Meshless Discretization of Generalized Laplace Operator For Anisotropic Heterogeneous Media

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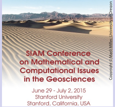
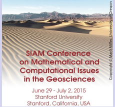


Outline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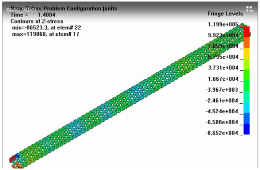
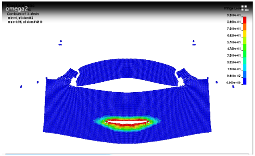
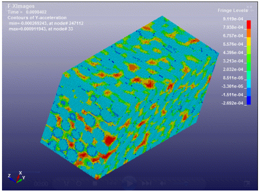
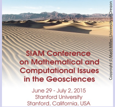
Introduction









Introduction

Literature

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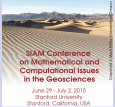
- Gingold, R.A., Monaghan, J.J. [1977]Smoothed particle hydrodynamics: theory and application to non-spherical stars, *Monthly Notices Royal Astronom. Soc.*, **181**, 135.
- Liu, M. B., Liu, G. R. [2003] Particle Hydrodynamics: A Meshfree Particle Method. World Scientific Publishing Co. Pte. Ltd. 449 p.
- OTHER:
  Belytschko, Li, Liu, Atluri, Liszka, Yagawa, Yamada, Nayroles, Johnson, Morris, Williams, Zhu, Fox, Chen, Chaniotis, Duarte, Oden, Brookshaw, Schwaiger



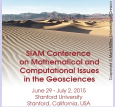
Introduction

- *Fundamental Relations* $\mathbf{r}', \mathbf{r} \in \mathbb{R}^3$:

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \langle \mathbf{A}(\mathbf{r}'), \delta(\mathbf{r}-\mathbf{r}')\rangle = \int_{\Omega,\mathbf{r}\in\Omega} \mathbf{A}(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')d\mathbf{r}' \\ \langle \mathbf{1}, \delta(\mathbf{r}-\mathbf{r}')\rangle &= \int_{\Omega,\mathbf{r}\in\Omega} \delta(\mathbf{r}-\mathbf{r}')d\mathbf{r}' = 1 \end{aligned} \tag{1}$$

- *Set of Kernel Functions* $\{W(\mathbf{r}-\mathbf{r}', h)\} \in C^0(\Omega)$:

$$\begin{aligned} &\lim_{h\to 0} W\{W(\mathbf{r}-\mathbf{r}', h)\} = \text{weakly} = \delta(\mathbf{r}-\mathbf{r}') \\ &\int_{\Omega,\mathbf{r}\in\Omega} W(\mathbf{r}-\mathbf{r}', h)d\mathbf{r}' = 1 \end{aligned} \tag{2}$$

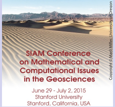


Introduction

- *Basic Equalities*:

$$\mathbf{A}(\mathbf{r}) = \lim_{h \to 0} \int_{\Omega, \mathbf{r} \in \Omega} \mathbf{A}(\mathbf{r}') W(\mathbf{r} - \mathbf{r}', h) d\mathbf{r}'$$

$$\mathbf{A}(\mathbf{r}) = \int_{\Omega, \mathbf{r} \in \Omega} \mathbf{A}(\mathbf{r}') W(\mathbf{r} - \mathbf{r}', h) d\mathbf{r}' + O(h^2) = \\ = \sum_{J \in \Omega_{\mathbf{r},h}} \mathbf{A}(\mathbf{r}_J) W(\mathbf{r} - \mathbf{r}_J, h) V_J + O(h^2), \forall h \in \Omega_h \tag{3}$$



Introduction

Kernel Function

$$W(z,h) = \frac{\Xi}{h^D} \begin{cases} 1 - \frac{3}{2}z^2 + \frac{3}{4}z^3, & 0 \le z \le 1 \\ \frac{1}{4}(2-z)^3, & 1 \le z \le 2 \\ 0, & z > 2 \end{cases} \tag{4}$$

where: $z = \left\| \mathbf{r} - \mathbf{r}' \right\|_2 / h$

$\Xi = \frac{3}{2}, \frac{10}{7\pi}, \frac{1}{\pi}$ in 1D, 2D and 3D respectively.

