$A = \begin{bmatrix} 19 & -9 & 0 \\ -9 & 19 & -9 \\ 0 & -18 & 19 \end{bmatrix},$$

and

$$\mathbf{b} = \begin{bmatrix} 2e^{1/3} + 18 \\ 2e^{2/3} \\ 2e \end{bmatrix}.$$

3. (a) The quadratic Lagrangian interpolation polynomial, with nodes x_0,x_1 and x_2 is given by

$$L_2(x) = f(x_0)L_{02}(x) + f(x_1)L_{12}(x) + f(x_2)L_{22}(x).$$

Approximating $f(x)$ by $L_2(x)$ and integration over x from $x_0 = a$ to $x_2 = b$ gives:

$$\begin{aligned}
\int_a^b f(x)\,\mathrm{d}x &\approx \int_a^b L_2(x)\,\mathrm{d}x \\
&= \int_a^b f(a)L_{02}(x) + f\left(\frac{a+b}{2}\right) L_{12}(x) + f(b)L_{22}(x)\,\mathrm{d}x \\
&= f(a)\int_a^b L_{02}(x)\,\mathrm{d}x + f\left(\frac{a+b}{2}\right)\int_a^b L_{12}(x)\,\mathrm{d}x + f(b)\int_a^b L_{22}(x)\,\mathrm{d}x \\
&= f(a)\frac{b-a}{6} + f\left(\frac{a+b}{2}\right)\frac{2(b-a)}{3} + f(b)\frac{b-a}{6} \\
&= \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right).
\end{aligned}$$

(b) A polynomial $p(x)$ of degree 3 or lower has the general form and derivatives

$$\begin{aligned}
p(x) &= ax^3 + bx^2 + cx + d, \\
p'(x) &= 3ax^2 + 2bx + c, \\
p''(x) &= 6ax + 2b, \\
p^{(3)}(x) &= 6a, \\
p^{(4)}(x) &= 0.
\end{aligned}$$

From the last line follows $m_4 = 0$, which after substitution in the upperbound gives

$$\left|\int_a^b f(x)\mathrm{d}x - I\right| = 0,$$

i.e. the approximation is exact.

(c) Applying Simpson's rule with $a = 0$, $b = 1$ and $f(x) = x^4$ results in

$$\begin{aligned}
\int_0^1 x^4\mathrm{d}x &\approx \frac{1}{6}\left(0^4 + 4\left(\frac{1}{2}\right)^4 + 1^4\right) \\
&= \frac{1}{6}\left(0 + \frac{1}{4} + 1\right) \\
&= \frac{1}{6}\frac{5}{4} \\
&= \frac{5}{24}.
\end{aligned}$$

Also

$$\int_0^1 x^4\mathrm{d}x = \frac{1}{5},$$

so the absolute value of the truncation error equals

$$\left|\int_0^1 x^4\mathrm{d}x - I\right| = \left|\frac{1}{5} - \frac{5}{24}\right| = \left|-\frac{1}{120}\right| = \frac{1}{120}.$$