DELFT UNIVERSITY OF TECHNOLOGY
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TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS
(WI3097TU WI3097Minor WI3197Minor AESB2210 AESB2210-18 CTB2400)
Tuesday August 13th 2019, 13:30-16:30

Number of questions: This is an exam with 11 open questions, subdivided in 3 main questions.

Answers All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will not give points.

Tools Only a non-graphical, non-programmable calculator is permitted. All other tools are not permitted.

Assessment In total 20 points can be earned. The final not-rounded grade is given by $P/2$, where P is the number of points earned.

1. We consider the following method

$$\begin{cases} w^*_{n+1} = w_n + \Delta t f(t_n, w_n) \\ w_{n+1} = w_n + \Delta t \left(a_1 f(t_n, w_n) + a_2 f(t_{n+1}, w^*_{n+1})\right) \end{cases}$$

for the integration of the **initial value problem** $y' = f(t, y)$, $y(t_0) = y_0$. The constants a_1 and a_2 satisfy $a_1 + a_2 = 1$.

(a) Show that the *local truncation error* of the above method has order $\mathcal{O}(\Delta t)$ in general. For *which value(s)* of a_1 and a_2 will the above method have a local truncation error of order $\mathcal{O}(\Delta t^2)$? ($3\frac{1}{2}$ pt.)

(b) Demonstrate that for *general values of* a_1 and a_2 the *amplification factor* is given by

$$Q(\lambda \Delta t) = 1 + \lambda \Delta t + a_2 (\lambda \Delta t)^2.$$

($1\frac{1}{2}$ pt.)

(c) Consider $\lambda < 0$ and $1 - 8a_2 < 0$. Show that the above method is *stable* for all $\Delta t > 0$ satisfying

$$\Delta t \leq \frac{-1}{a_2 \lambda}.$$

(2 pt.)

(d) We consider the following *system of non-linear differential equations*:

$$\begin{cases} x_1' &= \cos x_1 - 2x_2 + t, \\ x_2' &= \frac{1}{2}x_1 - x_2^2, \\ x_1(0) &= \pi, \\ x_2(0) &= 1. \end{cases} \qquad (1)$$

We choose $a_1 = a_2 = \frac{1}{2}$. For which *values* of Δt is the method applied to (1) stable at $t = 0$? ($1\frac{1}{2}$ pt.)

(e) We again choose $a_1 = a_2 = \frac{1}{2}$. *Perform one time step* with the given method and $\Delta t = 1$ to obtain an approximation of the solution of the system (1) at time $t = 1$. ($1\frac{1}{2}$ pt.)

please turn over

2. We consider the following boundary-value problem:

$$\begin{cases} -y''(x) + y(x) &= 2e^x, \quad x \in (0,1), \\ y(0) &= 2, \\ y'(1) &= 0. \end{cases} \tag{2}$$

The exact solution is given by

$$y(x) = e^x(2-x). \tag{3}$$

In this exercise we try to approximate this exact solution with a numerical method.

(a) *Show* that Equation (3) indeed constitutes the exact solution of Problem (2). (1 pt.)

(b) We solve the boundary value problem (3) using finite differences with a local truncation error of $\mathcal{O}(\Delta x^2)$, upon setting $x_j = j\Delta x$, $n\Delta x = 1$, where Δx denotes the uniform stepsize. After discretization we obtain the following formulas:

$$\begin{aligned} -\frac{w_2 - 2w_1}{(\Delta x)^2} + w_1 &= 2e^{\Delta x} + \frac{2}{\Delta x^2}, \\ -\frac{w_{j+1} - 2w_j + w_{j-1}}{(\Delta x)^2} + w_j &= 2e^{j\Delta x}, \qquad \text{for } j \in \{2, \dots, n-1\}, \\ -\frac{-2w_n + 2w_{n-1}}{(\Delta x)^2} + w_n &= 2e. \end{aligned}$$

Give (with arguments) the *derivation* of this scheme. (3 pt.)

(c) Choose $\Delta x = 1/3$ and derive the *system of equations* $A\mathbf{w} = \mathbf{b}$ with $\mathbf{w} = [w_1, \dots, w_n]^T$. Explicitly state A and $\mathbf{b}$ in your answer. (1 pt.)

3. To approximate $\int_a^b f(x)\,\mathrm{d}x$ Simpson's rule

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

can be used. Simpson's rule is based on the assumption $f(x) \approx L_2(x)$, with $L_2(x)$ the quadratic interpolatory polynomial with nodes $x_0 = a$, $x_1 = \frac{a+b}{2}$ and $x_2 = b$.

We know furthermore the following integrals:

$$\int_{x_0}^{x_2} L_{k2}(x)\mathrm{d}x = \begin{cases} \frac{1}{6}(x_2 - x_0) & \text{if } k \in \{0,2\}, \\ \frac{2}{3}(x_2 - x_0) & \text{if } k = 1, \end{cases}$$

where $L_{k2}(x)$ is the quadratic Lagrange basis polynomial of node x_k.

(a) Give a *derivation* of Simpson's rule. (2 pt.)

(b) Given is that an upperbound for the truncation error of Simpson's rule I is

$$\left|\int_a^b f(x)\,\mathrm{d}x - I\right| \leq \frac{1}{2880} m_4 (b-a)^5,$$

where $m_4 = \max_{a \leq x \leq b} |f^{(4)}(x)|$.
Show that Simpson's rule is exact for polynomials of degree 3 and lower. ($1\frac{1}{2}$ pt.)

(c) *Approximate* $\int_0^1 x^4\,\mathrm{d}x$ with Simpson's rule and give the absolute value of the *truncation error* in this approximation. ($1\frac{1}{2}$ pt.)

For the answers of this test we refer to:
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html