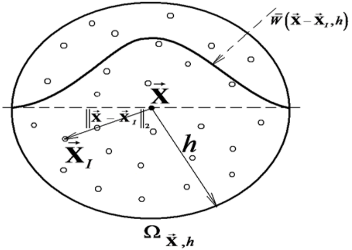


Figure: Neighboring particles of a Kernel support.

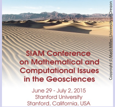
Generalized Laplace Operator For Anisotropic Heterogeneous Media

- *Generalized Laplace Operator*:

$$\mathbf{L}u = -\nabla\left(\mathbf{M}\left(\mathbf{r}\right)\nabla u\left(\mathbf{r}\right)\right) - g\left(\mathbf{r}\right),\ \ \mathbf{r} \in \Omega \subset \mathbb{R}^3 \quad (5)$$

- *Anisotropic Heterogeneous Media*:

$$\begin{pmatrix} M_{xx}\left(\mathbf{r}\right) & M_{xy}\left(\mathbf{r}\right) & M_{xz}\left(\mathbf{r}\right) \\ M_{xy}\left(\mathbf{r}\right) & M_{yy}\left(\mathbf{r}\right) & M_{yz}\left(\mathbf{r}\right) \\ M_{xz}\left(\mathbf{r}\right) & M_{yz}\left(\mathbf{r}\right) & M_{zz}\left(\mathbf{r}\right) \end{pmatrix} \quad (6)$$



Meshless Discretization For Anisotropic Heterogeneous Media

- *Mobility Decomposition*:

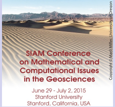
$$\begin{aligned} \mathbf{M}(\mathbf{r}) &= \mathbf{M}^S(\mathbf{r}) + \mathbf{M}^D(\mathbf{r}), \\ \mathbf{M}^S(\mathbf{r}) &= \tfrac{1}{3}\mathrm{tr}\left[\mathbf{M}\right]\cdot\mathbf{I},\ \ \mathrm{tr}\left[\mathbf{M}^D\right] = 0 \end{aligned} \tag{7}$$

- *Velocity Decomposition*:

$$\begin{aligned} \mathbf{V}(\mathbf{r}) &= \mathbf{V}^S(\mathbf{r}) + \mathbf{V}^D(\mathbf{r}) \\ \mathbf{V}^S(\mathbf{r}) &= -\mathbf{M}^S(\mathbf{r})\,\nabla p(\mathbf{r}), \\ \mathbf{V}^D(\mathbf{r}) &= -\mathbf{M}^D(\mathbf{r})\,\nabla p(\mathbf{r}) \end{aligned} \tag{8}$$

- *Divergence Decomposition*:

$$\nabla\mathbf{V}(\mathbf{r}) = \nabla\mathbf{V}^S(\mathbf{r}) + \nabla\mathbf{V}^D(\mathbf{r}) \tag{9}$$



Meshless Finite Difference Method

Seibold (2006)

- Construct neighbors (choose more neighbors than constraints)
- Select unique stencil satisfying additional requirements
- Solve optimization problem
- Compute monotone stencil
- Not a flexible way given pre-existing geology

Special Case

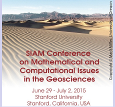
$$\mathbf{L}p = -\nabla^2 p(\mathbf{r}) - g(\mathbf{r})$$

$$\sum_s T_{SI}^M (\mathbf{r}_S - \mathbf{r}_I) = \mathbf{0}$$

$$\sum_s T_{SI}^M (\mathbf{r}_S - \mathbf{r}_I)(\mathbf{r}_S - \mathbf{r}_I) = \mathbf{I}$$

General Case

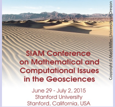
$$2\nabla(\phi(\mathbf{r})\nabla p(\mathbf{r})) = \nabla^2(\phi(\mathbf{r})p(\mathbf{r})) + \phi(\mathbf{r})\nabla^2 p(\mathbf{r}) - p(\mathbf{r})\nabla^2\phi(\mathbf{r})$$



SPH Discretization

- *Brookshaw (1985)*:

$$\langle \nabla \left(\mathbf{m}\left(\mathbf{r}\right) \nabla \mathbf{F}\left(\mathbf{r}\right)\right) \rangle =$$
$$\sum_{\Omega_{\mathbf{r},h}} V_{\mathbf{r}_J} \left[\mathbf{F}\left(\mathbf{r}_J\right) - \mathbf{F}\left(\mathbf{r}_I\right)\right] \frac{\left(\mathbf{r}_J - \mathbf{r}_I\right) \cdot \left(m_J + m_I\right) \nabla W\left(\mathbf{r}_J - \mathbf{r}_I, h\right)}{\left\|\mathbf{r}' - \mathbf{r}\right\|^2}$$



SPH Discretization

- *Schwaiger (2008)*:

$$\left(\frac{\Gamma_{kk}^{-1}}{n}\right)^{-1} \langle \nabla \left(\mathbf{m}\left(\mathbf{r}_I\right) \nabla \mathbf{F}\left(\mathbf{r}_I\right)\right)\rangle =$$

$$\sum_{\Omega_{\mathbf{r},h}} V_{\mathbf{r}_J} \left[\mathbf{F}\left(\mathbf{r}_J\right) - \mathbf{F}\left(\mathbf{r}_I\right)\right] \frac{\left(\mathbf{r}_J - \mathbf{r}_I\right)\cdot\left(m_J + m_I\right)\nabla W\left(\mathbf{r}_J - \mathbf{r}_I, h\right)}{\left\|\mathbf{r}' - \mathbf{r}\right\|^2}$$

$$-\left\{\left[\langle \mathbf{m}\left(\mathbf{r}_I\right)\mathbf{F}\left(\mathbf{r}_I\right)\rangle_\alpha - \mathbf{F}\left(\mathbf{r}_I\right)\langle \mathbf{m}\left(\mathbf{r}_I\right)\rangle_\alpha + \mathbf{m}\left(\mathbf{r}_I\right)\langle \mathbf{F}\left(\mathbf{r}_I\right)\rangle_\alpha\right]\mathbf{N}^\alpha\right\}$$

$$\langle \mathbf{F}\left(\mathbf{r}_I\right)\rangle_\alpha = \sum_{\Omega_{\mathbf{r},h}} V_{\mathbf{r}_J}\left[\mathbf{F}\left(\mathbf{r}_J\right) - \mathbf{F}\left(\mathbf{r}_I\right)\right]\nabla_\alpha W\left(\mathbf{r}_I - \mathbf{r}_J\right)$$

$$\mathbf{A} = \left[\sum_{\Omega_{\mathbf{r},h}} V_{\mathbf{r}_J}\left[\mathbf{r}_J - \mathbf{r}_I\right]\nabla_\alpha W\left(\mathbf{r}_I - \mathbf{r}_J\right)\right]^{-1}, \quad \nabla_\alpha^* W = \mathbf{A}_{\alpha\beta}\nabla_\beta W$$

Modified SPH Discretization*

- *Spherical Part*:

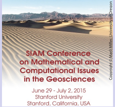
$$-\left(\frac{\Gamma_{kk}^{-1}}{n}\right)^{-1}\left\langle \nabla \mathbf{v}^{S}\left(\mathbf{r}\right)\right\rangle =$$

$$\sum_{\Omega_{\mathbf{r},h}} V_{\mathbf{r}_J}\cdot M^{eff}\cdot\left[\mathbf{F}\left(\mathbf{r}_J\right)-\mathbf{F}\left(\mathbf{r}_I\right)\right]\frac{\left(\mathbf{r}_J^{\alpha}-\mathbf{r}_I^{\alpha}\right)\cdot\overline{\nabla_{\alpha}W}\left(\mathbf{r}_J-\mathbf{r}_I,h\right)}{\left\|\mathbf{r}'-\mathbf{r}\right\|^{2}}-$$

$$-\left(\sum_{\Omega_{\mathbf{r},h}} V_{\mathbf{r}_J}\cdot M^{S}_{eff}\cdot\left[\mathbf{F}\left(\mathbf{r}_J\right)-\mathbf{F}\left(\mathbf{r}_I\right)\right]\overline{\nabla_{\alpha}W}\left(\mathbf{r}_J-\mathbf{r}_I,h\right)\right)\widetilde{\mathbf{N}}^{\alpha}$$

$$M^{S}_{eff}=\left(\frac{M^{S}\left(\mathbf{r}_J\right)\cdot M^{S}\left(\mathbf{r}_I\right)}{M^{S}\left(\mathbf{r}_J\right)+M^{S}\left(\mathbf{r}_I\right)}\right)$$

*Lukyanov, Vuik, JCP, To be submitted.



Modified SPH Discretization†

- *Deviatoric Part*:

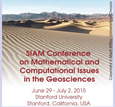
  Lukyanov(2010), Lukyanov (2012)

$$\langle \mathbf{v}^D_\gamma(\mathbf{r}_I)\rangle = -\mathbf{M}^D_{\gamma\alpha}\langle p(\mathbf{r}_I)\rangle_{,\alpha}$$

$$\langle p(\mathbf{r}_I)\rangle = \sum_{\Omega_{\mathbf{r},h}} V_{\mathbf{r}_J}\left[\langle p(\mathbf{r}_J)\rangle - \langle p(\mathbf{r}_I)\rangle\right]\overline{\nabla W}(\mathbf{r}_J-\mathbf{r}_I,h)$$

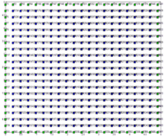
$$\langle \nabla \mathbf{v}^D_\gamma(\mathbf{r})\rangle_{,\gamma} = \sum_{\Omega_{\mathbf{r},h}} V_{\mathbf{r}_J}\left[\langle \mathbf{v}^D_\gamma(\mathbf{r}_J)\rangle - \langle \mathbf{v}^D_\gamma(\mathbf{r}_I)\rangle\right]\overline{\nabla_\gamma W}(\mathbf{r}_J-\mathbf{r}_I,h)$$

† Lukyanov, Vuik, JCP, To be submitted.

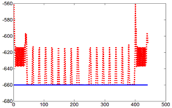


Results

- *For Deviatoric Scheme*:

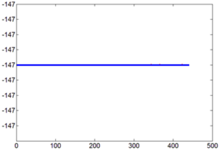


$$\bar{\mathbf{K}}_{ij}(\vec{\mathbf{X}}_I)=\begin{bmatrix}6 & 3\\ 3 & 5\end{bmatrix} \quad p\left(\vec{\mathbf{X}}\right)=110\cdot x\cdot y$$

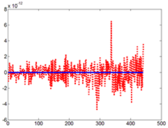


$$\bar{\mathbf{K}}_{ij}(\vec{\mathbf{X}}_I)=\begin{bmatrix}10+10x+6y & 2+2x+2y\\ 2+2x+2y & 4+4x+y\end{bmatrix}$$

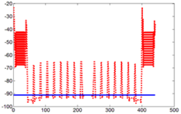
$$p\left(\vec{\mathbf{X}}\right)=11x+5y+17$$

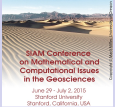


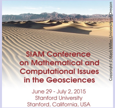
$$\bar{\mathbf{K}}_{ij}(\vec{\mathbf{X}}_I)=\begin{bmatrix}6 & 3\\ 3 & 5\end{bmatrix} \quad p\left(\vec{\mathbf{X}}\right)=11x+5y+17$$



$$\bar{\mathbf{K}}_{ij}(\vec{\mathbf{X}}_I)=\begin{bmatrix}6 & 3\\ 3 & 5\end{bmatrix} \quad p\left(\vec{\mathbf{X}}\right)=0.5\cdot\left(11x^2+5y^2+17\right)$$

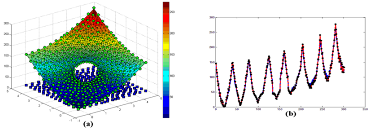






Results

- *For Full Scheme*:



Tensor	Particle Distribution	C_p	α_p	C_u	α_u
D	Uniform	0.348	1.991	0.325	1.891
D	Weakly Distorted	0.231	1.923	0.247	1.873
D	Highly Distorted	0.257	1.732	0.257	1.638
N	Uniform	0.391	1.990	0.301	1.872
N	Weakly Distorted	0.272	1.919	0.216	1.803
N	Highly Distorted	0.293	1.727	0.225	1.612

Table: Convergence rates for the relatively simple Dirichlet problem $\|p - p_h\| \leq C_p h^{\alpha_p}$ and $\|u - u_h\| \leq C_u h^{\alpha_u}$.

Monotonicity Issue

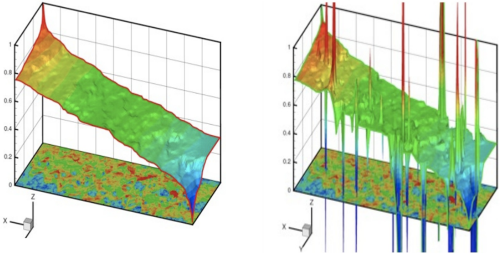


Figure: Monotonicity Issue (H. Hajibeygi, 2014).

Monotonicity Issue: Bouchon (2006)

Theorem

Let $\mathbf{A} = [a_{ij}]$ and $\tilde{\mathbf{A}} = [\tilde{a}_{ij}]$ be two real square matrices of dimension n, with the following properties:

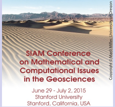
- $\mathbf{A} = [a_{ij}]$ is an irreducibly diagonal dominant M-matrix
- $\tilde{\mathbf{A}} \cdot \mathbf{I}$

If $\left\|\tilde{\mathbf{A}}\right\|_{\infty} < Cm(\mathbf{A})$ with $C = \dfrac{1}{(\eta)^{M} \cdot M \cdot e}$ then the matrix $\mathbf{A} + \tilde{\mathbf{A}}$ is monotone. Moreover $\left(\mathbf{A} + \tilde{\mathbf{A}}\right)^{-1} >> 0$

Where

$$\|\mathbf{A}\|_{\infty} = \sup_{x \neq 0} \frac{\|\mathbf{A}x\|}{\|x\|} = \max_{i=1,\dots,n} \left(\sum_j |a_{ij}|\right)$$

$$m(\mathbf{A}) = \min_{i=1,\dots,n} (|a_{ii}|), \quad \eta(\mathbf{A}) = \max_{i,j=1,\dots,n} \left(\frac{|a_{ii}|}{|a_{ij}|}\right)$$



Take Away

- It is required to have further numerical analysis of the method.
- Due to the meshfree particle nature of the method, it is not always straightforward to directly apply the techniques that were developed for mesh-based Eulerian or Lagrangian methods.
- The issues related to the stability, accuracy and convergence are understood for uniformly distributed particles and some times for only one-dimensional cases.
- It is not yet very well clear how the particle irregularity affects the accuracy of the solution.
- Monotonicity issue has to be investigated.